Semigroup Forum Vol. 16 (1978) 89-95

RESEARCH ARTICLE

CONVERGENCE RATES OF ERGODIC LIMITS FOR SEMIGROUPS AND COSINE FUNCTIONS*

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Communicated by Karl H. Hofmann

Suppose $\{E_t | t \geq 0\}$ is a uniformly bounded strongly continuous semigroup of operators on the Banach space X and denote its generator by L. We show the averages $(1/T) \int_0^T E_t\,dt$ converge in the strong operator topology as $T \to \infty$ at the rate $\mathcal{O}(1/T)$ on the direct sum $R(L) \oplus K(L)$, that they converge on the closed subspace $\overline{R(L)} \oplus K(L)$ to a bounded projection, and this subspace is all of X when X is reflexive. For cosine functions we also show the first two results in general, and the third for X reflexive.

1. INTRODUCTION

Let L be the generator of a uniformly bounded strongly continuous semigroup of linear operators $\{E_t | t \geq 0\}$ on the real or complex Banach space X. We denote by R(B) and K(B) the range and kernel, respectively, of the operator B. Our first interest is the (ergodic) limit as $T \to \infty$ of the average given by the strong integral $A_T \equiv (1/T) \int_0^T E_t\,dt$. We show A_T converges at the rate $\mathcal{O}(1/T)$ on the direct sum $R(L) \oplus K(L)$, that it converges on the closed subspace $X_0 \equiv \overline{R(L)} \oplus K(L)$ to a bounded projection, and that $X_0 = X$ when X is reflexive.

* Research supported in part by the National Science Foundation.

0037-1912/78/0016-0089 $01.40

The semigroup $\{E_t \mid t \geq 0\}$ is the exponential function by which the solution of the Cauchy problem

$$u'(t) = Lu(t),\ t > 0,\ u(0) = x$$

is represented in the form $u(t) = E_t x$ (see [3,5]). In a similar manner, the solution of the second order Cauchy problem

$$v''(t) = Av(t),\ t \in \mathbb{R},\quad v(0) = x,\ v'(0) = y$$

is represented in the form $v(t) = C_t x + S_t y$, where we assume that A is the generator of a uniformly bounded strongly continuous cosine function $\{C_t \mid t \in \mathbb{R}\}$ on X. The corresponding sine function is given by the strong integral $S_t = \int_0^t C_s\, ds$; see e.g. [1] or [7] for details. We show that the strong integral average $(1/T) \int_0^T C_t\, dt$ converges at the rate $\mathcal{O}(1/T)$ on $R(A) \oplus K(A)$, that it converges on the closed subspace $X_1 \equiv \overline{R(A)} \oplus K(A)$ to a bounded projection, and that $X_1 = X$ when X is reflexive.

2. SEMIGROUPS

Our key result is

<u>THEOREM</u> 1. *For* $x = y + z$ *in the direct sum* $R(L) \oplus K(L) \subseteq X$,

$$\|A_T x - z\| = \mathcal{O}(1/T) \quad \textit{as} \quad T \to \infty.$$

PROOF. Let $y = Lw$. Then

$$A_T x - z = A_T y = (1/T) \int_0^T \frac{d}{dt} (E_t w)\, dt = (E_T w - w)/T.$$

And since $A_T y \longrightarrow 0$ for y in $R(L)$ while $A_T z = z$ for z in $K(L)$, the direct sum is well defined. ▯

<u>COROLLARY</u> 1. $X_0 \equiv \overline{R(L)} \oplus K(L)$ *is closed in* X, *and for each* $x = y + z \in X_0$,

$$\|A_T x - z\| \to 0 \quad \textit{as} \quad T \to \infty.$$

PROOF. First we note the simple fact that the set of x in X, for which $A_T x$ is strongly convergent, is closed. Next we define the projection P by

$$(X_0 \subseteq)\ \overline{R(L) \oplus K(L)} \ni x \xrightarrow{P} \text{strong} \lim_{T\to\infty} A_T x \in K(L)$$

and note that $\|P\| \le \sup_{t\ge 0} \|E_t\| < \infty$. Therefore

$$P(\overline{R(L)}) = \{0\}$$

(so $\overline{R(L)} \cap K(L) = \{0\}$ and the direct sum is well defined), and if $y_n \in \overline{R(L)}$, $z_n \in K(L)$ and $y_n + z_n \longrightarrow v \in X$ then $P(y_n + z_n) = z_n \longrightarrow Pv$, so $\{z_n\}$, and therefore $\{y_n\}$, is Cauchy which proves that $\overline{R(L)} \oplus K(L)$ is closed. □

With a further assumption we obtain

COROLLARY 2. (THE MEAN ERGODIC THEOREM FOR REFLEXIVE BANACH SPACES) *If* X *is reflexive then* $\overline{R(L)} \oplus K(L) = X$, *so that for each* x *in* X, $A_T x$ *converges strongly as* $T \to \infty$.

PROOF. Using the canonical identification of X with its double dual X^{**} we have for $L = L^{**}$ [2],

$$K(L)^{\perp} = \overline{R(L^*)},\ \overline{R(L)}^{\perp} = K(L^*)\ ,$$
$$K(L^*)^{\perp} = \overline{R(L)},\ \overline{R(L^*)}^{\perp} = K(L)\ ,$$

where $Y^{\perp}$ denotes the annihilator of $Y \subseteq X$ in X^*. We also need the standard facts [4] that, since X is reflexive, $\{E_t^* \mid t \ge 0\}$ constitutes a uniformly bounded, strongly continuous semigroup on X^*, with generator L^*. Now assuming that $\overline{R(L)} \oplus K(L) \ne X$, we have

$$K(L)^{\perp} \cap \overline{R(L)}^{\perp} \supseteq [\overline{R(L)} \oplus K(L)]^{\perp} \ne \{0\}.$$

But then $\overline{R(L^*)} \cap K(L^*) \ne \{0\}$ which contradicts (1) applied to $\{E_t^*\}$. □

REMARKS AND EXAMPLES. (a) The convergence rate $O(1/T)$ can be lost in taking the closure of $R(L) \oplus K(L)$. For example, let $X = C_0[1,\infty)$, the bounded continuous functions on $[1,\infty)$, vanishing at ∞, with the supremum norm, and take $(E_t x)(s) = x(t + s)$. Then $L = d/ds$ and $x_0(s) \equiv 1/s$ satisfies $x_0 \in \overline{R(L)} = X$ but $\|A_T x_0\| = (1/T)\log(1 + T)$.

(b) It is well known (and easily verified) that $A_T x$ converges at the rate $O(1/T)$ for those x in $\cup\{R(I - E_t) \mid t \geq 0\}$. However the identity $L \int_0^T E_t\, x\, dt = E_T x - x$ shows $\cup\{R(I - E_t) \mid t \geq 0\} \subseteq R(L)$. This containment can be proper. For example, consider $(E_t x)(s) = \exp(ist) \cdot x(s)$ on $L^2(R,ds)$ and $x_0(s) = s \cdot \exp(-s^2)$.

3. COSINE FUNCTIONS

Our corresponding results for cosine and sine functions are given in the following theorem.

<u>THEOREM</u> 2. *For* $x = y + z$ *in the direct sum* $R(A) \oplus K(A) \subseteq X$,

$$\left\|\frac{1}{T} \int_0^T C_t\, x\, dt - z\right\| = O\left(\frac{1}{|T|}\right),$$

$$\left\|\frac{1}{T} \int_0^T S_t\, x\, dt - \left(\frac{T}{2}\right) z\right\| = O\left(\frac{1}{|T|}\right) \quad \text{as} \quad |T| \to \infty.$$

PROOF. Since C_t is even and S_t is odd, it suffices to verify the above for $T \to \infty$. Setting $y = Aw$, we integrate the identity $\frac{d}{dt} C_t w = AS_t w = S_t y$ to obtain

$$\frac{1}{T} \int_0^T S_t y\, dt = \frac{1}{T}(C_T w - w) = O\left(\frac{1}{T}\right).$$

Since $\frac{d^2}{dt^2} C_t w = C_t Aw$ is bounded on $t \geq 0$ we obtain from [4] the estimate

$$\sup_{t\geq 0} \|S_t(Aw)\| \leq 4 \sup_{t\geq 0} \|C_t\, Aw\| \cdot \sup \|C_t w\|.$$

This shows $S_t y$ is bounded on $t \geq 0$ and the definition of S_t then gives

$$\frac{1}{T}\int_0^T C_t\, y\,dt = \frac{1}{T} S_T y = O(\frac{1}{T}).$$

Since $z \in K(A)$, $\frac{d}{dt} C_t z = AS_t z = 0$, so $C_t z = z$ and $S_t z = t z$ for $t \in \mathbb{R}$. Thus we obtain

$$\frac{1}{T}\int_0^T C_t\, z\, dt = z\ , \quad \frac{1}{T}\int_0^T S_t\, z\, dt = (\frac{T}{2})\ z.$$

Since $\{S_t\}$ is uniformly bounded on each point in $R(A)$, the preceding computation shows that $R(A) \cap K(A) = \{0\}$. ▯

By replacing the semigroup in the proof of Corollary 1 by the cosine function we obtain a proof of

COROLLARY 3. $X_1 \equiv \overline{R(A)} \oplus K(A)$ *is closed in* X *and for each* $x = y + z \in X_1$

$$\left\|\frac{1}{T}\int_0^T C_t\, x\, dt - z\right\| \to 0 \quad \text{as} \quad |T| \to \infty.$$

By restricting ourselves to reflexive spaces we obtain a mean ergodic theorem for cosine functions.

COROLLARY 4. *If* X *is reflexive then* $\overline{R(A)} \oplus K(A) = X$, *so for each* $x \in X$, $\frac{1}{T}\int_0^T C_t\, x\, dt$ *converges strongly as* $T \to \infty$.

PROOF. A generates a uniformly bounded strongly continuous semigroup given by

$$E_t x = (\pi t)^{-1/2} \int_0^\infty e^{-s^2/4t}\, C_s\, x\, ds$$

(cf. [1]). The result now follows from the proof of Corollary 2. ▯

REMARKS AND EXAMPLES. (c) Theorem 2 shows that $\{S_t\}$ is pointwise bounded on $R(A)$ and unbounded on non-zero points in $K(A)$. The example of $A \equiv 0$ on $X = \mathbb{R}$ shows we may have $S_t x$ bounded if and only if $x \in R(A)$. Moreover, even when $K(A) = \{0\}$, we do not necessarily have $\{S_t\}$ pointwise bounded on $\overline{R(A)}$. For example, on the Hilbert space $X = L^2(\mathbb{R})$ the cosine function of contractions $C_t x(s) = (1/2)(x(s+t) + x(s-t))$ has generator $A = \frac{d^2}{ds^2}$ with $\overline{R(A)} = X$ and $K(A) = \{0\}$. The functions $x_n \in X$ defined as $1/\sqrt{2n}$ on $[-n,n]$ and zero elsewhere on $\mathbb{R}$ satisfy $\|x_n\| = 1$ and $\|S_t x_n\| \geq n$ for $t \geq 2n$. If $\{S_t x\}$ were bounded for each $x \in \overline{R(A)} = X$, then by the uniform boundedness principle we would have a contradiction.

(d) Frequently one can show A generates a cosine function on X by verifying that $L \equiv \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ generates a semigroup on a suitable product space; cf. [7, Proposition 2.6]. However, an example in [6] shows this reduction to the semigroup case is not always possible, so the results of Section 3 cannot be obtained in general from their counterparts in Section 2.

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Received on August 22, 1977 and in final form January 27, 1978.