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LETTER TO THE EDITOR

Fractal symmetry in an Ising model

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Abstract. We give an example of a short-range Ising model with a unique ground state with two unusual properties: it has some continuous spectrum, and it has fractal symmetry.

From experience with models, and from the vast amount of data concerning real solids, it seems there is a strong tendency for low-temperature matter to be highly ordered and, in fact, crystalline. Understanding this phenomenon from the statistical mechanics of particles (electrons and nuclei) interacting through Coulomb forces or, using molecules for the particles, through short-range forces, is a well known and important unsolved problem [1–5]. See [6] for a recent review.

Part of the difficulty of the problem is that one can only get detailed results with a handful of models. It is only recently, for example, that simple nearest-neighbour lattice gas models have been found (see [6]) which do not have crystalline ground states; they have 'quasicrystalline' ground states in a manner appropriate to real quasicrystals. Even these models are highly ordered, and are very close to being crystalline. The purpose of this letter is to give an example of a short-range Ising model which has a unique ground state which is much further from the crystalline examples than ever found before: two of these features of the ground state are that it has some continuous spectrum, and that it has perfect fractal symmetry in the sense that the structure of the ground state is invariant under certain scale changes.

Our model is an Ising spin system on the one-dimensional lattice $\boldsymbol{Z}$. The spin variable at site j of $\boldsymbol{Z}$ will be denoted σ_j, and can attain the values ± 1. The Hamiltonian is a translation-invariant four-body interaction of the general form

$$H = \sum_{j,k,m,n} J_{j,k,m,n}(\sigma_j + \sigma_k)^2(\sigma_m + \sigma_n)^2$$

with a specific non-negative coupling J. To be more precise,

$$H = \sum_{j=-\infty}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} V(j, p, r)$$

$$= \sum_{j=-\infty}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \exp[-(r+p)^2](\sigma_j + \sigma_{j+2^r})^2(\sigma_{j+(2p+1)2^r} + \sigma_{j+(2p+2)2^r})^2.$$

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We will show that the ground state of this model is non-degenerate, has both discrete and continuous spectra, and has perfect fractal symmetry. The ground state will in fact be shown to be the Morse configuration (see [7]):

$$\ldots 1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1, \ldots.$$

The fractal symmetry of this configuration can be seen, in some sense (but this will have to be refined below), by noting that if one looks only at the first, third, fifth spin etc in the above list, one recovers precisely the original full list.

To define this non-periodic configuration completely we will need some notation. First we define the map S by $S(1) = (1, -1)$, $S(-1) = (-1, 1)$. Then the Morse configuration $\sigma^{\mathrm{M}} = \{\sigma_j^{\mathrm{M}} | -\infty < j < \infty\}$ is defined inductively by: $(\sigma_0^{\mathrm{M}}, \sigma_1^{\mathrm{M}}) = S(1)$, $(\sigma_{2^{j+1}}^{\mathrm{M}}, \ldots, \sigma_{2^{j+2}-1}^{\mathrm{M}}) = (S(\sigma_{2^j}^{\mathrm{M}}), \ldots, S(\sigma_{2^{j+1}-1}^{\mathrm{M}}))$ for $j \geq 0$, and $\sigma_j^{\mathrm{M}} = \sigma_{-j-1}^{\mathrm{M}}$ for $j < 0$.

As we will see, we must work not just with this ground-state configuration σ^{M} but with what we will now distinguish as the ground state itself, namely the ensemble, or probability distribution on the set of all configurations, that one gets from the canonical ensemble of the model by taking the limit as temperature goes to zero. (From here on, a *configuration* will always mean some specific set of spin values at all the sites of $\boldsymbol{Z}$, and by a *state* we will always refer to some statistical ensemble of configurations.) Instead of showing directly that σ^{M} is a ground-state configuration, we will instead show that the ground state is what we call the Morse state m^{M}, which is the unique ensemble in which n-point correlations are computed by taking translation averages in σ^{M}. (That is, the probability of some pattern of spin values, for example the pattern where two consecutive spin values of 1 are followed five sites to the right by -1, is given, in the state m^{M}, by the fraction of times that the pattern appears in the spin configuration σ^{M}.) We now state our result in a theorem.

Theorem. The model has only one translation-invariant ground state and it is the Morse state m^{M}.

Proof. Thinking of the Morse configuration in the set Γ of all (doubly infinite) configurations of 1s and -1s, equiped with the usual mathematical structure, we are interested in the set M which consists of all configurations in Γ that can be obtained as limits of lattice translations of σ^{M}: the so-called orbit closure of σ^{M}. (This set M will be shown to have probability 1 with respect to the ground state of the model. See [7] for a useful characterisation of M.) One of the main steps in our argument is the following technical lemma.

Lemma. The configurations σ in M are precisely those which satisfy the following condition:

Condition M: Either $\sigma_j \neq \sigma_k$ or $\sigma_m \neq \sigma_n$ whenever $k = j + 2^r$, $n = m + 2^r$ *and*

$$m = j + (2p+1)2^r \text{ for any } r \geq 0 \text{ and } p \geq 0.$$

Postponing the proof of this lemma temporarily, we continue with the proof of the theorem. It is known [8] that there is only one possible translation-invariant probability distribution, namely m^{M}, on M, and it then follows from the lemma that m^{M} is characterised among translation-invariant probability distributions on Γ by giving probability zero to all sets $A(j, p, r)$ for j in $\boldsymbol{Z}$, $p \geq 0$ and $r \geq 0$, where

$$A(j, p, r) = \{\sigma \in \Gamma | [\sigma_j + \sigma_{j+2^r}]^2 [\sigma_{j+(2p+1)2^r} + \sigma_{j+(2p+2)2^r}]^2 \neq 0\}.$$

Now consider any translation-invariant ground state of our model, that is any (weak-*) limit m, as temperature goes to zero, of finite-temperature equilibrium states. (More generally, m could be any translation-invariant probability distribution on the so-called ground-state configurations of our model; see [6, 9].) We want to show that $m = m^{\mathrm{M}}$. We know [9] that m must have minimum energy density, i.e. it must satisfy

$$\sum_{p=0}^{\infty}\sum_{r=0}^{\infty} m[A(0,p,r)]V(0,p,r)$$

$$= \inf_{m'} \sum_{p=0}^{\infty}\sum_{r=0}^{\infty} m'[A(0,p,r)]V(0,p,r)$$

$$= \sum_{p=0}^{\infty}\sum_{r=0}^{\infty} m^{\mathrm{M}}[A(0,p,r)]V(0,p,r) = 0$$

where the infimum is over all translation-invariant probability distributions m' on Γ. It follows immediately that $m = m^{\mathrm{M}}$, and thus our proof will be complete once we prove the lemma. As one might expect from the fact that the structure of σ^{M} is somewhat complicated, our argument will now need to get rather technical.

Proof of the lemma. Let B be some t-tuple $(b_1, b_2, \ldots, b_t)$ of spin values, i.e. 1s and -1s, and then let $C(B)$ be the $(2t+1)$-tuple $(c_1, c_2, \ldots, c_{2t+1})$ where $c_j = b_j$ for $1 \leqslant j \leqslant t$, $c_j = b_{j-t} (= c_{j-t})$ for $t+1 \leqslant j \leqslant 2t$ and $c_{2t+1} = b_1 (= c_1)$. It is known [8] that M is precisely the set of all configurations σ for which the following condition holds:

Condition G–H: The pattern $C(B)$ never appears in σ for any t or any B.

So we must show that condition M is equivalent to condition G–H. Assume σ satisfies condition M, and assume σ contains some $C(B)$ where B contains $t = (2p+1)2^r$ terms for some $p \geqslant 1$ and $r \geqslant 0$: $B = (b_1, \ldots, b_{(2p+1)2^r})$. Because condition M holds and $c_{j+t} = c_j$ for $1 \leqslant j \leqslant t+1$, we must have $c_{1+2^r} = -c_1$. Repeating this argument, we find that $c_{1+2p2^r} = c_1$. But then again, using the fact that $c_{j+t} = c_j$ for $1 \leqslant j \leqslant t+1$, we find that condition M fails when $j = 1+2p2^r$, $k = 1+(2p+1)2^r$, $m = 1+(4p+1)2^r$ and $n = 1+(4p+2)2^r$. This proves the implication for cases where B contains $t = (2p+1)2^r$ terms for some $p \geqslant 1$ and $r \geqslant 0$. Now consider the cases where B contains $t = 2^r$ terms for some $r \geqslant 0$. Using the fact that $c_1 = c_{1+2^r} = c_{1+2^r+2^r}$ we see that again condition M fails, now when $p = 0$ and so $k = m$. This completes the implication: condition M implies condition G–H. Now assume that σ satisfies condition G–H. This implies [8] that σ belongs to M, and therefore [7] that σ contains precisely the same patterns as σ^{M}. So we only need to prove that σ^{M} satisfies condition M. From the definition of σ^{M} we note that $\sigma^{\mathrm{M}}_{2j+1} = -\sigma^{\mathrm{M}}_{2j}$ for all j. This implies that in the (one-sided) Morse sequence $\sigma^{+\mathrm{M}} = \{\sigma^{+\mathrm{M}}_j | j \geqslant 0\}$, where $\sigma^{+\mathrm{M}}_j \equiv \sigma^{\mathrm{M}}_j$ for $j \geqslant 0$, of the two subsequences consisting of the terms at distance 2 apart, $\sigma(1)^{+\mathrm{M}}_j \equiv \sigma^{+\mathrm{M}}_{2j}$ and $\sigma(2)^{+\mathrm{M}}_j \equiv \sigma^{+\mathrm{M}}_{2j+1}$, one is a spread-out copy of the Morse sequence $(\sigma(1)^{+\mathrm{M}}_j = \sigma^{+\mathrm{M}}_j)$ and the other is a spread-out copy of the negative of the Morse sequence $(\sigma(2)^{+\mathrm{M}}_j = -\sigma^{+\mathrm{M}}_j)$. From the construction of σ^{M} this is also true of the four sub-sequences consisting of the terms of $\sigma^{+\mathrm{M}}$ which are separated by distance 4, and in general the 2^r sub-sequences with terms separated by distance 2^r for any $r \geqslant 0$. (The fact that this argument only works for the $j \geqslant 0$ terms of σ^{M} is of some significance, as will be noted below.) Now we use the fact that the configuration σ^{M} is 'almost periodic' (see [7]) to see that the patterns which appear in it are precisely the same as those that appear in the sequence $\sigma^{+\mathrm{M}}$ (and they appear with the same frequencies). Now the fact noted above, that $\sigma^{\mathrm{M}}_{2j+1} = -\sigma^{\mathrm{M}}_{2j}$ for all j, implies that any two pairs of pairs of identical adjacent terms (such as $\{\sigma_j^{\mathrm{M}}, \sigma_{j+1}^{\mathrm{M}}\}$ and $\{\sigma_k^{\mathrm{M}}, \sigma_{k+1}^{\mathrm{M}}\}$, where $\sigma_j^{\mathrm{M}} = \sigma_{j+1}^{\mathrm{M}}$ and $\sigma_k^{\mathrm{M}} = \sigma_{k+1}^{\mathrm{M}}$) must as pairs require a translation by an even number of sites to coincide (that is, $j-k$ must be even), never an odd number of sites. And the above argument shows this same fact to be true for those pairs of pairs of like terms which instead of consisting of adjacent terms have the terms a distance 2^r apart for some $r \geqslant 0$. This proves that σ^{M} satisfies condition M, and completes the proof of the lemma, and therefore the theorem.

Note that in our proofs no use is made of the magnitude of $V(0, p, r)$ when it is positive; this shows that models with the desired properties exist with any desired rate of decay in p and r. (It may also be of interest that our proof of the lemma is easily altered to prove that condition G–H is equivalent to condition M', which is the same as condition M but with 'either $\sigma_j \neq \sigma_k$ or $\sigma_m \neq \sigma_n$' replaced by 'either $\sigma_j \neq \sigma_k$ or $\sigma_m \neq \sigma_n$ or $\sigma_j \neq \sigma_m$'.) Finally we need to justify our claims about the spectrum and fractal symmetry. The spectrum of m^{M} (defined as usual as the spectrum of lattice translation, acting as a unitary operator on the Hilbert space $L_2(\Gamma)$ with respect to the measure m^{M}) is well known; see, for example, [10]. And the fractal symmetry, which can now be stated in the sense that all correlation functions

$$\langle \sigma_{j_1} = a_1, \sigma_{j_2} = a_2, \ldots, \sigma_{j_n} = a_n \rangle$$

computed in the state m^{M} are invariant under the transformation whereby distances between sites are doubled, follows exactly as in the second half of the proof of the lemma.

In summary we note that we have exhibited a statistical mechanical model of standard type (an Ising model), with translation-invariant (four-body) interactions of arbitrarily short range, for which the ground state is unique and has at least two properties of significance not found in previous standard statistical mechanical models (see, however, [11], which uses many-body interactions of all orders, not just four-body): the ground state has some continuous spectrum (as well as a dense set of eigenvalues)—of interest for example for its connection with the idea of 'turbulent crystals' (see [12])—and it has fractal symmetry—which is of interest as, in some sense, a derivation of this important symmetry from traditional microscopic laws. In this regard it may be of note that the exact symmetry is not found in any of the actual ground-state configurations of the model, such as σ^{M}, but in the state m^{M}. That is, the invariance of σ^{M} discussed at the beginning of this letter actually only holds for either the half space $j \geqslant 0$ or the half space $j \leqslant 0$ (there is a necessary 'defect' of sorts at some site, in this case at $j = 0$) and to get the symmetry for the whole space $\boldsymbol{Z}$ we are forced to consider the symmetry on the ground *state* m^{M} and its correlation functions, instead of any ground state *configuration* such as σ^{M}.

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