RASHOMON

Pavages et rotations

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We have all seen how illuminating it can be to view something, even something one knows very well, from an unusual direction. I will describe here a mathematical example of this which is somewhat unusual in the number of views used, and, particularly, their diversity. The story I will tell is how the modelling of quasicrystals led, by a circuitous route, to a classification of the "generalized dihedral" groups, realizable as the subgroups of SO(3) generated by rotation about the x axis by $2\pi/p$ and rotation about the y axis by $2\pi/q$, where $p, q \geq 2$.

A discovery was made in 1984 [SBG] of an aluminum-manganese alloy[1] with a puzzling characteristic. What was odd about the alloy (since called a quasicrystal) was that when its atomic structure was probed by electron diffraction some of the diffraction patterns looked like those produced by ordinary (crystalline) solids – a large collection of dark spots surrounding a big central spot – except they had 10-fold rotational symmetry about the central spot, known to be impossible for ordinary crystals from the classification of the crystallographic groups. However a paper had been published two years previously by the crystallographer Alan Mackay [Mac] showing that if a material had an atomic structure associated in any simple way with a 3-dimensional version of the "kite[2]

[1] Alloy → alliage
[2] Kite → cerf-volant

& dart[3]" tilings (discussed below) of Roger Penrose, it would exhibit diffraction patterns with just such "forbidden" 10-fold rotational symmetry. This led to an explosion of work, continuing to this day, mostly by physicists, crystallographers and discrete geometers, intent on understanding structures like the Penrose tilings; the new subject is called "aperiodic tiling". To proceed it is necessary to give some background on the Penrose tilings and aperiodic tiling.

Around 1960 the philosopher Hao Wang was analyzing a class of of predicate calculus formulas, those which begin with the structure "For all x there exists y such that for all $z\ldots$", ending with a combination of predicates without any further quantifiers. To analyze such "AEA" formulas he invented [Wan] what he called the "domino game", as follows.

Imagine you have some finite collection $\mathcal{B}$ of unit square "basic tiles", for each of which the four edges are colored in some specific way, not necessarily all four the same color. The tiles are given with their edges parallel to some set of orthogonal axes, and you have access to an unlimited number of copies of each colored tile in $\mathcal{B}$ (similarly aligned).

The domino game consists of translating the tiles (without rotation or reflection), and abutting[4] them full edge to full edge (as is usual in a floor tiling) trying to fill up the whole Euclidean plane, but only allowing edges with the same color to abut. For some collections $\mathcal{B}$ of colored-edge tiles this would be easy to do – for instance if $\mathcal{B}$ contained only one tile, all edges black, while for some collections it is clearly impossible – for instance if $\mathcal{B}$ contained only one tile, three edges black and one edge white. (Remember we may not rotate the tiles, so the white edge cannot be placed against another tile.)

Wang considered the question of whether or not there could be an *algorithm* for deciding whether any possible finite set $\mathcal{B}$ of colored-edge tiles could be used as the basis for a tiling of the plane. He proved several things. He proved that if there could be no algorithm for this domino game then there was no algorithm for deciding whether any possible AEA formula is self contradictory (his real interest was AEA formulas of course). He also proved that if there was a set $\mathcal{B}$ of colored-edge tiles which could be used to tile the plane but not in any periodic way, then there could be no algorithm for the domino game and therefore no algorithm for AEA formulas. (If you associate a tile $b_{(j,k)}$ with each point (j,k) in the square lattice $\mathbb{Z}^2$, the tiling $\{b_{(j,k)} : j,k \in \mathbb{Z}^2\}$ is called periodic if there exist $J, K \in \mathbb{Z}$ such that $b_{(j,k)} = b_{(j+J,k+K)}$ for all $j,k \in \mathbb{Z}$.) In 1967 Wang's student Robert Berger published his Ph.D. thesis

[3]Dart → pointe de flèche

[4]To abut → abouter

[Ber], in applied mathematics, containing an elegant solution : an explicit example of a set $\mathcal{B}$ of colored-edge tiles (a bit more than 20,000 different tiles) with which one could tile the plane but only nonperiodically.

Wang had by then already proven the undecidability of AEA formulas by a slightly more circuitous route, but Berger's counterintuitive example took on a life of its own. In particular, over the next decade Berger's example was simplified more and more, continually reducing the number of different tiles needed for such an example. Then in 1977 a new sort of example was produced by Penrose [Gar], the "kite & dart" tilings. (Instead of requiring that abutting edges have matching colors, it is sometimes more convenient to incorporate small bumps and dents in the edges of tiles, as is done in jigsaw puzzle pieces, to enforce the requirement that only "matching " edges abut in a tiling.) The tiles used in kite & dart tilings are shown in Fig. 1, and a portion of such a tiling is shown in Fig. 2, but without showing the bumps and dents.

One feature of tilings such as those of Berger and Penrose is worth emphasizing. Although there are infinitely many kite & dart tilings distinct in the sense that no one is a translate of another, this is in a way misleading. The kite & dart tilings are all "locally identical" in that any pattern of tiles you find in any finite region of one kite & dart tiling appears in some region of any other kite & dart tiling; in that sense there is a (locally) unique but complicated structure being forced by the information contained in the bumpy edges of the tiles of Fig. 1.

The only essential difference between the tilings of Penrose and those

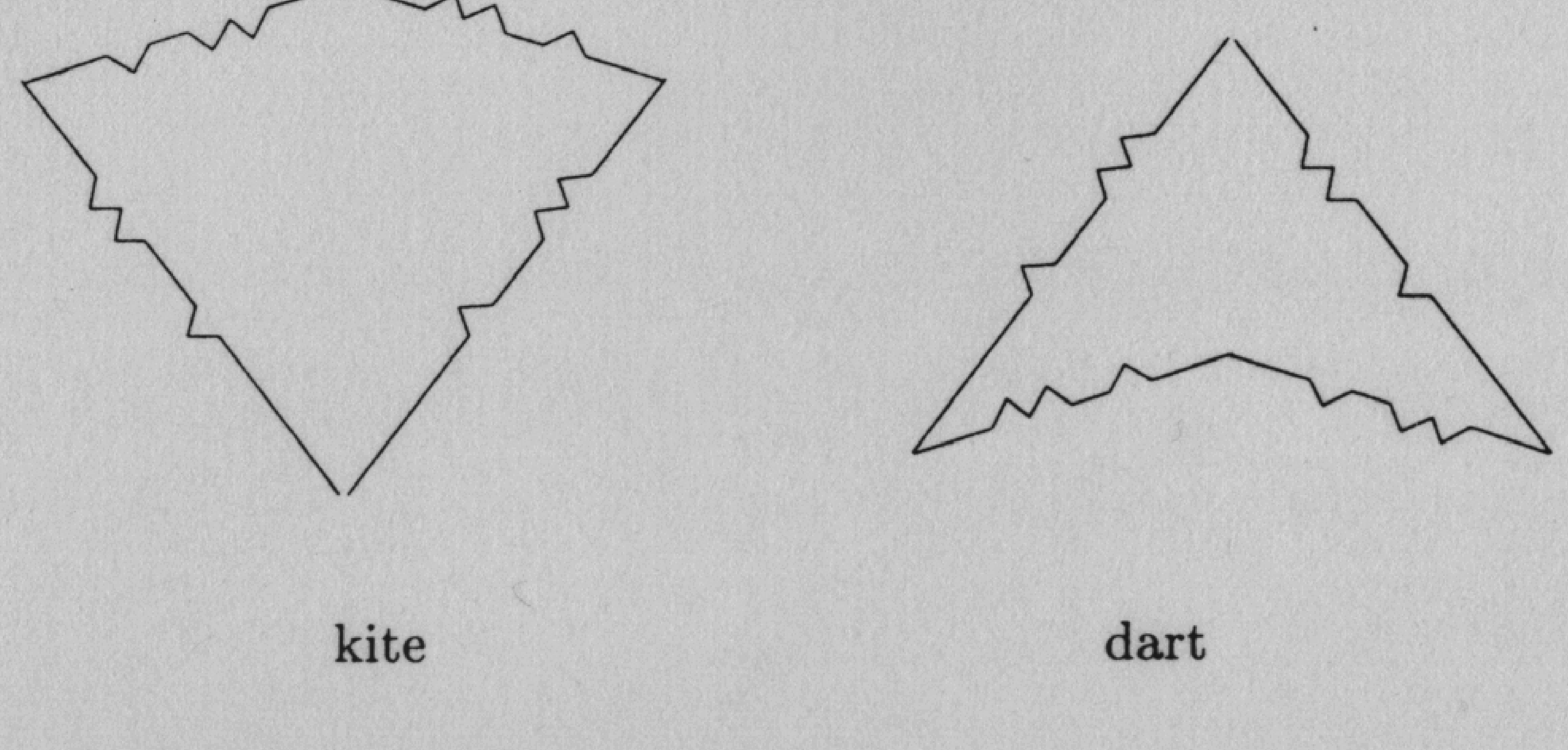

Figure 1. *The kite and dart tiles*

of Wang and Berger are that the shapes of the kite & dart tilings are polygons but not unit squares. This has no bearing on the complexity issues which motivated Wang. But the change in shapes was very significant, as we will see.

We next consider the "pinwheel[5]" tilings of the plane, and their connection with the above. It's hard to give you a list of the tiles used in pinwheel tilings, in the manner of Fig. 1 for kite & dart tilings. But it is perhaps sufficient to say that they consist of a large but finite number of different "versions" of a 1-2-$\sqrt{5}$ right triangle and its reflection; the versions differ by the addition of different patterns of bumps and dents on the edges. A portion of a pinwheel pattern is shown in Fig. 3, again without including the bumps and dents. In a pinwheel tiling each kind of tile appears in infinitely many rotational orientations, so in order to keep the set $\mathcal{B}$ of different tiles finite we allow rotation of the tiles when making the tilings. This was unnecessary for the Berger or Penrose tilings, since we were careful to include in $\mathcal{B}$ each tile in each orientation needed. We now digress to examine this detail.

The pinwheel tilings came about as follows [Ra1]. In 1990 I noticed that in all the known aperiodic tiling examples the tiles only appeared

[5]Pinwheel → moulinet/moulin à vent, jeu d'enfant

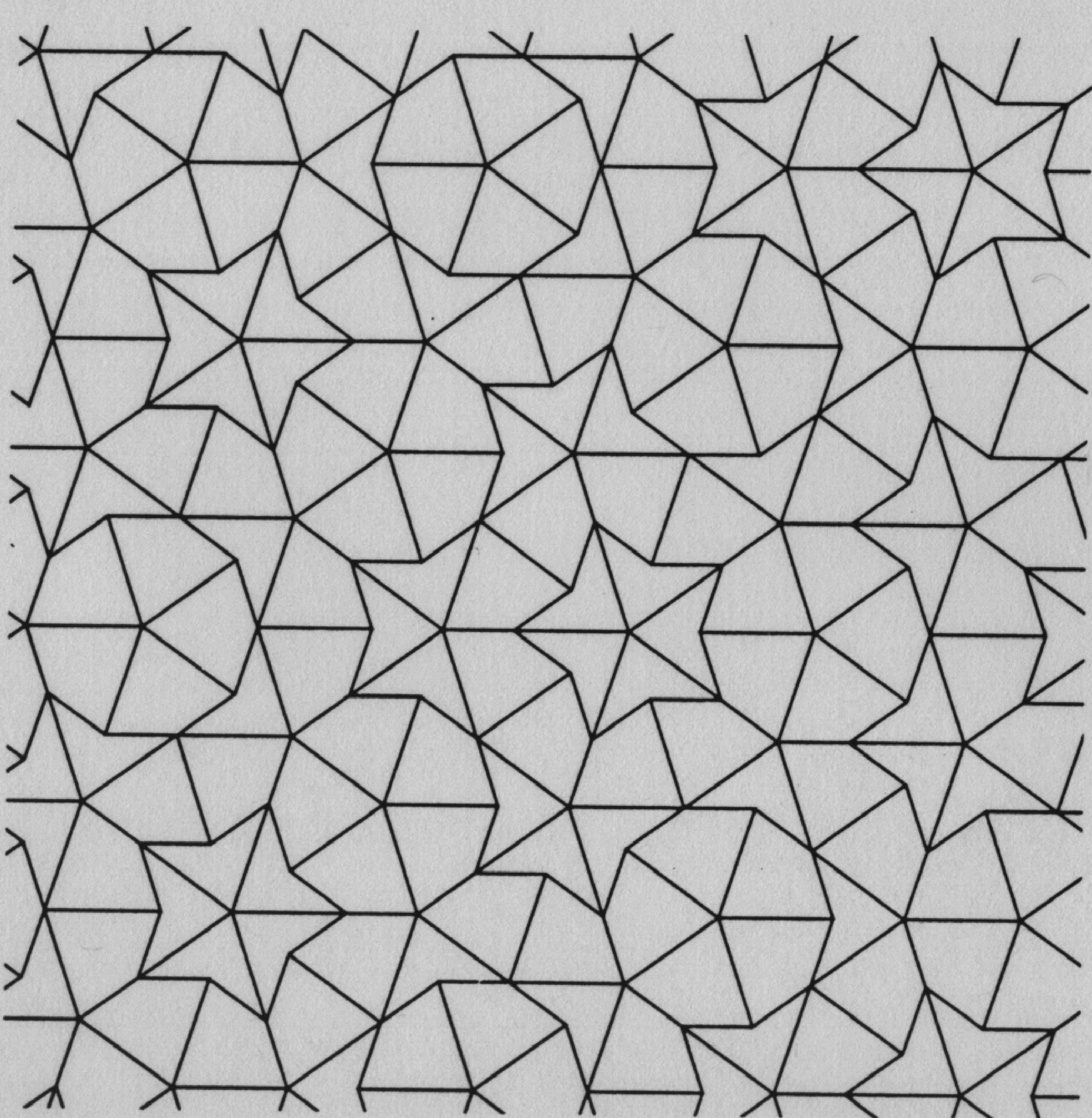

Figure 2. *A Penrose "kite and dart" tiling*

in finitely many orientations. In fact, most of the examples were produced using a technique due to N.G. de Bruijn [Bru], based on projection of a lattice in a high dimensional Euclidean space, which automatically implies this feature. But I wanted to use these tilings to model materials [Ra2]. The fact that the structure of the tiling is forced by requiring that edges fit together, like a jigsaw puzzle, is closely analogous to the condition of an energy-minimizing state of an ensemble of classical particles interacting through short range forces. (Imagine that two tiles put next to one another generate a potential energy of $+1$ if their abutting edges do not fit together and -1 if they do. A tiling will then correspond to an energy-minimizing state.) I was interested in understanding how this kind of minimization problem in models of solids led to the very restrictive and highly symmetric structures seen in real materials – either ordinary periodic crystals, or these new quasicrystals. And I noticed that the physics did not require any analogue of the "finite number of orientations" that was a feature of all the known aperiodic tilings used to model quasicrystals. So I tried to find an aperiodic tiling model – a $\mathcal{B}$ – which would allow infinitely many orientations. The fact is, it is difficult to find a collection of finitely many pairwise-noncongruent polygons copies of which can tile the plane using infinitely many orientations. And

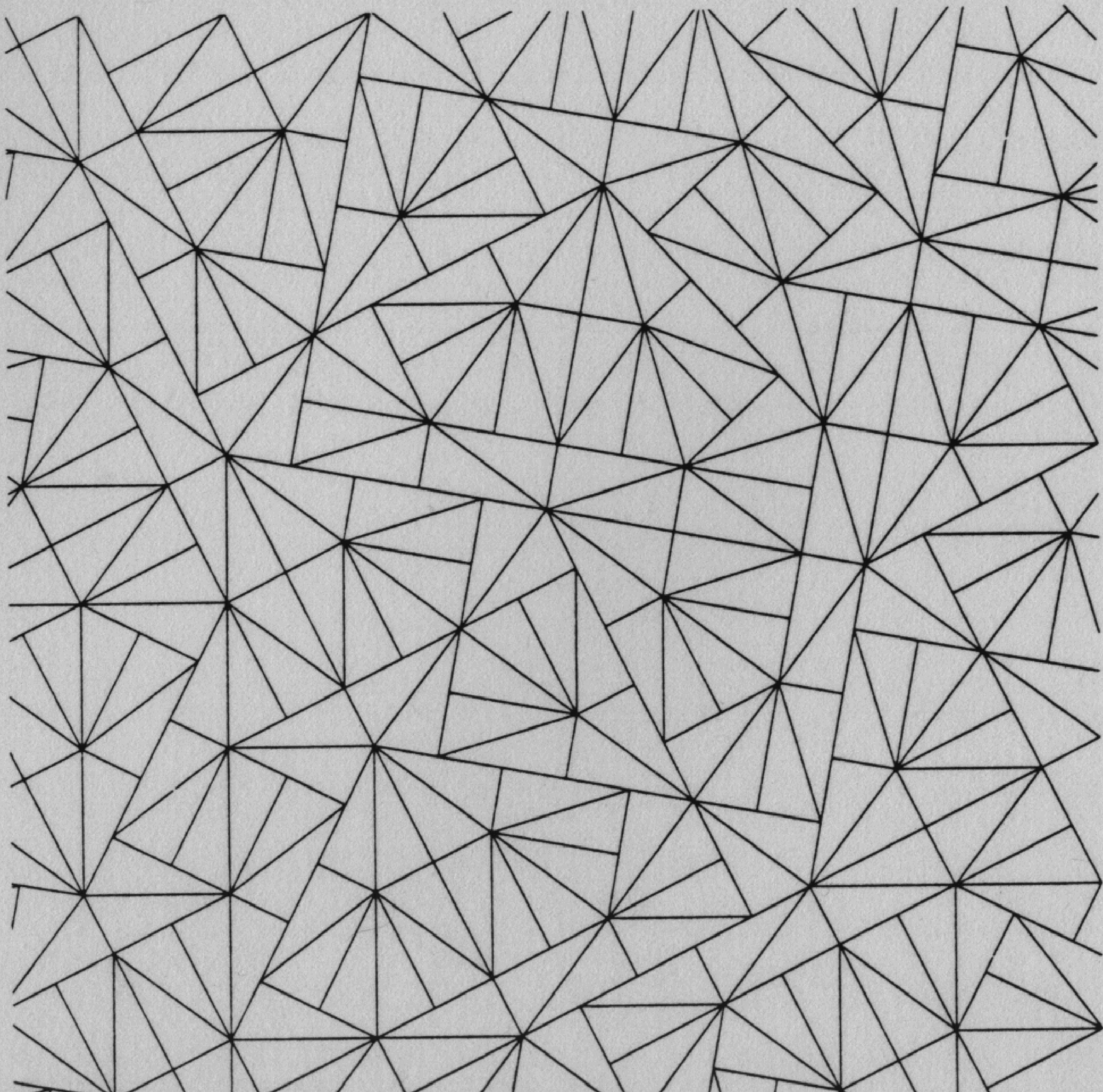

Figure 3. *A pinwheel tiling*

what I needed was such a collection which could *only* tile the plane that way! As a first step, John Conway and I came up with the following iterative procedure. Start with a 1-2-$\sqrt{5}$ right triangle, and break it up into 5 pieces as in Fig. 4.

Use this same rule to break up the 5 pieces, producing the 25 triangles of Fig. 5.

Note the triangle roughly in the middle (with the dark outline) which is similar to the outer edge of this collection of 25 triangles. There is a point P in the interior of that triangle which is the center of the similarity. Consider the compound process whereby you start with a single 1-2-$\sqrt{5}$ triangle, break it up into 25 triangles as in Fig. 5, then expand about the point P by a linear factor of 5. Repeat the break-up process on each of the triangles, but expand about the same point P. This process can be understood as adding more triangles around those already produced. So repeating it infinitely many times produces a pinwheel tiling of the plane, a portion of which appears in Fig. 3.

This solved the simpler question, of creating a tiling of the plane using

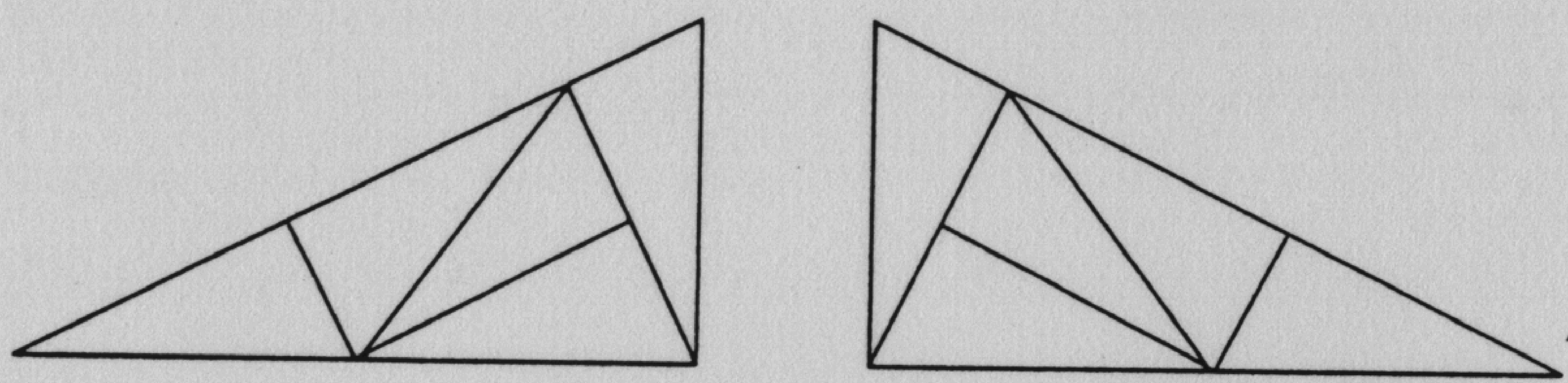

Figure 4. *The substitution for pinwheel tiling*

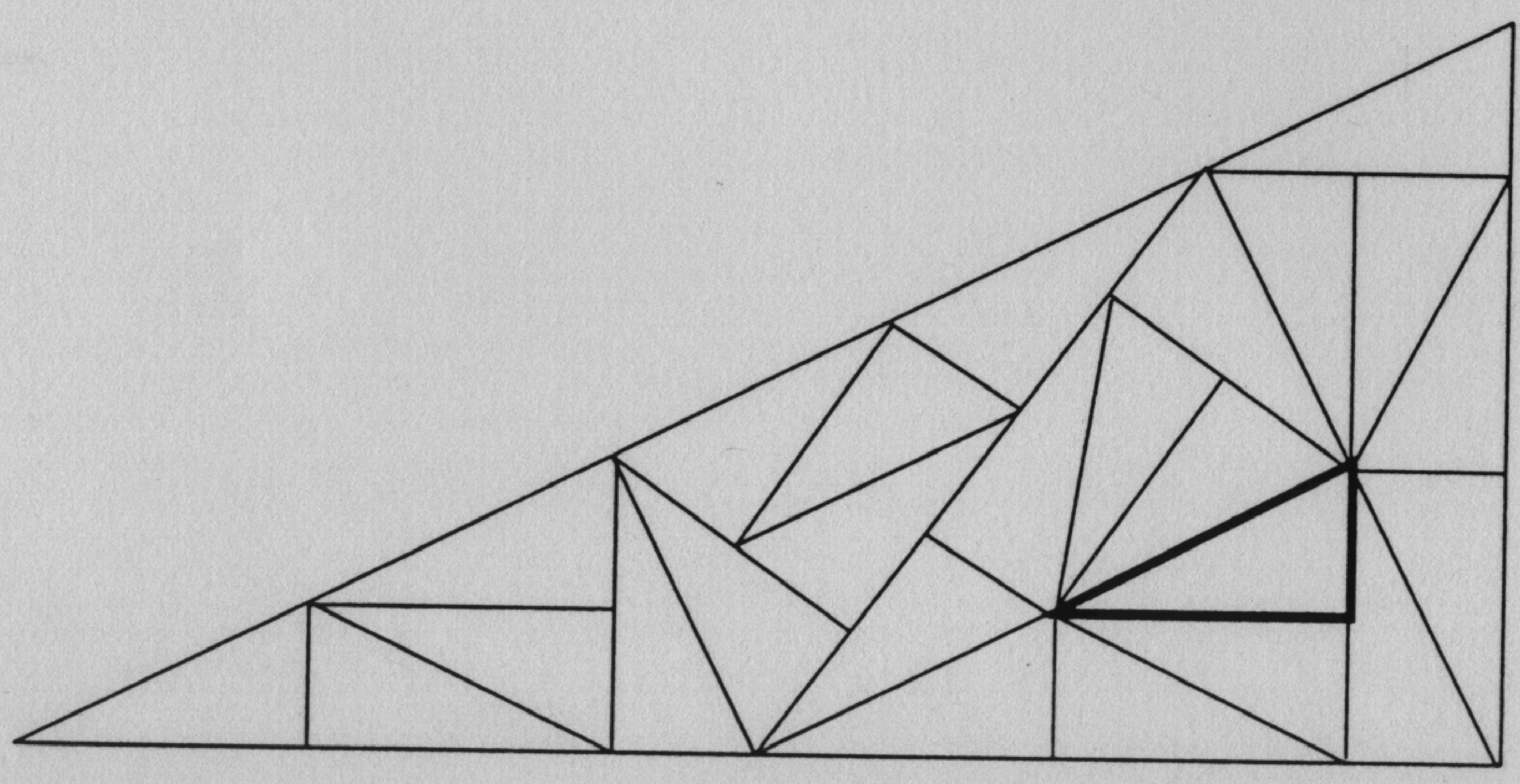

Figure 5. *The pinwheel similarity*

a finite number of pairwise-noncongruent polygons (in this construction 2 triangles ; 1 if you allow reflections which I'd rather not do) which uses the tiles in infinitely many orientations. ($\tan^{-1}(1/2)$ is irrational with respect to π, which implies the number of orientations.) But in order to be useful as a model for matter it was necessary to have an example which could *only* tile that way, a property certainly not shared by these 2 triangles! That was much harder, and was solved in [Ra1]. The point I want to make is that producing matching rules (edge bumps, or edge colors) for this iterative example was quite difficult – it takes about 30 pages in [Ra1] even to define them – but I felt I had good enough reason to think they existed to justify the effort. This was based on several things. First, if aperiodic tilings were to be used as models of physical systems there could be no restriction that tiles only appear in finitely many orientations. To me such a restriction was very artificial, offensive to the physics. So I avoided de Bruijn's projection technique. And second, there was a paper of Shahar Mozes in 1989 [Moz] which organized and vastly extended most of the previous work on Wang (colored-square-tile) tilings in a framework of ergodic theory, which used this iterative technique. Mozes showed that for almost any Wang tiling made by such an iterative process there was a way to make versions with bumps on the edges of the squares so that the squares could *only* tile the plane in the way made by iteration. So I decided that if aperiodic tiling is reasonable for modelling quasicrystals, as seemed to be the case from all the work following Mackay, and the physics should allow rotation invariance, and iterative (Wang) tilings automatically had matching rules, then the pinwheel iterative scheme should have matching rules. It was a mixture of intuitions drawn from a variety of separate research areas.

This still did not lead immediately to the generalized dihedral groups, which came about as follows. Thinking about the pinwheel as a model for a (2-dimensional) quasicrystal, I soon realized that the new feature, the infinitely many orientations, was in a practical sense invisible. As you look at larger and larger regions of a pinwheel tiling, the number of different orientations you see only grows logarithmically with the size of the region. Conway and I thought about this, and concluded that what was needed was a 3-dimensional iterative tiling, so that the noncommutativity of the rotations could allow algebraic growth in the number of orientations. That is the origin of the quaquaversal tilings [CoR]. The relative orientations of the tiles in the quaquaversal tilings was our first example of a generalized dihedral group. So the study of these groups first arose to prove that the growth rate in these tilings actually was algebraic, and then continued [RaS] to classify all such groups in part to distinguish between different tilings made by the iterative process.

Note that the generalized dihedral groups are basic mathematical objects ; they could have attracted attention and been analyzed many years ago. They finally came up naturally enough from 3-dimensional iterative tilings, but only through significant use of intuition and knowledge from distant research areas – ergodic theory and condensed matter physics in particular [Ra2]. And it was not just theorems that were used ; the theorem of Mozes was important, but just as important was the intuition drawn from condensed matter physics, in particular the role of symmetries in that subject. For me at least, it has often been indispensible to use the intuition developed in such an applied area to guide me in finding the right path to theorems in pure mathematics. Our proofs [CoR, RaS] about the generalized dihedral groups are ring theoretic ; but the proofs came much more quickly than realizing the importance of the groups themselves. To summarize : As in the movie of the title of this article there can be a significant advantage in considering various points of view of a complicated phenomenon, and it is not surprising that the further separated the worlds from which the views originate, the more useful is the contrast.

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