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Relations in *SO*(3) Supported by Geodetic Angles*

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Abstract. We consider rotations $A,\ B$ of finite order in $SO(3)$, about axes separated by an angle of restricted type, and attempt to classify the possible group relations between A and B. We show that the relations responsible for the symmetries of Platonic solids have consequences far beyond that simple geometric setting.

1. Introduction and Results

In two recent papers [1], [3] we classified certain subgroups of $SO(3)$, namely, those generated by two rotations, of order p and q, respectively, about perpendicular axes. Presentations were given for these groups, and an interesting feature emerged: for all these groups the only relators needed were based on symmetries of the cube.

In [4] we generalized these results to pairs of rotations of finite order whose axes are separated by an angle α which is a rational multiple of π—in other words, for which $e^{2i\alpha}$ is a root of unity. Again the only relators needed were based on symmetries of the cube, and, in particular, there are never any relations between generators of odd order.

This is surprising, since the group of rotational symmetries of the tetrahedron is generated by two rotations of order 3. Moreover, the group of rotational symmetries of the icosahedron (or dodecahedron) may be generated in several distinct ways by two rotations of odd order: either two rotations of order 3, two of order 5, or one

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of order 3 and one of order 5. In each of these cases the angle α between the axes of rotation is not, of course, a rational multiple of π. Instead, α is in many cases an angle whose squared trigonometric functions are rational. This means that $e^{2i\alpha}$ is algebraically simple in a different way. Instead of treating $e^{2i\alpha} = e^{i\pi}$ as a (square) root of unity, we can think of it as satisfying a quadratic polynomial with integer coefficients.

Angles whose squared trigonometric functions are rational are called "geodetic" and were introduced in [2]. If an angle α is geodetic, then $x = \exp(2i\alpha)$ is at worst a quadratic irrational, since $x + x^{-1} + 2 = 4\cos^2(\theta)$ is rational. Geodetic angles are denoted by the squares of their sines: $\angle(a/c)$ denotes $\sin^{-1}(\sqrt{a/c})$.

In [2] we found all additive relations between geodetic angles. Equivalently, we classified all relations between rotations in the plane by geodetic angles. In this paper we extend this analysis to three dimensions, analyzing relations between rotations of finite order about axes that are separated by geodetic angles, as in Platonic solids.

A basic feature of our method is to employ only those relators coming from Platonic symmetry groups. In a sense the main result we report is that although this is not sufficient for all these groups, it does go surprisingly far. With only a few exceptions, the relations that can occur are direct consequences of symmetries of Platonic solids.

Given a geodetic angle $\angle(a/c)$, where a and c are relatively prime positive integers with $a < c$, we let $A = R_x^{2\pi/p}$ be a rotation about the x axis by angle $2\pi/p$ and let $B = R_\ell^{2\pi/q}$ be a rotation about the ℓ axis by $2\pi/q$, where the ℓ axis is in the x–y plane, itself making an angle of $\angle(a/c)$ with the x axis.

Theorem 1. *The following conditions are necessary for there to be nontrivial relations between A and B*:

H1. *c is not divisible by any prime greater than* 5.
H2. *c is not divisible by* 2^3, 3^3, *or* 5^2.
H3. *If c is even, then both p and q are even.*
H4. *If c is divisible by* 3, *then p or q is divisible by* 3.
H5. *If c is divisible by* 3^2, *then both p and q are divisible by* 3.
H6. *If c is divisible by* 5, *then both p and q are divisible by* 5.
H7. *If p and q are both odd, then a is divisible by* 4.

Corollary. There is only a finite set of geodetic angles $\angle(a/c)$ that can support nontrivial relations between finite-order rotations A and B about axes separated by $\angle(a/c)$. These angles have $c \in \{1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, 180\}$.

It is a daunting task to understanding which of the geodetic angles of the corollary actually do support relations for which values of p and q, and to understand what the relations actually are. However, when p and q are of prime order, a complete classification is possible:

Theorem 2 (The Primordial Theorem). *Assume p and q are prime and that the group generated by A and B is not the free product of the obvious cyclic groups. The following* *examples occur, and are the only ones possible*:

(i) $\alpha = \angle\frac{1}{1}$, $p = 2$ *and/or* $q = 2$;
(ii) $\alpha = \angle\frac{8}{9}$, $p = q = 3$;
(iii) $\alpha = \angle\frac{4}{5}$, $p = q = 5$;
(iv) $\alpha = \angle\frac{3}{4}$, $p = q = 2$;
(v) $\alpha = \angle\frac{2}{3}$, $\{p, q\} = \{2, 3\}$;
(vi) $\alpha = \angle\frac{1}{2}$, $p = q = 2$;
(vii) $\alpha = \angle\frac{4}{9}$, $p = q = 3$;
(viii) $\alpha = \angle\frac{1}{3}$, $\{p, q\} = \{2, 3\}$;
(ix) $\alpha = \angle\frac{1}{4}$, $p = q = 2$.

Furthermore, the relations derive straightforwardly from those exhibited in Platonic solids.

The assumption of prime order in Theorem 2 is essential. Allowing composite orders we have the additional

Example. If $c = 6$, $a = 5$, $p = 6$, and $q = 2$, then

$$A^2BA^3BA^2BA^4BA^3BA^4B$$

is the identity. (This follows from the "accidental" fact that $A^2BA^3BA^2$ commutes with B.)

2. Representations into $PGL(2, \mathbb{C})$

The tools for addressing these questions are explicit representations of our rotations in $SO(3) \sim PSU(2) \subset PGL(2, \mathbb{C})$. The rotation R_z^θ is represented by

$$\begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \sim \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & \bar{v} \end{pmatrix}, \tag{1}$$

where $v = \exp(i\theta)$, and where we may choose whichever one of the three equivalent matrices is most convenient. The rotation R_x^θ is represented by

$$\begin{aligned} &\begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \\ &\qquad = \begin{pmatrix} (v^{1/2} + \bar{v}^{1/2})/2 & (v^{1/2} - \bar{v}^{1/2})/2 \\ (v^{1/2} - \bar{v}^{1/2})/2 & (v^{1/2} + \bar{v}^{1/2})/2 \end{pmatrix} \sim \begin{pmatrix} v+1 & v-1 \\ v-1 & v+1 \end{pmatrix}. \end{aligned} \tag{2}$$

Next we need to represent B. Let $C = R_z^{\angle(a/c)}$, so that $B = CR_x^{2\pi/q}C^{-1}$. Let $b = c - a$, and let $\sigma = \sqrt{b} + i\sqrt{a} = \sqrt{c}\exp(i\angle(a/c))$. We represent

$$\begin{aligned} C &= \begin{pmatrix} \sigma/\sqrt{c} & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} \sigma & 0 \\ 0 & \sqrt{c} \end{pmatrix}, \\ C^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & \sigma/\sqrt{c} \end{pmatrix} \sim \begin{pmatrix} \sqrt{c} & 0 \\ 0 & \sigma \end{pmatrix}. \end{aligned} \tag{3}$$

As a result, we can represent rotations about the ℓ axis by

$$R_\ell^\theta = CR_x^\theta C^{-1} = \begin{pmatrix} \sigma & 0 \\ 0 & \sqrt{c} \end{pmatrix} \begin{pmatrix} v+1 & v-1 \\ v-1 & v+1 \end{pmatrix} \begin{pmatrix} \sqrt{c} & 0 \\ 0 & \sigma \end{pmatrix}. \tag{4}$$

Now suppose we have a relation involving A and B. After conjugating by some power of A or B this relation can always be put in the form

$$\begin{aligned} \tau &= A^{x_1} B^{y_1} \cdots A^{x_n} B^{y_n} \\ &= R_x^{\theta_1} R_\ell^{\varphi_1} \cdots R_x^{\theta_n} R_\ell^{\varphi_n} \\ &= R_x^{\theta_1} C R_x^{\varphi_1} C^{-1} \cdots R_x^{\theta_n} C R_x^{\varphi_n} C^{-1} \\ &= \begin{pmatrix} v_1+1 & v_1-1 \\ v_1-1 & v_1+1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{c} \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} w_1+1 & w_1-1 \\ w_1-1 & w_1+1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} \sqrt{c} & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad \cdots \begin{pmatrix} v_n+1 & v_n-1 \\ v_n-1 & v_n+1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{c} \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} w_n+1 & w_n-1 \\ w_n-1 & w_n+1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} \sqrt{c} & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{5}$$

where $1 < x_k < p$, $1 < y_k < q$, $\theta_k = 2\pi x_k/p$, $\varphi_k = 2\pi y_k/q$, $v_k = \exp(i\theta_k)$, and $w_k = \exp(i\varphi_k)$. Conjugating τ by $\left(\begin{smallmatrix} \sqrt{c} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we get that a multiple of the identity matrix can be written as the alternating product of matrices of the form

$$\begin{aligned} \tilde{A}_k &= \begin{pmatrix} \sqrt{c} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_k+1 & v_k-1 \\ v_k-1 & v_k+1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{c} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{c}(v_k+1) & c(v_k-1) \\ v_k-1 & \sqrt{c}(v_k+1) \end{pmatrix} \end{aligned} \tag{6}$$

and

$$\begin{aligned} \tilde{B}_k &= \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_k+1 & w_k-1 \\ w_k-1 & w_k+1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \\ &= \begin{pmatrix} \sigma(w_k+1) & \sigma^2(w_k-1) \\ w_k-1 & \sigma(w_k+1) \end{pmatrix}. \end{aligned} \tag{7}$$

To understand $\tilde{A}_k$ and $\tilde{B}_k$ better we do a change-of-basis from the standard basis $\binom{1}{0}$, $\binom{0}{1}$ to

$$e_1 = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}, \qquad e_2 = (\alpha + \sqrt{c}) \begin{pmatrix} \sqrt{c} \\ 1 \end{pmatrix}.$$

Equivalently, we conjugate by M^{-1}, where

$$M = \begin{pmatrix} \alpha & c + \alpha\sqrt{c} \\ 1 & \alpha + \sqrt{c} \end{pmatrix}. \tag{8}$$

We compute

$$
\begin{aligned}
\tilde{A}_k e_1 &= \begin{pmatrix} (v-1)\sqrt{c}(\sqrt{c}+\sigma)+2\sigma\sqrt{c} \\ (v_k-1)(\sqrt{c}+\sigma)+2\sqrt{c} \end{pmatrix} = 2\sqrt{c}e_1 + (v_k-1)e_2, \\
\tilde{A}_k e_2 &= \begin{pmatrix} 2cv_k(\sqrt{c}+\sigma) \\ 2\sqrt{c}v_k(\sqrt{c}+\sigma) \end{pmatrix} = 2\sqrt{c}v_k e_2, \\
\tilde{B} e_1 &= \begin{pmatrix} 2\sigma^2 w_k \\ 2\sigma w_k \end{pmatrix} = 2\sigma w e_1, \\
\tilde{B} e_2 &= (\sqrt{c}+\sigma)\begin{pmatrix} \sigma(\sqrt{c}+\sigma)(w_k-1)+2\sigma\sqrt{c} \\ (\sqrt{c}+\sigma)(w_k-1)+2\sigma \end{pmatrix} \\
&= (\sqrt{c}+\sigma)^2(w_k-1)e_1 + 2\sigma e_2 \\
&= 2\sigma(\sqrt{b}+\sqrt{c})(w_k-1)e_1 + 2\sigma e_2,
\end{aligned}
\tag{9}
$$

so that in this basis A_k and B_k are represented by

$$
\begin{aligned}
\hat{A}_k &= M^{-1}\tilde{A}_k M = \begin{pmatrix} 2\sqrt{c} & 0 \\ v_k-1 & 2\sqrt{c}v_k \end{pmatrix}; \\
\hat{B}_k &= M^{-1}\tilde{B}_k M = \begin{pmatrix} 2\sigma w_k & 2\sigma(\sqrt{b}+\sqrt{c})(w_k-1) \\ 0 & 2\sigma \end{pmatrix} \sim \begin{pmatrix} w_k & (\sqrt{b}+\sqrt{c})(w_k-1) \\ 0 & 1 \end{pmatrix}.
\end{aligned}
\tag{10}
$$

We will consistently use the last expression for $\hat{B}_k$. Notice that the expressions (10) for $\hat{A}_k$ and $\hat{B}_k$ involve rational integers, roots of unity, and the square roots of b and c. These are all elements of the cyclotomic ring of integers $\mathbb{Z}[\zeta_N]$, where ζ_N is a primitive Nth root of unity and N is any common multiple of 8, b, c, p, and q. We therefore do all our analysis in this ring.

3. Some Algebraic Facts about Cyclotomic Rings of Integers

Lemma 1. *Let $p_n(z)$ be the minimal polynomial for a primitive nth root of unity, $n \geq 2$. Then*

$$
p_n(1) = \begin{cases} P, & \text{if } n \text{ is a power of a rational prime } P; \\ 1, & \text{otherwise.} \end{cases}
\tag{11}
$$

Proof. If n is a prime, then $p_n(z) = 1+z+\cdots+z^{n-1}$ so $p_n(1) = n$. If n is composite, then $p_n(z) = (1+z+\cdots+z^{n-1})/\Pi p_m(z)$, where $1 < m < n$ and m ranges over the factors of n. The two formulas for $p_n(1)$ when n is composite then follow by induction on the number of factors of n. □

Lemma 2. *If z and z' are both primitive nth roots of unity, then $z-1$ and $z'-1$ divide one another in $\mathbb{Z}[\zeta_N]$, for any multiple N of n.*

Proof. Since z and z' are both primitive nth roots of unity, there exist integers r and s such that $z' = z^r$ and $z = z'^s$. However, then $z'-1 = (z-1)(1+z+\cdots+z^{r-1})$ and $z-1 = (z'-1)(1+z'+\cdots+z'^{s-1})$. □

Lemma 3. *Let $n|N$ and let ζ_n be a primitive nth root of unity. If n has distinct prime factors, then $\zeta_n - 1$ is a unit in the ring $\mathbb{Z}[\zeta_N]$. If $n = P^k$, P a rational prime, then $[\zeta_n - 1]^{P^k - P^{k-1}} = P \times (unit)$.*

Proof. $p_n(1) = \Pi_j(1 - v_j) = \pm\Pi_j(v_j - 1)$, where v_j ranges over all the primitive nth roots of unity. If n has distinct prime factors, then $p_n(1) = 1$ and $\zeta_n - 1$, which divides $p_n(1)$, is a unit. If $n = P^k$, by Lemma 2 and induction each of the $P^k - P^{k-1}$ terms in the product $\Pi_j(1 - v_j)$ are equal, up to multiplication by a unit, so $P = p_n(1) = (\zeta_n - 1)^{P^k - P^{k-1}} \times$ unit. □

For each rational prime P which divides N choose the largest power k such that P^k divides N. $\zeta_{P^k} - 1$ is irreducible in $\mathbb{Z}[\zeta_N]$, so for each element $x \in \mathbb{Z}[\zeta_N]$ we associate a unique power of P as follows. Let y be the largest rational integer such that $(\zeta_{P^k} - 1)^y$ divides x and associate to x the power $y/(P^k - P^{k-1})$ of P, sometimes described by saying x is divisible by $y/(P^k - P^{k-1})$ powers of P.

4. Proof of Theorem 1

We are now ready for the proof of Theorem 1. We show that if any of the conditions H1–H7 is not met it is impossible for a product $\hat{\tau} = \hat{A}_1\hat{B}_1 \cdots \hat{A}_n\hat{B}_n$ to be a multiple of the identity. The strategy is the same for each condition: assuming the condition is violated, we find a rational prime P such that the matrix element $\hat{\tau}_{22}$ is divisible by fewer powers of P than is $\hat{\tau}_{11}$, so $\hat{\tau}_{22} \neq \hat{\tau}_{11}$, so $\hat{\tau}$ is not a multiple of the identity, and neither is τ. In particular, we show that the term

$$(\hat{A}_1)_{21}(\hat{B}_1)_{12} \cdots (\hat{A}_n)_{21}(\hat{B}_n)_{12} = (\sqrt{b} + \sqrt{c})^n \, \Pi_{k=1}^n (v_k - 1)(w_k - 1) \tag{12}$$

in the expansion of $\hat{\tau}_{22}$ is divisible by fewer powers of P than any other term, and fewer than any term in the expansion of $\hat{\tau}_{11}$. The point is that all other terms in the expansion of $\hat{\tau}_{22}$, and all terms in the expansion of $\hat{\tau}_{11}$, take the same form as (12) but with pairs of off-diagonal matrix elements replaced by diagonal matrix elements. Each such replacement exchanges a factor of $(\sqrt{b} + \sqrt{c})(v_k - 1)(w_l - 1)$ for $2\sqrt{c}$ times a unit (either v_k or w_l).

So to prove the theorem we need only show that if any of the conditions is violated, then $2\sqrt{c}$ is divisible by more powers of P than is $(\sqrt{b} + \sqrt{c})(v_k - 1)(w_l - 1)$, for every possible v_k and w_l. In other words, the comparison is always

$$(\sqrt{b} + \sqrt{c})(v_k - 1)(w_l - 1) \quad \text{versus} \quad 2\sqrt{c}. \tag{13}$$

Suppose that c is divisible by a rational prime P. Since a and c are relatively prime in $\mathbb{Z}$, the Euclidean algorithm shows that a is not divisible by any powers of P. Then $(\sqrt{c} - \sqrt{b})(\sqrt{c} + \sqrt{b}) = c - b = a$ implies the same for $(\sqrt{c} + \sqrt{b})$. If p (or q) is divisible by P, then $v_k - 1$ (or $w_l - 1$) has at most $1/(P - 1)$ powers of P, while if p (or q) is not divisible by P, then $v_k - 1$ (or $w_l - 1$) has no powers of P.

If $P > 5$, then the left-hand side of (13) has at most $2/(P - 1) < \frac{1}{2}$ powers of P, while the right hand has at least half a power of P (since $P|c$), and there cannot be a relation. This establishes condition H1.

If $P \leq 5$ there is still a limit to the number of powers of P on the left-hand side of (13). If the right-hand side exceeds this number, then there can be no relation. If $P = 5$ there is at most half a power of 5 on the left-hand side, but if 5^2 divides c there is at least one power of P on the right. If $P = 3$ there is at most one power of 3 on the left, but if 3^3 divides c there are at least one and a half powers on the right. If $P = 2$ there are at most two powers of 2 on the left, but if 2^3 divides c there are at least two and a half powers on the right. This establishes H2.

If c is even, then there are at least one and a half powers of 2 on the right. However, if either p or q is odd, then there can only be one power of 2 on the left. This establishes H3.

If c is divisible by 3, then the right has at least half a power of 3, and we need either p or q divisible by 3 to obtain half a power of 3 on the left. If c is divisible by 3^2, then the right is divisible by 3, and we need both p and q divisible by 3 to obtain one power of 3 on the left. This establishes H4 and H5.

If c is divisible by 5, then the right has at least half a power of 5. Since $v_k - 1$ and $w_l - 1$ each have at most a quarter of a power of 5, to get half a power of 5 on the left requires both p and q to be divisible by 5. This establishes H6.

Finally, suppose p and q are odd, so that $v_k - 1$ and $w_l - 1$ are not divisible by any powers of 2. The right-hand side has an explicit power of 2 (regardless of c), so to obtain relations we must have $(\sqrt{c} + \sqrt{b})$ divisible by 2. However, in that case $\sqrt{c} - \sqrt{b} = \sqrt{c} + \sqrt{b} - 2\sqrt{b}$ is also divisible by 2, so $a = (\sqrt{c} + \sqrt{b})(\sqrt{c} - \sqrt{b})$ must be divisible by 4. □

5. Platonic Relations and the Angle-Dividing Trick

Let $G_\alpha(p, q)$ be the subgroup of $SO(3)$ generated by rotations A and B of order p and q, respectively, about axes separated by an angle α. We say an angle α *supports relations* if, for some integers p and q, the group $G_\alpha(p, q)$ is not the free product of the obvious $\mathbb{Z}_p$ and $\mathbb{Z}_q$ subgroups. The corollary to Theorem 1 says that only a finite number of geodetic angles support relations. In this section we consider the angles that do.

We begin with the Platonic relations known since antiquity. The symmetry group of each Platonic solid is finite (and in particular not free), so the angle between any two axes of symmetry must support relations.

The tetrahedron has an axis of threefold symmetry (a *triad*) through each vertex, passing through the middle of the opposite face. It also has axes of twofold symmetry (*dyads*) passing from the middle of each edge through the middle of the opposite edge. The triad axes make angles of $\angle\frac{8}{9}$ with one another and $\angle\frac{2}{3}$ with the dyad axes. The dyad axes are perpendicular to one another.

The cube has triad axes through each pair of opposite vertices, fourfold (*quad*) axes through opposite faces and dyad axes through opposite edges. The angles $\angle\frac{1}{3}$, $\angle\frac{2}{3}$, $\angle\frac{8}{9}$, $\angle\frac{3}{4}$, and $\angle\frac{1}{2}$ appear as dyad-triad, quad-triad, triad-triad, dyad-dyad, and dyad-quad angles, respectively. The angles $\angle\frac{2}{3}$ and $\angle\frac{8}{9}$ appear both in the tetrahedron and the cube. This reflects the fact that one can embed a tetrahedron in a cube, making the tetrahedral group a subgroup of the cube group.

In the icosahedron one finds the angles from the tetrahedron repeated once again, and the additional angles $\angle\frac{1}{4}$, $\angle\frac{4}{5}$, and $\angle\frac{4}{9}$, corresponding to dyad-dyad, pentad-pentad and triad-triad angles, respectively. The angles between other pairs of axes support relations, of course, but are either not new or not geodetic, and so are not listed.

The angles appearing in Platonic solids form a short list, but from them we can construct many angles that support relations, thanks to

Theorem 3 (The Angle-Dividing Trick). *If the angle α supports relations, then so does α/n for every positive integer n.*

Proof. Suppose $G_\alpha(p', q')$ is not a free product and let $\beta = \alpha/n$, $n \geq 2$. Let $p = q = 2p'q'$. We will show that $G_\beta(p, q)$ is not a free product of the obvious $\mathbb{Z}_p$ and $\mathbb{Z}_q$ subgroups.

The group $G_\beta(p, q)$ contains rotations of order p about the x and ℓ axes. Since $p = q$ is even, it contains a rotation $B^{q/2}$ by π about the ℓ axis. Then $C = B^{q/2}AB^{q/2}$ is a rotation of order p about an axis ℓ' making an angle 2β with the x axis. Conjugating B by $C^{p/2}$ gives a rotation D of order p about an axis ℓ'' making an angle 3β with the x axis. See Fig. 1. Continuing in this way we construct a rotation Z of order p about an axis making an angle of $n\beta = \alpha$ with the x axis. However, by assumption there are relations between $Z^{2p'}$ and $A^{2q'}$, hence relations between A and B. □

Note that, although the proof used $p = q = 2p'q'$, smaller values will typically suffice. In particular, if α supports relations between two rotations of order p', then $\alpha/2$ supports relations between a rotation of order p' and a rotation of order 2. For example, $\angle\frac{8}{9}$ supports triad-triad relations, while $\angle\frac{1}{3} = \frac{1}{2}\angle\frac{8}{9}$ supports a triad-dyad relation.

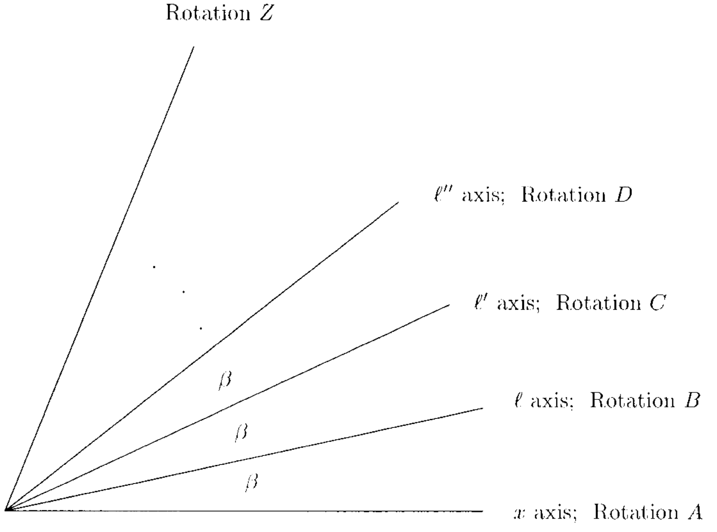

Fig. 1. Some rotations.

6. Proof of Theorem 2

We know from Theorem 1 that, for $\angle(a/c)$ to support relations, c must be of the form $2^s 3^t 5^u$, where $0 \le s \le 2,\ 0 \le t \le 2,\ 0 \le u \le 1$.

(a) If $c = 1$, then $a = 1$ and the axes are orthogonal. Since a is not divisible by 4, either p or q is even (hence equal to 2) and the group in question is a finite dihedral group. This can be viewed as coming from the cube group by the angle-dividing trick.

For the remainder we can assume $c > 1$.

(b) If $u = 1$, then $p = q = 5$, so $c = 5$ and $a = 4$. Angle $\angle\frac{4}{5}$ is indeed an angle between two pentad axes of the icosahedron.

For the remainder we can assume $u = 0$.

(c) If $s \neq 0$, then $p = q = 2$ and $c = 2$ or 4, giving $\angle\frac{1}{2} = \pi/4,\ \angle\frac{3}{4} = \pi/3$, and $\angle\frac{1}{4} = \pi/6$. The first is a quad-dyad angle in the cube group, the second is a dyad-dyad angle in the cube group, and the third is half the second, hence obtainable by the angle-dividing trick.

For the remainder we can assume $s = 0$, so $c = 3$ or 9.

(d) If $c = 3$, then p or $q = 3$. If the other was odd, then $a = 4$ and we would have a contradiction ($a > c$). So one of p, q is 3, the other is 2, and the two possible angles are $\angle\frac{1}{3}$ and $\angle\frac{2}{3}$. Each is a dyad-triad angle, the first from the cube and the second from the tetrahedron.

(e) If $c = 9$, then $p = q = 3$, so $a = 4$ or 8. Angle $\angle\frac{4}{9}$ is the angle between two triad axes of the icosahedron, while $\angle\frac{8}{9}$ appears as the angle between two triad axes of the tetrahedron (also of the cube and of the icosahedron). □

7. Which Angles Support Relations?

Theorems 1–3 largely determine which rational or geodetic angles can support relations. All rational angles support relations, and the relations are simple: they derive from the symmetries of the cube, extended by the angle-dividing trick. By contrast, only a finite number of geodetic angles support relations. Some of these relations come from symmetries of Platonic solids, while others (such as the example with $\angle\frac{5}{6}$) do not appear to.

The question naturally arises of how to understand which angles in general can support relations. We present here two results that illuminate the scope of the problem:

Theorem 4. *If the angle α supports nontrivial relations, then $e^{2i\alpha}$ is algebraic.*

Proof. Let $N = pq$. Every element of the group $G_\alpha(p, q)$ can be built from the rotations $R_x^{2\pi/N}$, R_z^{α}, and $R_z^{-\alpha}$, which we represent as

$$\begin{pmatrix} \zeta_N + 1 & \zeta_N - 1 \\ \zeta_N - 1 & \zeta_N + 1 \end{pmatrix}, \qquad \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}, \tag{14}$$

respectively, where $z = e^{i\alpha}$. The matrix elements of any purported relator τ are therefore polynomials in z with coefficients in $\mathbb{Z}[\zeta_N]$. τ is a multiple of the identity if and only if z satisfies the polynomial equations

$$\tau_{11} - \tau_{22} = \tau_{12} = \tau_{21} = 0. \tag{15}$$

If τ is a nontrivial relator, then $\tau_{11} - \tau_{22}$, τ_{12}, and τ_{21} cannot all be the zero polynomial, as we know that there are some angles (e.g., $\angle\frac{1}{11}$) that do not satisfy (15). Thus z (and therefore z^2) is algebraic. □

Although every angle that supports relations is algebraic, the relations themselves, and thus the minimal polynomials, can be arbitrarily complicated:

Theorem 5. *Fix p and q, and let τ be an arbitrary* (*but fixed*) *word in A and B, rotations of order p and q, respectively, about axes separated by α. For a dense set of angles $\alpha \in [0, 2\pi]$, a power of τ is a relator.*

Proof. For any α, τ is a rotation by some angle θ about some axis. The angle θ depends real analytically on α, and can be recovered from the trace of τ, represented as a matrix in $SO(3)$. Whenever θ is a rational multiple of π, a power of τ is a rotation by a multiple of 2π, hence is the identity. If θ is not constant as α varies, the denseness of the α's that give rational θ's follows from the denseness of the rational numbers and the analytic dependence of θ on α. If θ is constant, it equals the value for $\alpha = 0$, which is a multiple of $2\pi/pq$. In that case a power of τ is a relation for all α. □

8. Conclusions

We have been discussing the group relations that exist between a pair of elements of $SO(3)$, more specifically, a pair of rotations of finite order about axes separated by an angle α of restricted type. Our method for analyzing such group relations involves algebraic relations of $e^{2i\alpha}$. In previous papers we solved all cases where $e^{2i\alpha}$ is a root of unity, and in this paper we consider the next simplest situation, where $e^{2i\alpha}$ is a quadratic irrational—a class with natural geometric significance.

It is plausible that this effort could be extended to algebraic angles of higher order, but Theorem 5 suggests that progressively higher orders will yield progressively more complicated (and more difficult) results. In principal it seems to us unlikely that an understanding of such groups could be attained for arbitrary α, but that other reasonable subclasses might be profitably attempted.

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