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An Algebraic Invariant for Substitution Tiling Systems

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Abstract. We consider tilings of Euclidean spaces by polygons or polyhedra, in particular, tilings made by a substitution process, such as the Penrose tilings of the plane. We define an isomorphism invariant related to a subgroup of rotations and compute it for various examples. We also extend our analysis to more general dynamical systems.

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1. Introduction, Definitions and Statement of Results

This paper concerns tilings of Euclidean spaces by polygons or polyhedra, more specifically, tilings made by a 'substitution process'. Given a substitution rule, the set of resultant tilings is a topological space with an action of the Euclidean group, hence a dynamical system. We develop here an algebraic invariant that helps determine when two tiling systems are equivalent *as dynamical systems*. In this introduction we define the notions of 'substitution tiling system' and of equivalence between two such systems, and state what the invariant is. In subsequent sections we analyze the invariant, in particular we show its use in distinguishing between substitution tilings.

Our eventual goal is to associate certain groups to substitution tilings of Euclidean m-space. These groups, subgroups of $\mathrm{SO}(m)$, are generated by the relative orientations of tiles in the tilings, depend on the specific tiling x and on some specific choices (indexed by an integer j), and are denoted $\mathcal{O}_j(x)$. Although $\mathcal{O}_j(x)$ depends on j and x, the dependence is quite controlled. If x and x' are different tilings with the same substitution rule we will show that, under some mild hypotheses, $\mathcal{O}_j(x)$ and $\mathcal{O}_{j'}(x')$ are conjugate as subgroups of $\mathrm{SO}(m)$. Even without the mild hypotheses, they are conjugate (in $\mathrm{SO}(m)$) to subgroups of one another, a condition we call 'c-equivalence'. So we can associate to a substitution tiling system the

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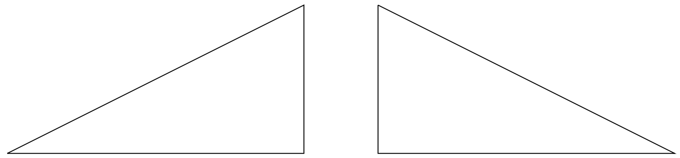

Figure 1. Two 'pinwheel' tiles.

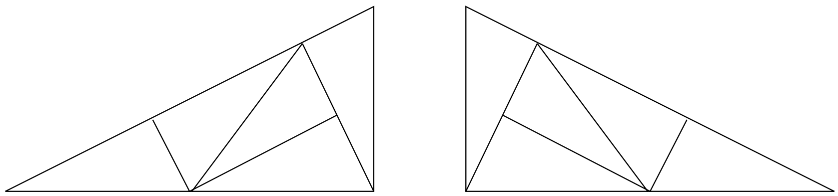

Figure 2. The substitution for pinwheel tilings.

common conjugacy class (or c-equivalence class) of the groups associated to the tilings in the system.

What will remain, then, is to show that this conjugacy (or c-equivalence) class can be considered an invariant in a natural sense. That is, we will show that two substitution tiling systems that are equivalent as dynamical systems have the same class of groups. We will do this by finding a dynamical description of the class. For each $\varepsilon > 0$ and each tiling x we will define a group $\mathcal{O}^\varepsilon(x)$ using dynamical information only. For ε sufficiently small, and for almost every x, we show that $\mathcal{O}^\varepsilon(x)$ is conjugate to (or c-equivalent to) $\mathcal{O}_j(x)$ for some, and hence all, choices j. The class of $\mathcal{O}^\varepsilon(x)$ is thus the same as the class of $\mathcal{O}_j(x)$. Since $\mathcal{O}^\varepsilon(x)$ is defined using data that is preserved by dynamical equivalence, the class of $\mathcal{O}^\varepsilon(x)$ is a dynamical invariant.

Note that the group $\mathcal{O}_j(x)$ depends only on the geometry of the tiling x. Since the class of $\mathcal{O}_j(x)$ is the same for every tiling x with the given substitution rule, we can obtain information about a substitution tiling system by looking at any single tiling in it. So if two substitution tilings x and x' give rise to groups $\mathcal{O}_j(x)$ and $\mathcal{O}_{j'}(x')$ that are not conjugate (or c-equivalent), then x and x' cannot belong to equivalent substitution tiling systems.

Before defining substitution tiling systems in general, we present an example. Hopefully, the general definitions will be clearer with this example in mind. The 'pinwheel' tiling of the plane [Ra1] is made as follows. Consider the triangles of Figure 1. Divide one of them into five small triangles as in Figure 2 and expand the figure about the origin by a linear factor of $\sqrt{5}$, producing 5 triangles congruent to the originals.

Repeat this two-step procedure ϕ simultaneously for all the triangles of the figure, then again, an infinite number of times, producing a (pinwheel) tiling C of the plane, a portion of which appears in Figure 3. Such tilings have a hierarchical structure which is of interest for various reasons; in particular it leads to interesting behavior of the relative orientations of tiles within a tiling [Ra3]. For background

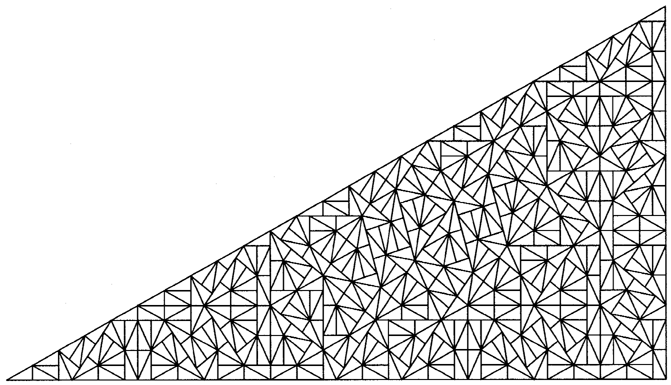

Figure 3. Part of a pinwheel tiling.

on related recent work see [AnP, CEP, DwS, G-S, Kel, Ken, LaW, Min, Moz, Ra3, Rob, Sad, Sch, Sen, Sol, Tha] and references therein.

Substitution Tiling Systems

With the pinwheel example in mind, we now address substitution tiling systems in general. Let $\mathcal{A}$ be a nonempty finite collection of polyhedra in m (typically 2 or 3) dimensions. Let $X(\mathcal{A})$ be the set of all tilings of Euclidean space by congruent copies, which we will call tiles, of the elements of (the 'alphabet') $\mathcal{A}$. We label the 'types' of tiles by the elements of $\mathcal{A}$. We endow $X(\mathcal{A})$ with the metric

$$d(x, y) \equiv \sup_n \frac{1}{n} m_H[B_n(\partial x), B_n(\partial y)], \tag{1}$$

where $B_n(\partial x)$ denotes the intersection of two sets: the closed ball B_n of radius n centered at the origin of the Euclidean space and the union ∂x of the boundaries ∂a of all tiles a in x. m_H is the Hausdorff metric on compact sets defined as follows. Given two compact subsets A and B of $\mathbb{R}^m$, $m_H[A, B] = \max\{\tilde{d}(A, B), \tilde{d}(B, A)\}$, where

$$\tilde{d}(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|, \tag{2}$$

with $\|w\|$ denoting the usual Euclidean norm of w. Although the metric d depends on the location of the origin, the topology induced by d is translation invariant. A sequence of tilings converges in the metric d if and only if its restriction to every compact subset of $\mathbb{R}^m$ converges in m_H. It is not hard to show [RaW] that $X(\mathcal{A})$ (which is automatically nonempty in our applications) is compact and that the natural action of the connected Euclidean group G_E on $X(\mathcal{A})$, $(g, x) \in G_E \times X(\mathcal{A}) \to T^g x \in X(\mathcal{A})$, is continuous.

A 'substitution tiling system' is a closed subset $X_\phi \subset X(\mathcal{A})$ satisfying some additional conditions. To understand these conditions we first need the notion of

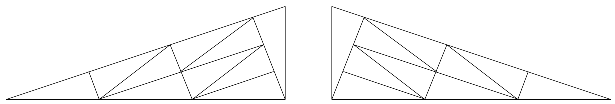

Figure 4. The substitution for pinwheel variant tilings.

'patches'. A patch is a (finite or infinite) subset of an element $x \in X(\mathcal{A})$; the set of all patches for a given alphabet will be denoted by W. Next we need, as for the pinwheels, an auxiliary 'substitution function' ϕ, a map from W to W, with the following properties:

(i) There is some constant $|\phi| > 1$ such that, for any $g \in G_E$ and $x \in X$, $\phi[T^g x] = T^{\phi(g)} x$, where $\phi(g)$ is the conjugate of g by the similarity of Euclidean space consisting of stretching about the origin by $|\phi|$.
(ii) For each tile $a \in \mathcal{A}$ and for each $n \geqslant 1$, the union of the tiles in $\phi^n a$ is congruent to $|\phi|^n a$, and these tiles meet full face to full face.
(iii) For each tile $a \in \mathcal{A}$, ϕa contains at least one tile of each type.

Condition (ii) is quite strong. It is satisfied by the pinwheel tilings only if we add additional vertices at midpoints of the legs of length 2, creating boundaries of 4 edges. A similar (minor) adjustment is needed for other examples in this paper. Even with such adjustments however, condition (ii) is not satisfied by the kite and dart tilings [Gar], or those which mimic substitution tilings using so-called edge markings [G-S, Moz, Ra3]. It is to handle such examples that we introduce the general development of Section 3.

DEFINITION. For a given alphabet $\mathcal{A}$ of polyhedra and substitution function ϕ the 'substitution tiling system' is the pair $\{X_\phi, T\}$, where $X_\phi \subset X(\mathcal{A})$ is the compact subset of those tilings x with the property that every finite subpatch of x is congruent to a subpatch of $\phi^n a$ for some $n > 0$ and some $a \in \mathcal{A}$, and T is the natural action of G_E on X_ϕ. (For simplicity we often refer to X_ϕ as a substitution tiling system.)

One planar example of a substitution tiling system is based on the pinwheel substitution of Figure 2. A slight variant of the pinwheel is defined by the 1-3-$\sqrt{10}$ right triangle and its reflection, and the substitution of Figure 4.

Two further special conditions which we will occasionally impose are:

(iv) A tiling in X_ϕ can only be tiled in one way by supertiles of level n, for any $n \geqslant 1$.
(v) For every $a \in \mathcal{A}$, there exists $n_a > 0$ such that $\phi^{n_a} a$ contains a tile a', of the same type as a, and parallel to a.

We note here that with the convention that patches of the form $\phi^n a$ are called 'supertiles' of 'level' n and 'type' a, it is easy to show by a diagonal argument that, for each $n \geqslant 0$, each tiling x is tiled by supertiles of level n [Ra3]. A supertile of level 4 for the pinwheel is shown in Figure 3.

Finally, let

$$S^{\varepsilon}(x) = \{y \in X_{\phi}: d(\phi^n y, \phi^n x) < \varepsilon \text{ for all } n \geqslant 0, \text{and } d(\phi^n y, \phi^n x) \underset{n\to\infty}{\longrightarrow} 0\}.$$

We call such a family of sets a 'local contracting direction (at x)'.

Our goal is to define a notion of equivalence for substitution tiling systems, and an invariant for that equivalence. For the equivalence we use:

DEFINITION. The substitution tiling systems (X_{ϕ^1}, T^1) and (X_{ϕ^2}, T^2) are 'equivalent' if there are subsets $Y_j \subset X_{\phi^j}$, invariant under T^j and of measure zero with respect to all translation invariant Borel probability measures on X_{ϕ^j}, and a one-to-one, onto, Borel bimeasureable map $\tau: X_{\phi^1} - Y_1 \to X_{\phi^2} - Y_2$, such that:

(a) $\tau \circ T^1 = T^2 \circ \tau$;
(b) for each $x \in X^1 - Y_1$, $\varepsilon > 0$ and $\varepsilon' > 0$, there exist $\tilde{\varepsilon} > 0$ and $\tilde{\varepsilon}' > 0$ such that $\tau[S^{\tilde{\varepsilon}}(x)] \subset S^{\varepsilon}(\tau x)$, and $\tau^{-1}[S^{\tilde{\varepsilon}'}(\tau x)] \subset S^{\varepsilon'}(x)$.

We call such a map τ an 'isomorphism'.

This notion of equivalence is stronger than simply intertwining the actions of G_E. This is appropriate; it has been known at least since [CoK] that substitution subshifts show almost none of their richness if considered merely as subshifts. So in classifying tilings that have a hierarchical structure we make some feature of that hierarchical structure part of our notion of equivalence.

To define an invariant we extract information from the local contracting directions. Since the local contracting directions are preserved by equivalence, such information is manifestly invariant. We define here the invariant. In later sections we relate it to directly computable quantities (the $\mathcal{O}_j(x)$) and demonstrate its use in distinguishing between tiling dynamical systems.

Consider G_E as the semidirect product of $\mathrm{SO}(m)$ with $\mathbb{R}^m$, with $g = (r, t)$ denoting a rotation r about the origin followed by a translation t. Then consider, for any substitution tiling system X_{ϕ} and $\varepsilon > 0$:

$$R^{\varepsilon}(x) = \{r \in \mathrm{SO}(m): \text{there exists } t \text{ such that } T^{(r,t)}x \in S^{\varepsilon}(x)\}.$$

Now let $\mathcal{O}^{\varepsilon}(x)$ be the subgroup of $\mathrm{SO}(m)$ generated by $R^{\varepsilon}(x)$. The corollary to Theorem 2 shows that the conjugacy class of $\mathcal{O}^{\varepsilon}(x)$ is independent of x and ε (when small enough) for substitution tiling systems satisfying (iv) and (v). The conjugacy class of $\mathcal{O}^{\varepsilon}(x)$ is therefore an invariant of the tiling dynamical system, not just a feature of the individual tiling x.

2. The Group of Relative Orientations

The group $\mathcal{O}^{\varepsilon}(x)$ generated by $R^{\varepsilon}(x)$ is not directly computable. In this section we remedy this by constructing, for a substitution tiling system, a more easily

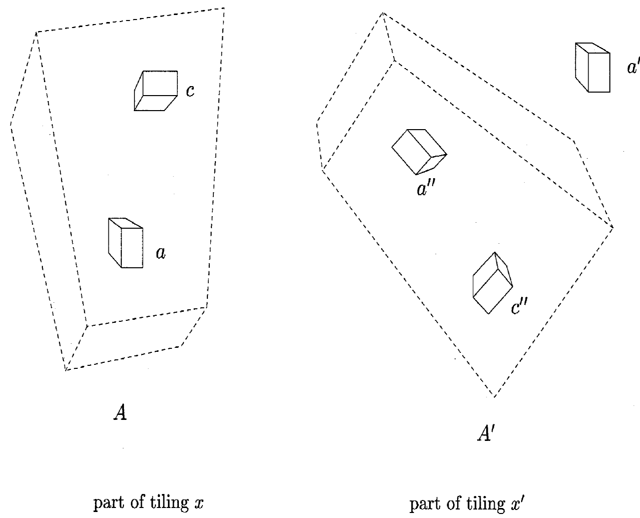

Figure 5.

computable group $\mathcal{O}_j(x)$ related to the relative orientations of the tiles in the single tiling x. The group $\mathcal{O}^\varepsilon(x)$ is then shown to be conjugate to $\mathcal{O}_j(x)$.

Given a tiling x and some tile a of type j in it, let $R_j(a, x) \subset \mathrm{SO}(m)$ be the set of relative orientations with respect to a of the tiles of type j in x; that is, $R_j(a, x)$ is the set of rotations of x which bring a tile of type j parallel to (the fixed) a. The group generated by $R_j(a, x)$ is easily seen to be generated by the relative orientations between *all* pairs of tiles of type j in x; in particular it is independent of a, and we denote it by $\mathcal{O}_j(x)$. Furthermore,

LEMMA 1. *If x' has a tile a' of type j parallel to the tile a in x, then $\mathcal{O}_j(x) = \mathcal{O}_j(x')$. For any $\tilde{x}$, $\mathcal{O}_j(\tilde{x})$ is conjugate to $\mathcal{O}_j(x)$.*

Proof. First note that $\mathcal{O}_j(x)$ is generated by the relative orientations between a and all other tiles of type j in x. So consider the generator g of $\mathcal{O}_j(x)$ which is the relative orientation of a tile c with respect to a in x. We will show that $g \in \mathcal{O}_j(x')$, from which it follows that $\mathcal{O}_j(x) \subset \mathcal{O}_j(x')$. By symmetry, we would then have $\mathcal{O}_j(x') \subset \mathcal{O}_j(x)$, and hence $\mathcal{O}_j(x) = \mathcal{O}_j(x')$.

From the definition of substitution tilings, the tiles a and c can be thought of as belonging to some supertile A of level n (although not all of A need exist in x). Since x' is tiled by supertiles of level n, there is a supertile A' of level n in x' containing a pair of tiles, a'' and c'', which have the same positions relative to A' as do a and c relative to A. See Figure 5.

Let g' be the relative orientation of c'' with respect to a''. Then $g = R^{-1}g'R$ where R is the relative orientation of A' with respect to A. But R is then also the relative orientation of a'' with respect to a, which is the same as that of a'' with respect to a', and R is thus an element of $\mathcal{O}_j(x')$. But then g is an element of $\mathcal{O}_j(x')$, as claimed.

If $\tilde{x}$ is any tiling at all, $\mathcal{O}_j(x)$ and $\mathcal{O}_j(\tilde{x})$ are conjugate by an element of $\mathrm{SO}(m)$, namely a rotation which makes a tile of type j in $\tilde{x}$ parallel to one in x. □

Finally we consider the dependence of $\mathcal{O}_j(x)$ on j.

DEFINITION. Two subgroups of $\mathrm{SO}(m)$ are 'c-equivalent' if each is conjugate (in $\mathrm{SO}(m)$) to a subgroup of the other. (Note that in SO(2) c-equivalence is the same as identity.)

LEMMA 2. *For any tilings $x, \tilde{x} \in X_\phi$ and tile types j and k, $\mathcal{O}_j(x)$ is c-equivalent to $\mathcal{O}_k(\tilde{x})$.*

Proof. By Lemma 1 it is sufficient to show that $\mathcal{O}_j(x)$ and $\mathcal{O}_k(x)$ are c-equivalent. Consider any two tiles a and b of type j in x, and let g be the relative orientation of b with respect to a. After one substitution a and b give rise to tiles a' and b' of type k in the tiling ϕx. The relative orientation of b' with respect to a' is again g, since g takes each part of b onto the corresponding part of a. Applying this construction to all the generators of $\mathcal{O}_j(x)$, we see that $\mathcal{O}_j(x)$ is a subgroup of $\mathcal{O}_k(\phi x)$. Similarly, $\mathcal{O}_k(x)$ is a subgroup of $\mathcal{O}_j(\phi x)$. But $\mathcal{O}_j(x)$ and $\mathcal{O}_k(x)$ are conjugate to $\mathcal{O}_j(\phi x)$ and $\mathcal{O}_k(\phi x)$, respectively, so $\mathcal{O}_j(x)$ and $\mathcal{O}_k(x)$ are conjugate to subgroups of each other. □

LEMMA 3. *Assume ϕ satisfies (v). Then $\mathcal{O}_j(x) = \mathcal{O}_k(x)$.*

Proof. Let $n = \Pi_a n_a$. Since the tilings defined by ϕ^n are the same as those defined by ϕ we can, without loss of generality, assume $n = 1$, so that ϕx contains tiles parallel to every tile of x. Then, by Lemma 1, $\mathcal{O}_j(x) = \mathcal{O}_j(\phi x)$ and $\mathcal{O}_k(x) = \mathcal{O}_k(\phi x)$. But we have shown that $\mathcal{O}_k(x) \subset \mathcal{O}_j(\phi x)$ and $\mathcal{O}_j(x) \subset \mathcal{O}_k(\phi x)$, so $\mathcal{O}_j(x) = \mathcal{O}_k(x)$. □

To summarize: From Lemmas 1 and 2 we can associate a subgroup of $\mathrm{SO}(m)$ to any substitution tiling system, uniquely defined up to c-equivalence. If the substitution tiling system satisfies (v), Lemma 3 shows that the group is uniquely defined up to conjugacy.

Before we can use these groups as an invariant for equivalence of substitution tiling systems we must refer to the relative orientations in a more fundamental way. Our next goal is to connect this group with the invariant introduced at the end of Section 1. The essential observation is that, if tilings $x \neq y$ agree in some neighborhood of the origin in Euclidean space, then ϕx and ϕy will agree in a larger neighborhood of the origin, so we typically expect $d(\phi x, \phi y) < d(x, y)$. We are thus led to a quantity introduced earlier. For each x in the substitution tiling system X_ϕ and for each $\varepsilon > 0$, consider:

$$S^\varepsilon(x) = \{y \in X_\phi : d(\phi^n y, \phi^n x) < \varepsilon \text{ for all } n \geqslant 0, \\ \text{and } d(\phi^n y, \phi^n x) \underset{n\to\infty}{\longrightarrow} 0\}. \tag{3}$$

THEOREM 1. *Assume a substitution tiling system X_ϕ.*

(a) *Given any $\varepsilon > 0$ there exists $N > 0$ such that $B_N(\partial y) = B_N(\partial x)$ implies $y \in S^\varepsilon(x)$.*

(b) *There exists $\varepsilon > 0$ such that, for every $x \in X_\phi$, $y \in S^\varepsilon(x)$ and every tile $a \in x$ that meets the origin, there is a tile $a' \in y$ that exactly coincides with a.*

Proof. (a) is immediate from the form of the metric. The proof of (b) requires the following two lemmas.

LEMMA 4. *For every $N > 0$ and every neighborhood U of the identity in G_E there exists $\varepsilon > 0$ with the following property: Let $x, x' \in X(\mathcal{A})$ be any two tilings with $d(x, x') < \varepsilon$, and let a be a tile of x that is contained in B_N. Then x' contains a tile a' of the form $T^g a$ where $g \in U$.*

Proof. Let $z > 0$ be such that for each $b \in \mathcal{A}$ some ball of diameter z lies in the interior of b. Fix some $\delta \in (0, z/3)$ and define the heart $h_\delta(b)$ of $b \in \mathcal{A}$ as $\{p \in b\colon \|p - q\| > \delta$ for all $q \in \partial b\}$. By the corridor $C_\delta(x)$ of a tiling x we mean the complement of $\cup_{b\in x} h_\delta(b)$. Let D be the largest of the diameters of all $b \in \mathcal{A}$. Without loss of generality, we can assume $N > D$.

With this notation we note that if $d(x, x') < \delta/(N + D)$ we have $B_N(\partial x) \subset C_\delta(x')$ and $B_N(\partial x') \subset C_\delta(x)$. So if $\delta \sim 0$ each tile in x in B_N is closely approximated by some tile in x' and vice versa. In particular it now follows that for small enough ε, if $d(x, x') < \varepsilon$ then the tiles $a' \in x'$ must be of the same type as the tiles $a \in x$ they approximate, and in fact satisfy $a' = T^g a$ with $g \in U$. □

LEMMA 5. *For each $N > 0$ there is a neighborhood U_N of the identity in G_E with the following property: If a is a tile in B_N and g_1 and g_2 are distinct elements of U_N, then $T^{g_1}a$ and $T^{g_2}a$ overlap but are distinct. In particular, it is impossible for $T^{g_1}a$ and $T^{g_2}a$ to both be tiles in the same tiling.*

Proof. This follows from the continuity of the action of G_E on tiles, and the fact that polyhedra do not admit infinitesimal symmetries. □

We now return to the proof of Theorem 1. Pick $N > D$ and let U_N be as in Lemma 5. Pick a smaller bounded neighborhood $U \subset U_N$ of the identity of G_E with the property that $(r, t) \in U$ implies $(r, |\phi| t) \in U_N$. Then pick ε small enough that Lemma 4 applies.

Let a be a tile of x containing the origin. By Lemma 4 there is a tile a' in y of the form $T^g a$ with $g = (r, t) \in U \subset G_E$. We will show that $t \neq 0$ implies that, for some n, $d(\phi^n x, \phi^n y) > \varepsilon$, while $t = 0$, $r \neq 0$ implies that $\lim_{n\to\infty} d(\phi^n x, \phi^n y) \neq 0$. This will complete the proof.

Note that $\phi^n a' = T^{(r, |\phi|^n t)}\phi^n a$. If $t \neq 0$, pick n such that $(r, |\phi|^n t)$ is outside the neighborhood U but in U_N. Let $\tilde{a}$ be a tile of $\phi^n a$ containing the origin. Then there is a tile $\tilde{a}' = T^{(r, |\phi|^n t)}\tilde{a}$ in $\phi^n y$. But by Lemma 5 this means there cannot be a tile of the form $T^g \tilde{a}$ in y with $g \in U$. By Lemma 4 this means that $d(\phi^n x, \phi^n y) \geqslant \varepsilon$.

If $t = 0$ then $\phi^n a' = T^{(r,0)}\phi^n a$. If $r \neq 0$, for every tile $\tilde{a} \in \phi^n a$ containing the origin there is a tile $\tilde{a}' \in \phi^n a'$ overlapping it and with relative orientation r, which implies that the distance between $\phi^n x$ and $\phi^n y$ will not go to zero. □

Recall the following quantity from Section 1

$$R^\varepsilon(x) = \{r \in \mathrm{SO}(m)\colon \text{there exists } t \text{ such that } T^{(r,t)}x \in S^\varepsilon(x)\}. \tag{4}$$

Let $\mathcal{O}^\varepsilon(x)$ be the group generated by $R^\varepsilon(x)$. Assuming ε small enough for Theorem 1(b), we see that every $r \in R^\varepsilon(x)$ is the relative orientation of a tile of x with respect to a corresponding tile of x near the origin. By Theorem 1(a), if C is a region of x containing B_N, and if C' is any region of x congruent to C, then $R^\varepsilon(x)$ includes the relative orientation of C' to C.

Consider the following property.

PROPERTY F. *The subset of tilings x, for which every fixed finite ball B of Euclidean space is contained in some supertile of finite level in x, is of full measure for every translation invariant measure on X_ϕ.*

We will prove that Property F holds for a large class of interesting systems, at least those satisfying condition (iv). This assumption, which implies that ϕ is a homeomorphism on X_ϕ, is satisfied by all known nonperiodic examples. In fact it is automatically true for a system that contains nonperiodic tilings and in which the tiles only appear in finitely many orientations in any tiling [Sol].

If a tiling contains two or more regions each tiled by supertiles of level n for all $n \geqslant 0$, we call these regions supertiles of infinite level. Recall that any tiling is tiled by supertiles of any finite level n. If a ball in a tiling x fails to lie in *any* supertile of any level n, then x is tiled by two or more supertiles of infinite level, with the offending ball straddling a boundary. (One can construct a pinwheel tiling with two supertiles of infinite level as follows. Consider the rectangle consisting of two supertiles of level $n-1$ in the middle of a supertile of level n. For each $n \geqslant 1$ orient such a rectangle with its center at the origin and its diagonal on the x-axis, and fill out the rest of a (non-pinwheel) tiling x_n by periodic extension. By compactness this sequence has a convergent subsequence, which will be a pinwheel tiling and which will consist of two supertiles of infinite level.)

We now use the above to prove:

LEMMA 6. *For a substitution tiling system satisfying (iv), let S be the set of tilings in which some ball does not lie within a supertile of any level n. S has zero measure with respect to any translation invariant measure on X_ϕ.*

Proof. We only give the proof for dimension $m = 2$. Note first that the boundary of a supertile of infinite level must be either a line, or have a single vertex, since it is tiled by supertiles of all levels and therefore cannot contain a finite edge.

Furthermore, for a given substitution system there is a constant K such that no tiling in it contains more than K vertices of supertiles of infinite level; specifically, one can take $K = 2\pi/p$ where p is the smallest angle of any of the vertices of the tiles.

Next we fix some orthogonal coordinate system in the plane and decompose S into disjoint subsets as follows. Let $C = [0, 1) \times [0, 1)$ be the 'half open' unit edge square in $\mathbb{R}^2$. Let C_t be the translate of C by the vector t. Let S' be the subset of S consisting of tilings containing vertices of supertiles of infinite level. For $x \in S'$ we choose a vertex $V(x)$ by lexicographic order: we choose that vertex which in the given coordinate system has the largest first coordinate; if there is more than one with that coordinate we choose the one with the largest second coordinate. Then we decompose $S' = \bigcup_{t \in \mathbb{Z}^2} S'(t)$, where $x \in S' \cap S'(t)$ if $V(x) \in C_t$. It is easy to see that each $S'(t)$ is measurable, and that they are translates of one another so they must have zero measure with respect to any translation invariant measure.

The tilings $x \in S - S'$ contain two supertiles of infinite level, each occupying a half plane. Next we decompose $S - S' = \cup_{j,k \in \mathbb{Z}} \cup \sigma_j \cup \sigma'_k$ where $x \in S - S' \cap \sigma_j$ if the boundary between the supertiles of infinite level crosses the first axis in $[j, j+1)$, and $x \in S - S' \cap \sigma'_k$ if the boundary between the supertiles of infinite level is parallel to the first axis and crosses the second axis in $[k, k+1)$. Note that all sets σ_j are translates of one another, and all sets σ'_k are translates of one another, so $S - S'$ has zero measure with respect to any translation invariant measure. □

THEOREM 2. *For any substitution tiling system X_ϕ satisfying (iv), there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, and for almost every tiling $x \in X_\phi$, $\mathcal{O}^\varepsilon(x)$ is c-equivalent to $\mathcal{O}_j(x)$ for some (and therefore any) j. Up to conjugacy, $\mathcal{O}_j(x)$ is independent of x. Furthermore, if ϕ satisfies (v) then $\mathcal{O}^\varepsilon(x) = \mathcal{O}_j(x)$ for some (and therefore any) j.*

Proof. From Theorem 1(b) it follows that, for small ε, $\mathcal{O}^\varepsilon(x)$ is contained in $\mathcal{O}_j(x)$, where j is the type of any of the tiles of x which meet the origin. On the other hand, let N correspond to ε in Theorem 1(a). By Lemma 6, for almost every x there is some n such that the tiles which intersect B_N are contained in some supertile b of level n in x. Let k be the type of b. It follows from Theorem 1(a) that $\mathcal{O}_k(x')$ is a subgroup of $\mathcal{O}^\varepsilon(x)$, where $x' = \phi^{-n}x$. But $\mathcal{O}_k(x')$ and $\mathcal{O}_j(x)$ are c-equivalent, so $\mathcal{O}_j(x)$ is conjugate to a subgroup of $\mathcal{O}_k(x')$, and therefore is conjugate to a subgroup of $\mathcal{O}^\varepsilon(x)$. So $\mathcal{O}_j(x)$ and $\mathcal{O}^\varepsilon(x)$ are c-equivalent. By Lemma 1, $\mathcal{O}_j(x)$ is, up to conjugacy, independent of x. If ϕ satisfies (v), then $\mathcal{O}_k(x') = \mathcal{O}_k(x) = \mathcal{O}_j(x)$. Since $\mathcal{O}_j(x) = \mathcal{O}_k(x') \subset \mathcal{O}^\varepsilon(x) \subset \mathcal{O}_j(x)$, $\mathcal{O}^\varepsilon(x) = \mathcal{O}_j(x)$. □

COROLLARY 1. *For each substitution tiling system satisfying (iv) the group $\mathcal{O}^\varepsilon(x)$ is uniquely defined up to c-equivalence, for almost all tilings x, and all small enough ε, thus the c-equivalence class of the group is an invariant for equivalence. Furthermore, among substitution tiling systems also satisfying (v), the conjugacy class of this subgroup of* $\mathrm{SO}(m)$ *is an invariant for equivalence.*

3. Abstract Substitution Systems

In going from Lemma 2 to Theorem 2 we see that we can associate with each substitution tiling system a c-equivalence class of subgroups of $\mathrm{SO}(m)$ in a reasonably fundamental way. We are now ready to relax the hypotheses.

DEFINITION. A 'substitution (dynamical) system' is a quadruple $(X, T, \phi, |\phi|)$ consisting of a compact metric space X on which there is a continuous action $T\colon (g, x) \in G_E \times X \to T^g x \in X$ of G_E and a homeomorphism $\phi\colon X \to X$ such that $\phi[T^g x] = T^{\phi(g)} x$ for all x, where $\phi(g)$ is the conjugate of g by the similarity of Euclidean space consisting of stretching about the origin by $|\phi| > 1$.

Substitution tiling systems are special cases of substitution systems. The map ϕ is not intrinsic to the substitution tiling system (X_ϕ, T) since, for tiling systems, ϕ and ϕ^k lead to the same set of tilings; so equivalence of such systems should not be required to intertwine the actions of the maps ϕ. The objects $S^\varepsilon(x)$, $R^\varepsilon(x)$ and $\mathcal{O}^\varepsilon(x)$ are well-defined in our abstract setting. Motivated by the last section, we use the following notion of equivalence.

DEFINITION. The substitution systems $(X^1, T^1, \phi^1, |\phi^1|)$ and $(X^2, T^2, \phi^2, |\phi^2|)$ are 'equivalent' if there are subsets $Y_j \subset X^j$, invariant under T^j and of measure zero with respect to all translation invariant Borel probability measures on X_{ϕ^j}, and a one-to-one, onto, Borel bimeasureable 'isomorphism' $\tau\colon X^1 - Y_1 \to X^2 - Y_2$, such that $\tau \circ T^1 = T^2 \circ \tau$. Furthermore, τ must respect the 'local contracting directions' $S^\varepsilon(x)$. Respecting the local contracting directions means that, for each $x \in X^1 - Y_1$, $\varepsilon > 0$ and $\varepsilon' > 0$, there exist $\tilde{\varepsilon} > 0$ and $\tilde{\varepsilon}' > 0$ such that $\tau[S^{\tilde{\varepsilon}}(x)] \subset S^\varepsilon(\tau x)$, and $\tau^{-1}[S^{\tilde{\varepsilon}'}(\tau x)] \subset S^{\varepsilon'}(x)$.

It is easy to see that for the special case of substitution tiling systems this notion of equivalence reduces to that previously defined. We will now introduce an invariant for equivalence which reduces to the class of subgroups of $\mathrm{SO}(m)$ we found for substitution tiling systems. We note that this allows us to generalize our discussion of substitution tiling systems to include tiling systems which do not quite fit the conditions of Section 2. In particular, our analysis applies to the various versions of Penrose tilings of the plane, such as the kite and dart tilings, both the substitution version and the version with edge markings [Gar, Ra3], and to the various tilings discussed in [G-S, Moz].

We will need to introduce a few more definitions. Given two subgroups G_1 and G_2 of $\mathrm{SO}(m)$ we write $G_1 \prec G_2$ if G_1 is conjugate (by an element of $\mathrm{SO}(m)$) to a subgroup of G_2. The binary relation $\prec$ lifts in an obvious way to a partial ordering on the set of c-equivalence classes. We denote by 'lower bound' to a set $\tilde{S}$ of subgroups of $\mathrm{SO}(m)$ any c-equivalence class of groups G each of which satisfies $G \prec S$ for all $S \in \tilde{S}$. It is almost immediate that $\mathcal{O}^\varepsilon(x) \prec \mathcal{O}^{\varepsilon'}(x)$ if $\varepsilon < \varepsilon'$. For each $x \in X$ we define $\hat{\mathcal{O}}(x)$ as the set of all lower bounds of the family $\{\mathcal{O}^\varepsilon(x)\colon \varepsilon > 0\}$; it is nonempty since it contains $\{e\}$. Note that the set $\hat{\mathcal{O}}(x)$ is an invariant for substitution systems – if τ is an isomorphism then $\hat{\mathcal{O}}(\tau x) = \hat{\mathcal{O}}(x)$ for almost every x. For substitution tiling systems, the sets $\hat{\mathcal{O}}(x)$ have unique greatest elements which are constant for almost every x with respect to every translation invariant measure. In the latter case, where $\hat{\mathcal{O}}(x)$ has an almost everywhere constant greatest element, we denote this greatest element by $[\mathcal{O}](X)$. Note that $[\mathcal{O}](X)$ is a c-equivalence class, unlike $\mathcal{O}^{\varepsilon}(x)$, which is a specific group. We have thus generalized the analysis of substitution tiling systems to the more general setting.

As with substitution tiling systems, we can avoid the use of c-equivalence classes for systems with a special property.

PROPERTY P. *For almost every x there exists an $\varepsilon > 0$ such that, if $0 < \varepsilon' < \varepsilon$, then $\mathcal{O}^{\varepsilon'}(x) = \mathcal{O}^{\varepsilon}(x)$.*

Note that, by Theorem 2, any substitution tiling system that satisfies (v) also satisfies Property P. If a substitution system satisfies Property P, we can define $\mathcal{O}(x)$ to be $\mathcal{O}^{\varepsilon}(x)$ for ε sufficiently small. If the conjugacy class of $\mathcal{O}(x)$ is almost everywhere constant, we define $[\mathcal{O}]_0(X)$ to be that conjugacy class. The previously defined $[\mathcal{O}](X)$ is, of course, the c-equivalence class of $[\mathcal{O}]_0(X)$.

THEOREM 3. *Suppose $(X^1, T^1, \phi^1, |\phi^1|)$ and $(X^2, T^2, \phi^2, |\phi^2|)$ are equivalent substitution systems, with the notation of the definition. Then if $(X^1, T^1, \phi^1, |\phi^1|)$ satisfies Property P so does $(X^2, T^2, \phi^2, |\phi^2|)$. Furthermore, for almost every $x \in X^1$, $\mathcal{O}(\tau x) = \mathcal{O}(x)$. In particular, if $\mathcal{O}(x)$ is almost everywhere constant up to conjugacy then $\mathcal{O}(\tau x)$ is almost everywhere constant up to conjugacy and $[\mathcal{O}]_0(X^2) = [\mathcal{O}]_0(X^1)$.*

Proof. Let x be a generic point of X^1. Since $(X^1, T^1, \phi^1, |\phi^1|)$ has Property P we can find $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, $\mathcal{O}^{\varepsilon}(x) = \mathcal{O}^{\varepsilon_0}(x) = \mathcal{O}(x)$. From the equivalence we can find $\tilde{\varepsilon}$ such that $\mathcal{O}^{\tilde{\varepsilon}}(\tau x) \subset \mathcal{O}^{\varepsilon_0}(x)$. Now let $\varepsilon_0' = \tilde{\varepsilon}$. We will show that, for any $0 < \varepsilon' \leqslant \varepsilon_0'$, $\mathcal{O}^{\varepsilon'}(\tau x) = \mathcal{O}^{\epsilon_0'}(\tau x) = \mathcal{O}(x)$. From this it will follow that (X', ϕ') has Property P and that $\mathcal{O}(\tau x) = \mathcal{O}(x)$.

Fix any $0 < \varepsilon' \leqslant \varepsilon_0'$. Since τ is an isomorphism there exists $\tilde{\varepsilon}' > 0$ such that $\mathcal{O}^{\tilde{\varepsilon}'}(x) \subset \mathcal{O}^{\varepsilon'}(\tau x)$. But $\mathcal{O}^{\varepsilon'}(\tau x) \subset \mathcal{O}^{\epsilon_0'}(\tau x) \subset \mathcal{O}^{\varepsilon_0}(x)$. If $\mathcal{O}^{\tilde{\varepsilon}'}(x) = \mathcal{O}^{\varepsilon_0}(x)$ then all the inclusions must be equalities, and we are done. So it suffices to show $\mathcal{O}^{\tilde{\varepsilon}'}(x) = \mathcal{O}^{\varepsilon_0}(x)$. If $\tilde{\varepsilon}' \leqslant \varepsilon_0$ this follows from the definition of ε_0. But if $\tilde{\varepsilon}' > \varepsilon_0$ then $O^{\varepsilon_0}(x) \subset O^{\tilde{\varepsilon}'}(x)$, so $\mathcal{O}^{\tilde{\varepsilon}'}(x) = \mathcal{O}^{\varepsilon_0}(x)$. □

4. Examples and Analysis of the Invariant

For pinwheel $[\mathcal{O}]_0(X_{\phi_p})$ is the group generated by rotations by $\pi/2$ and 2 arctan $(\frac{1}{2})$; for the variant of pinwheel $[\mathcal{O}]_0(X_{\phi_v})$ is the group generated by rotations by $\pi/2$ and $2\arctan(\frac{1}{3})$ [RaS]. It is clear that these are distinct, so the substitution tiling systems are not equivalent.

Rotations appear in more interesting ways in 3-dimensional tilings, for example the quaquaversal and dite and kart substitution tiling systems, defined in [CoR] and

[RaS] respectively. These systems both satisfy (v) and therefore Property P. Let R_x^θ be a rotation about the x axis by an angle θ, with similar notation for other axes. If we denote by $G(p,q)$ the subgroup of SO(3) generated by $R_x^{2\pi/p}$ and $R_y^{2\pi/q}$, it can be shown [CoR, RaS] that $[\mathcal{O}]_0(X_{\phi_q})$ is the conjugacy class of $G(6,4)$ for the quaquaversal tilings and $[\mathcal{O}]_0(X_{\phi_{d\&k}})$ is the conjugacy class of $G(10,4)$ for the dite and kart tilings. We shall see that $G(6,4)$ and $G(10,4)$ are not conjugate (indeed not even c-equivalent) by using the following obvious fact: if the groups G and G' are conjugate (or c-equivalent) and one of them has an element of order m (finite or infinite) then the other must have an element of order m.

Structure Theorem for G(p,q) [RaS]

(a) If $p, q \geqslant 3$ are odd, then $G(p,q)$ is isomorphic to the free product

$$\mathbb{Z}_p * \mathbb{Z}_q = \langle \alpha, \beta \colon \alpha^p, \beta^q \rangle. \tag{5}$$

(b) If $p \geqslant 4$ is even and $q \geqslant 3$ is odd, then $G(p,q)$ has the presentation

$$\langle \alpha, \beta \colon \alpha^p, \beta^q, (\alpha^{p/2}\beta)^2 \rangle. \tag{6}$$

(c) If $p \geqslant 4$ is even and $q = 2s$, $s \geqslant 3$ odd, then $G(p,q)$ has the presentation

$$\langle \alpha, \beta \colon \alpha^p, \beta^q, (\alpha^{p/2}\beta)^2, (\alpha\beta^s)^2 \rangle. \tag{7}$$

(d) If 4 divides both p and q, then $G(p,q) = G([p,q],4)$, where $[p,q]$ denotes the least common multiple of p and q.
(e) If 4 divides m, then $G(m,4)$ has the presentation

$$\langle \alpha, \beta, : \alpha^m, \beta^4, (\alpha^{m/2}\beta)^2, (\alpha\beta^2)^2, (\alpha^{m/4}b)^3 \rangle. \tag{8}$$

In cases (a), (b) and (c), the isomorphism between the abstract presentation and $G(p,q)$ is given by $\alpha \mapsto R_x^{2\pi/p}$, $\beta \mapsto R_y^{2\pi/q}$. In case (e) the isomorphism is similar.

THEOREM 4. (a) *If 4 does not divide both p and q then the orders of elements of finite order in $G(p,q)$ are* {factors of p} $\cup$ {factors of q}; (b) *If 4 divides both p and q then the orders of elements of finite order in $G(p,q)$ are* {factors of $[p,q]$} $\cup$ {3}.

COROLLARY 2. *If p and q are not both divisible by 4, and p' is not a factor of p or q, then $G(p,q)$ and $G(p',q')$ are not c-equivalent.*

COROLLARY 3. *The quaquaversal and dite and kart systems are not equivalent.*

Proof of the theorem. (a) Assume $g \in G(p,q)$ has finite order $\neq 1$. We know g can be expressed in one of the forms $g = A^{a_1}B^{b_1}A^{a_2}\ldots A^{a_{n+1}}$, $g = B^{b_1}A^{a_1}B^{b_2}\ldots B^{b_{n+1}}$, $g = B^{b_1}A^{a_1}B^{b_2}\ldots A^{a_n}$ or $g = A^{a_1}B^{b_1}A^{a_2}\ldots B^{b_n}$, with all

$0 < a_j < p$, $0 < b_j < q$ and $n \geqslant 1$. Within the class of $g' \in G(p,q)$ which are conjugate to g with respect to $G(p,q)$, we assume that n is minimal. Assume $n \geqslant 2$; we will obtain a contradiction. Since one could conjugate with A^{a_1}, the form $A^{a_1}B^{b_1}A^{a_2}\dots A^{a_n}$ can be exchanged for $B^{b_1}A^{a_2}\dots A^{a_n}$ and since one could conjugate with B^{b_1}, the form $B^{b_1}A^{a_1}B^{b_2}\dots B^{b_n}$ can be exchanged for $A^{a_1}B^{b_2}\dots B^{b_n}$, and the form $B^{b_1}A^{a_1}B^{b_2}\dots A^{a_n}$ for $A^{a_1}B^{b_1}A^{a_2}\dots B^{b_n}$. So we assume that the form is $A^{a_1}B^{b_1}A^{a_2}\dots B^{b_n}$. If $a_j = p/2$ (resp. $b_j = q/2$) for any j then we could use the relation $A^{p/2}B^b = B^{-b}A^{p/2}$ (resp. $B^{q/2}A^a = A^{-a}B^{q/2}$) to reduce the value of n; thus these values of a_j (or b_j) cannot occur. But then, by the structure theorem, g has infinite order, which is a contradiction. Thus n must equal 1, and g can be assumed to be of the form A^{a_1}, B^{b_1} or $A^{a_1}B^{b_1}$. Considering $A^{a_1}B^{b_1}A^{a_1}B^{b_1}\dots A^{a_1}B^{b_1}$, the only way $A^{a_1}B^{b_1}$ could have finite order is if $a_1 = p/2$ or $b_1 = q/2$, in which case g has order 2, and 2 is a factor of p or q. Finally, the elements A^{a_1} can have as orders any factor of p and the elements B^{b_1} can have as orders any factor of q.

(b) If p and q are divisible by 4 then $G(p,q) = G([p,q],4)$, so we consider $G(m,\mathbf{4})$ with m divisible by 4. Using the presentation (8), we can put any $g \in G(m,4)$ in the form $WST^{a_1}ST^{a_2}\dots ST^{a_n}E$ with $S = R_y^{2\pi/4}$, $T = R_x^{2\pi/m}$, $n \geqslant 0$, $a_j \neq km/\mathbf{4}$ and with both W and E in the cube group $G(\mathbf{4},\mathbf{4})$. Assume g has finite order $\neq 1$ and that in its conjugacy class (which of course all have the same order), the smallest value of n in the above representation is $\geqslant 2$. (We will obtain a contradiction to this.) By conjugation we eliminate W from g.

Now $G(4,4)$ can be partitioned: $G(4,4) = H_1 \cup H_1S \cup H_1SU$, where $U = R_x^{2\pi/\mathbf{4}}$ and H_1 is the 8 element subgroup generated by S^2 and U. In detail,

$$H_1 = \{1, U, U^2, U^3, S^2, S^2U, S^2U^2, S^2U^3\}. \tag{9}$$

Some power of g equals the identity element

$$(ST^{a_1}ST^{a_2}\dots ST^{a_n}E)(ST^{a_1}ST^{a_2}\dots ST^{a_n}E)(\dots) = e. \tag{10}$$

We consider the three cases: (i) $E \in H_1$; (ii) $E \in H_1SU$; (iii) $E \in H_1S$.

(i) The factor E in (10) is of the form $(S^2)^aU^b$ with $a = \mathbf{0}, 1$ and $b = \mathbf{0}, 1, 2, 3$. We alter (10) to

$$\begin{aligned}&[ST^{a_1}ST^{a_2}\dots ST^{a_n}U^{(-1)^a b}(S^2)^a]\dots\\&\quad\times[ST^{a_1}ST^{a_2}\dots ST^{a_n}U^{(-1)^a b}(S^2)^a][\dots] = e,\end{aligned} \tag{11}$$

or

$$\begin{aligned}&[ST^{a_1}ST^{a_2}\dots ST^{[a_n+(-1)^a bm/\mathbf{4}]}(S^{2a})]\\&\quad\times[ST^{a_1}ST^{a_2}\dots ST^{[a_n+(-1)^a bm/4]}(S^{2a})]\dots = e,\end{aligned} \tag{12}$$

and we know [RaS] this cannot be the case. So we cannot have $E \in H_1$.

(ii) E is now of the form $(S^2)^a U^b SU$ with $a = 0, 1$ and $b = 0, 1, 2, 3$. We now alter (10) to

$$ST^{a_1} ST^{a_2} \dots ST^{[a_n + (-1)^a bm/4]} (S^2)^a SUST^{a_1} ST^{a_2} \dots$$

$$\dots ST^{a_n} E \cdots = e. \tag{13}$$

Using $SUS = USU$, (13) becomes

$$ST^{a_1} ST^{a_2} \dots ST^{[a_n + (-1)^a (b+1)m/4]} S^{2a} ST^{a_1 + m/4} ST^{a_2} \dots$$

$$\dots ST^{[a_n + (-1)^a (b+1)m/4]} S^{2a} SU \cdots = e. \tag{14}$$

Again, we know [RaS] this cannot be the case. So we cannot have $E \in H_1 SU$.

(iii) We cannot have $E \in H_1 S$ and $n \geqslant 2$. For if we represent conjugacy by $\cong$

$$g \cong ST^{a_1} ST^{a_2} \dots ST^{a_n} E$$

$$= ST^{a_1} ST^{a_2} \dots ST^{a_n} (S^2)^a U^b S$$

$$\cong T^{a_1} ST^{a_2} \dots ST^{[a_n + (-1)^a bm/4]} (S^2)^{a+1}$$

$$\cong ST^{a_2} \dots ST^{[a_n + (-1)^a bm/4 + (-1)^{a+1} a_1]} (S^2)^{a+1}, \tag{15}$$

and g is conjugate to a word with smaller n.

Thus $n = 0$ or $n = 1$. $n = 0$ means $g \in G(4, 4)$, and these have orders $1, 2, 3, 4$. $n = 1$ means g is of the form $ST^{a_1} E$ where $a_1 \neq km/4$ and $E \in G(4, 4)$. We again consider the three cosets to which E may belong. As before we see that cases (i) and (ii) lead to infinite order for g. But in case (iii) g is conjugate to $T^{a_1} (S^2)^a U^b S^2 \cong T^c (S^2)^d$, which can have as orders the factors of m. □

5. Conclusion

We have been concerned with substitution tilings of Euclidean spaces, and have defined an invariant for them related to the group generated by the relative orientations of the tiles in a tiling. This feature is captured in an intrinsic way by means of a contractive behavior of the substitution. It is unrelated to other features of tiling systems, such as their topology, and we introduce the notion of substitution dynamical system to emphasize the features associated with the invariant.

To distinguish examples, for instance to distinguish the quaquaversal tilings from the dite and kart tilings, requires consideration of 2-generator subgroups of SO(3), in particular the orders of elements of such subgroups, which we analyze.

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