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Dynamics of Limit Models*

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Abstract. Using noncommutative integration theory, we show why certain singular behavior has been appearing in the dynamics of large quantum mechanical systems, and how to isolate the singularities.

§ 0. Introduction

The aim of this paper is to propose a solution for certain difficulties which arise when treating dynamics within the C^*-algebra formalism. The presence of these difficulties is evident from a phenomenon described by Thirring and Wehrl [1] and then others [2–6] for certain nonrelativistic quantum mechanical models. The phenomenon is that for these models the dynamics cannot be represented by a group of automorphisms of the C^*-algebra of observables, at least as far as all initial states are concerned. Rather, one finds that in different equilibrium representations the dynamics is represented by different automorphisms, and may even map the observables out into the von Neumann algebra which they generate. The difficulty that this leads to is that for a general initial state ϱ and observable A, the time dependent expectation value $\varrho(A_t)$ is not well defined.

The above phenomenon has been treated from one point of view by Dubin and Sewell in [2], and to that extent one knows how to extend the C^*-algebra formalism to cope with equilibrium Green's functions and the dynamics of initial states near thermal equilibrium. (For a discussion of the meaning of "near" see [7].) In this paper we will attack the complementary problem of describing the time development of initial states which are far from equilibrium.

We organize our argument as follows. In § 1 we note that an analogous problem occurs in classical mechanics, and we sketch the manner in which it can be overcome. In the next two sections we use and extend parts of Segal's noncommutative integration theory [8] to generalize

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the classical (abelian) solution to an abstract noncommutative domain, and in § 4 we relate our results to a broad class of "limit models" of physical interest, including the BCS model.

§ 1. Motivation from a Simple Classical Model

As we inferred in the introduction, even in classical mechanics time development cannot generally be represented by automorphisms of the C^*-algebra of observables $C(X)$ (the bounded continuous functions on X, which is all or part of phase space), or equivalently by homeomorphisms of phase space. For infinite volume systems with reasonable interactions this is one of the many consequences of Lanford's work [9, 10] where it is shown that, for certain initial states catastrophes occur in finite time, thus preventing any global time evolution for these states. However we need not resort to such difficult models for insight; the C^*-algebra formalism should be able to cope with the mundane dynamical system of N (hard, frictionless) billiard balls on a (rectangular) billiard table. In this model it is easy to see that whenever a ball hits a wall, thereby reversing a component of its momentum, some observable (i.e. bounded continuous function on phase space) is going to become discontinuous. Therefore the dynamics cannot be represented by automorphisms of the observables. But even more interesting is the fact that there are some initial states for which there is no canonical time evolution assignable at all. An example of this is the following. Assume that at time zero three balls labeled 1, 2 and 3 are spatially arranged so that each of the balls touches the other two, but no wall or other ball. Assume that for small negative time, balls 2 and 3 are stationary and that ball 1 has finite velocity directed towards the center of ball 2. We claim that there is no canonical dynamics past time zero. To see this, consider two slightly different initial configurations; one where ball 3 is shifted a distance ε in such a way that ball 1 hits ball 2 without touching ball 3, and another where ball 3 is shifted a distance ε in such a way that ball 1 hits ball 3 before ball 2. As $\varepsilon \to 0$, the two different situations lead to very different motions for positive time. In the first case ball 3 remains stationary and in the second it obtains a finite (not order ε) velocity. Clearly either motion is a possible assignment for positive time of the evolution of the original configuration, possible in the sense that it violates no physical principle. Fortunately however it can be shown (with some work!) that the set of initial configurations which, when evolved back or forward in time, never lead to such a singular configuration has full measure (i.e. the complement is of measure zero). Another feature of the model is that for each of these remaining (nonstationary) states there is a nonempty set of time of measure zero (the moments when simple collisions take place) for which no meaningful state is assignable, since the momentum of a colliding ball is ill-defined at the moment of collision.

In summary, the dynamics of the billiard ball model has two noteworthy features. First, there is a set of initial states of measure zero for which no canonical, global, time evolution is attributable. And second, for each of the remaining initial states there is a set of time of measure zero for which no canonical evolved state is assignable. For these reasons, it is advantageous to represent the dynamics of this system by a one parameter group of automorphisms of $L_\infty(X, dx)$ so that problems on sets of measure zero are automatically taken care of. But not only is it convenient to work this way; the main point of this section is that such an algebraic description still has within it essentially all the information of the more intuitive description (i.e. the evolution of the points in phase space), *and that this information can be retrieved* from the algebraic formalism. We will now sketch how this retrieval can be effected for the billiard ball model; some further comments will be made in § 4.

First we note that the evolution $\{f_t \,|\, t \in \mathbb{R}\}$ of each observable f in $C(X)$ can be considered an element of $L_\infty(\mathbb{R}, L_\infty(X, dx), dt)$, the equivalence classes of $L_\infty(X, dx)$-valued, essentially bounded dt-measurable functions on $\mathbb{R}$, which is isomorphic to $L_\infty(\mathbb{R} \times X, dt \times dx)$ from § 1.22 of [11]. Then all that is needed is elementary measure theory (Theorem 12.21 i) of [12]) to see that we can choose a representative of $f_{(\cdot)}(\cdot)$ and, evaluating it at almost every x in X obtain a function $f_{(\cdot)}(x)$ in $L_\infty(\mathbb{R})$. The essential uniqueness of this procedure, whereby we reconstruct the physical data $\{f_t(x) \,|\, \text{almost every } x \in X,\ t \in \mathbb{R}\}$, is then not hard to prove.

In summary, if the time evolution is represented by a group of automorphisms of $L_\infty(X, dx)$, one can reformulate the information in the useful form of time dependent expectation values, but only for almost every initial state and for these only for almost every instant of time. In the next two sections we generalize this procedure to the noncommutative domain.

§ 2. Mathematical Preliminaries

The terminology of this section is that of Sakai's book [11]. Throughout this section, M and N will be fixed countably decomposable W^*-algebras and μ (resp. ν) will be a semifinite (resp. finite) faithful normal trace on M^+ (resp. N^+). We assume that ν is normalized to one on the unit I of N. A (resp. B) will denote a fixed C^*-algebra which is dense in M (resp. N) when the latter is equipped with its s^*-topology, and contains the unit. For a general C^*-algebra D, S_D represents the set of states on D. We will denote the algebraic tensor product of C^*-algebras C and D $\left(\text{with elements of the form } \sum_i c_i \otimes d_i\right)$ by $C \odot D$ and the C^*-tensor product by $C \otimes D$. We will denote the W^*-tensor product of two W^*-algebras Q and R by $Q \mathbin{\overline{\otimes}} R$.

Let C and D be two C^*-algebras. It is well known (and follows easily e.g. from 1.22.1 of [11]) that given a state ϱ on D, the map

$$\varrho : e \in C \odot D \subseteqq C \otimes D \to [e](\varrho) \in C$$

defined by

$$\sum_i c_i \otimes d_i \to \sum_i c_i \varrho(d_i)$$

is linear and a contraction. It can therefore be uniquely extended by continuity to a map

$$\varrho : e \in C \otimes D \to [e](\varrho) \in C \tag{1}$$

which remains a linear contraction. Note that for φ in S_C, ϱ in S_D and e in $C \otimes D$,

$$\varphi \otimes \varrho(e) = \varphi([e](\varrho))$$

where $\varphi \otimes \varrho$ is the obvious state (1.22.1 of [11]) on $C \otimes D$. If furthermore C and D are W^*-algebras and φ (resp. ϱ) is in the predual C_* (resp. D_*), then by the definition of $C \mathbin{\overline{\otimes}} D$, $\varphi \otimes \varrho$ is in $(C \mathbin{\overline{\otimes}} D)_*$. Therefore for ϱ in D_* the map in (1) extends uniquely to a normal contraction

$$\varrho : e \in C \mathbin{\overline{\otimes}} D \to [e](\varrho) \in C \tag{1'}$$

which for φ in C_* satisfies

$$(\varphi \otimes \varrho)(e) = \varphi([e](\varrho)). \tag{2}$$

The maps in (1) and (1′) will play a central role in the remainder of this paper, and in a sense our technical results can be viewed as a procedure for extending these maps in a canonical way.

We begin by recalling some definitions introduced in [13]. A set $S \subseteqq S_B$ is said to contain *ν-almost every* state, or to be of *full ν-measure*, if there exists a sequence $\{P_n\}$ of projections P_n in N such that

a) $\{P_n\}$ is increasing,
b) $\nu(I - P_n) \xrightarrow[n \to \infty]{} 0$,
c) $S \supseteqq S(\{P_n\}) \equiv \overline{\bigcup_n \{\text{normal states } \varrho \text{ of } N \mid \operatorname{supp} \varrho \subseteqq P_n\}}$,

where the ϱ are considered as states on B and the closure is in the w^*-topology of S_B. A sequence $\{P_n\}$ of projections in N satisfying a) and b) is called an *exhaustion*, and a set $S \subseteqq S_B$ is said to be of *ν-measure zero* if its complement in S_B is of full ν-measure. For the connection with usual terminology see the Appendix and [13].

Proposition I. *Given an element a of $M \bar{\otimes} N$, let $\{a_n\}$ be any sequence with a_n in $M \bar{\otimes} N$ such that $a_n \xrightarrow[n\to\infty]{} a$ in the s^*-topology of $M \bar{\otimes} N$. Then there exists an exhaustion $\{P_j\} \subseteqq N$ and a subsequence $\{a_{n_k}\}$ such that for every fixed j, $[a_{n_k}](\varrho) \xrightarrow[k\to\infty]{} [a](\varrho)$ in the σ-topology of M, uniformly in normal states ϱ of N such that* $\operatorname{supp}\varrho \subseteqq P_j$.

Proof. Let $\{\varphi_m\}$ be a countable, norm-dense subset of the normal states of M (which exists from Chapter I, § 3, Proposition 1 of [14] and V.5.1. of [15]). By analogy with the map in (1′), we can define for each normal state φ of M and a in $M \bar{\otimes} N$, an element $\varphi[a]$ of N such that

$$\varrho(\varphi[a]) = \varphi \otimes \varrho(a)$$

for all normal states ϱ of N. From Theorem 14 of [8] we know that in the faithful standard representation of M defined by μ, every state in M_* is a vector state, which readily implies that

$$\nu(\varphi_m[a_n - a]^* \, \varphi_m[a_n - a]) \xrightarrow[n\to\infty]{} 0 \,.$$

Therefore from Corollary 13.1 of [8], and [13], there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ and an exhaustion $\{P_j\} \subseteqq N$ such that for every fixed m and j

$$\varphi_m([a_{n_k} - a](\varrho)) = \varrho(\varphi_m[a_{n_k} - a]) \xrightarrow[k\to\infty]{} 0$$

uniformly in normal states ϱ of N for which $\operatorname{supp}\varrho \subseteqq P_j$. And since the $\|a_{n_k}\|$ (and therefore the $\|[a_{n_k}](\varrho)\|$) are uniformly bounded by the Uniform Boundedness theorem, the proposition easily follows.

Definition I. Given a in $M \bar{\otimes} N$, there exists from Chapter I, § 3, cor. to Proposition 1 of [14] a sequence $\{a_n\}$ with a_n in $A \otimes B$ such that $a_n \xrightarrow[n\to\infty]{} a$ in the s^*-topology of $M \bar{\otimes} N$. Using Proposition I, we *assign the values* $\{[a](\varrho) \,|\, \varrho \in S(\{P_n\})\}$ where

$$\begin{aligned}[a](\varrho) &= \sigma\text{-}\lim_\alpha \sigma\text{-}\lim_k [a_{n_k}](\varrho_\alpha) = \sigma\text{-}\lim_k \sigma\text{-}\lim_\alpha [a_{n_k}](\varrho_\alpha) \\ &= \sigma\text{-}\lim [a](\varrho_\alpha) = \sigma\text{-}\lim_k [a_{n_k}](\varrho)\end{aligned}$$

and $\{\varrho_\alpha\}$ is any net of normal states of N such that $\operatorname{supp}\varrho_\alpha \subseteqq P_j$ for some fixed j, and as states on B $\varrho_\alpha \to \varrho$ in the w^*-topology. It is clear that this assignment is "essentially" unique, i.e. that it is independent, up to ν-measure zero, of the approximations $\{a_n\}$ of a and $\{\varrho_\alpha\}$ of ϱ which are used.

We will extend Definition I somewhat using

Proposition II. *Given a sequence $\{a_n\}$ with a_n in $M \overline{\otimes} N$, there exists a set $S \subseteqq S_B$ of full ν-measure and an assignment by means of Definition I of $\{[a_n](\varrho) \mid \varrho \in S\}$ for all n. If $\{a_n\}$ is Cauchy in norm, then $\{[a_n](\varrho)\} \subseteqq M$ is Cauchy in norm for every ϱ in S.*

Proof. It is clear from Theorem 6 of [8] and Proposition I that there exists *a* set S of full ν-measure for which we can simultaneously assign values $\{[b](\varrho) \mid \varrho \in S\}$ for the countable set of b's of the form $b = a_n$ or $b = a_n - a_m$. It is also clear from the proof of Proposition I that for all ϱ in S

$$\|[a_n](\varrho) - [a_m](\varrho)\| = \|[a_n - a_m](\varrho)\| \leqq \|a_n - a_m\|$$

which proves the proposition.

Definition II. Given a norm-separable subset H of $M \overline{\otimes} N$, choose a countable norm-dense subset H' of H. Then from Propositions I and II and Definition I, there exists a subset S of S_B of full ν-measure and an assignment of values $\{[a](\varrho) \mid \varrho \in S,\ a \in H\}$ which is "essentially" unique, i.e. independent, up to ν-measure zero, of the choice of H' or any other choice in the procedure.

§ 3. Application to Abstract Dynamics

We now apply the above machinery to the following abstract dynamical situation, using all the notation and assumptions of § 2 and also the assumption that B is norm-separable. Let Ω be either $\mathbb{Z}$ or $\mathbb{R}$ as a topological group, and assume that we are given a group homomorphism

$$\alpha : t \in \Omega \to \alpha^{(t)} \in \{*\text{-automorphisms of } N\}$$

which is continuous in the sense that for every a in N, the map

$$t \in \Omega \to \alpha^{(t)}(a) \in N$$

is continuous when N is in its σ-topology. Let A be the C^*-algebra $C(\Omega)$ of bounded, complex valued, continuous functions on Ω, let μ be Haar measure on Ω, and let M be $L_\infty(\Omega, \mu)$. Using 2.9.3 and 2.9.4 of [11] and elementary measure theory, it is easy to see that A, M and μ satisfy all our assumptions. From 1.22.3 of [11] we know that there is a natural $*$-algebraic isomorphism between $C(\Omega, B)$, the C^*-algebra of B-valued continuous functions on Ω, and $A \otimes B$. And from 1.22.13 of [11] we know that there is a natural W^*-isomorphism between $L_\infty(\Omega, N, \mu)$ and $M \bar{\otimes} N$, where $L_\infty(\Omega, N, \mu)$ is the space of equivalence classes of N-valued essentially bounded functions f on Ω which are measurable in the sense that for each ϱ in N_*, the map

$$f : t \in \Omega \to \varrho(f(t))$$

is μ-measurable. To keep the notation from becoming (even) more cumbersome, we promote these last two isomorphisms to identities.

It is now rather clear how we propose to use our machinery. For each b in B, we identify $\alpha^{(\cdot)}(b)$ with its equivalence class in $M \bar{\otimes} N$. In this way we obtain a norm-separable subset $\{\alpha^{(\cdot)}(b) \mid b \in B\}$ of $M \bar{\otimes} N$. Then from Definition II there exists a subset S of S_B of full ν-measure and an essentially unique assignment of values

$$\{[\alpha^{(\cdot)}(b)](\varrho) \in L_\infty(\Omega, \mu) \mid b \in B,\ \varrho \in S\}.$$

If necessary we could, for each ϱ in S, choose a representative of $[\alpha^{(\cdot)}(b)](\varrho)$ so as to get a bona fide function on Ω, but this is unnecessary for most purposes.

This is the procedure that we referred to in § 0 and § 1. We emphasize that it naturally displays the two features which we extracted from the billiard ball model; it assigns a time development only to ν-almost every initial state, and to these states it assigns a time development only up to sets of time of μ-measure zero. Thus it allows for the possibility of singular behavior on a set of initial states of ν-measure zero and for the other states on a set of time of μ-measure zero.

§ 4. Limit Models

In this section we implicitly define a class of physical models which we call *limit models*, apply our procedure as in § 3, and discuss the justification of our method. We use the same notation and assumptions as in § 3. We again assume that B, which we interpret as an algebra of observables, be norm-separable. In practice N will then be the von Neumann algebra generated by the GNS representation π_ν of B with respect to the (faithful tracial) state ν of B. We assume given a sequence of *approximate evolutions*, i.e. a sequence α_n of group homomorphisms

$$\alpha_n : t \in \Omega \to \alpha_n^{(t)} \in \{*\text{-automorphisms of } B\}$$

such that

$$\varrho(\alpha_n^{(t)}(b)) \text{ is continuous in } t \text{ for all } b \text{ in } B \text{ and } \varrho \text{ in } N_* . \tag{3}$$

$$\text{For each } b \text{ in } B,\ \alpha_n^{(t)}(b) \text{ is } s^*\text{-Cauchy in } N \text{ for } \mu\text{-a.e. } t \text{ in } \Omega . \tag{4}$$

Some examples will be given below. Now considering $\alpha_n^{(\cdot)}(b)$ to be in $M \bar{\otimes} N$ we first show that it is s^*-Cauchy in $M \bar{\otimes} N$. Given φ in $M_* \odot N_* \subseteq (M \bar{\otimes} N)_*$ of the form $\varphi = \sum_i \varphi_i \otimes \varrho_i$, we have

$$\begin{aligned}\varphi\{|\alpha_n^{(\cdot)}(b) - \alpha_m^{(\cdot)}(b)|^2\} \\ = \sum_i \int \varrho_i\{|\alpha_n^{(t)}(b) - \alpha_m^{(t)}(b)|^2\}\, \varphi_i(t)\, d\mu(t)\end{aligned} \tag{5}$$

where we have used (2). Now since for each i, $\varrho_i\{|\alpha_m^{(t)}(b) - \alpha_n^{(t)}(b)|^2\}$ is uniformly bounded in t and has limit zero as $m, n \to \infty$ for μ-a.e. t, from Lebesgue's Dominated Convergence theorem the RHS of (5) has limit zero as $m, n \to \infty$. Clearly this extends to all $\varphi \in (M \bar{\otimes} N)_*$, proving our assertion. We define $\alpha_\infty^{(\cdot)}(b)$ to be the s^*-limit of $\alpha_n^{(\cdot)}(b)$ in $M \bar{\otimes} N$. (Note that we do not assume that $\alpha_\infty^{(\cdot)}(b)$ comes from an automorphism of N as in § 3.) And now we proceed just as in § 3, assigning an evolution $[\alpha_\infty^{(\cdot)}(b)](\varrho)$ for each b in B and ν-almost every ϱ in S_B. This is our general procedure.

At this point we want to compare our procedure with another possible way of obtaining time dependent expectation values for limit models. Namely, we could assign $\varrho(b_t) = \lim_n \varrho(\alpha_n^{(t)}(b))$ whenever this limit exists. We show that both methods must agree for ν-almost every initial state as follows. From Proposition I we may assume that we used $\alpha_n^{(\cdot)}(b) \in M \bar{\otimes} N$ in our procedure, thus assigning $\sigma\text{-}\lim_k [\alpha_{n_k}^{(\cdot)}(b)](\varrho)$. But from § 21, Ex. (3) of [17], since $\varrho(\alpha_n^{(t)}(b))$ converges for μ-almost every t, its equivalence class $[\alpha_n^{(\cdot)}(b)](\varrho)$ converges in the s^*-topology, and the two limits obviously are in the same equivalence class, which furthermore must coincide with $\sigma\text{-}\lim_k [\alpha_{n_k}^{(\cdot)}(b)](\varrho)$.

We summarize the results of this section in

Proposition III. *Let B be a norm-separable C^*-algebra with unit, let ν be a faithful tracial state on B and let N be $\pi_\nu(B)''$. Let Ω be the topological group $\mathbb{Z}$ or $\mathbb{R}$ (interpreted as time) with Haar measure μ, and let $\{\alpha_n\}$ be a sequence of approximate evolutions of B, as defined above. Then there exists a set S of states on B, of full ν-measure, and a natural definition, for all ϱ in S and b in B, of "limiting time dependent expectation values" $[\alpha_\infty^{(t)}(b)](\varrho)$, which as functions of $t \in \Omega$ are in $\mathscr{L}_\infty(\Omega, \mu)$ and which coincide, for ν-a.e. ϱ and μ-a.e. t, with any limits of $\varrho(\alpha_n^{(t)}(b))$ which might exist.*

Although it takes a surprising amount of work to give a complete proof, it can be shown that the billard ball model of § 1 can be considered a limit model, where the hard wall "potentials" are approximated by any of a general class of sequences of bona fide potentials. It is useful however to look at the model this way, since then one can even see where the singularities come in. One can easily choose a sequence of approximating potentials which behave properly for the "nice" initial states but which will not yield any limit for positive time for the explicit initial state that we discussed in § 1. In other words the singular initial states are too sensitive to the specific approximate evolutions which are used. A similar situation holds for the singular set in time.

Another example of a limit model, aside from the lattice-spin models for which our procedure is unnecessary, is the BCS model [1]; our Condition (4) is proven in [16].

§ 5. Conclusion

Using noncommutative integration theory, we have shown how to extend the C^*-algebra formalism so as to cope with a larger class of dynamical models than has so far proved manageable, including the BCS model. It is hoped that a large class of continuous nonrelativistic Fermion models, such as those discussed in [5] for example, will also yield to our method. If so, it would be of interest to generalize our method so as to include Bose systems and dynamical models of mixed species, for which our assumption of a finite trace may be too strong.

Appendix

As justification for our terminology of sets of full measure, we include the following lemma. (This lemma should have been included in [13], as well as the remark that in [13] if $\mathfrak{A}$ is abelian then $\tilde{\alpha}$ maps pure (i.e. multiplicative) states to pure states.)

Lemma. *Assume that B is an abelian C^*-algebra, s^*-dense in the W^*-algebra N, and containing the unit. Let v be a faithful normal finite trace on N. Clearly we may assume that $B = C(X)$, $N = L_\infty(X, v)$, where X is compact Hausdorff, and $v =$ a finite regular Borel measure on X. Then if $S \subseteqq S_B$ is a set of full v-measure in the sense of § 2, S contains v-almost every pure state of B in the usual sense.*

Proof. Let $\{X_j\} \subseteqq N$ be an exhaustion such that $S \supseteqq S(\{X_j\})$. Let X_j' be a characteristic function in the equivalence class X_j, for the set $E_j \subseteqq X$. We now show that, for every j, v-almost every point in E_j (and therefore of $\bigcup_j E_j$ and therefore of X) is in S. In fact, for each j let F_j be the set of points x in E_j for which there exists an open subset of X which contains x but only intersects E_j in a set of ν-measure zero. Let $G_j = E_j \setminus F_j$. First we show that $G_j \subseteqq S$. Given g in B, y in G_j and $\varepsilon > 0$, there exists an open subset O_y of X, containing y and such that $|g(x) - g(y)| < \varepsilon$ for all x in O_y. Let h be the characteristic function of $O_y \cap E_j$ and $f = h/\nu(O_y \cap E_j)$. Defining the states:

$$\delta_y : a \in B \to a(y), \qquad \varrho_f : a \in B \to \int f a \, d\nu,$$

we have

$$|(\delta_y - \varrho_f)(g)| \leqq \int |g(y) - g| f \, d\nu < \varepsilon$$

which proves that $G_j \subseteqq S$. It only remains to show that $\nu(F_j) = 0$. Since F_j is clearly measurable, and ν is regular, it suffices to show that $\nu(Y) = 0$ for all compact subsets Y of F_j. But for each point y in such a Y, there exists an open subset O_y of X such that $\nu(O_y \cap Y) = 0$. Since Y is compact there exists a finite subcover of Y using such O_y's, which implies that $\nu(Y) = 0$, and which completes the proof.

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