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SPACE TILINGS AND LOCAL ISOMORPHISM

ABSTRACT. We prove for a large class of tilings that, given a finite tile set, if it is possible to tile Euclidean n-space with isometric copies of this set, then there is a tiling with the 'local isomorphism property'.

1. INTRODUCTION AND STATEMENT OF RESULTS

The subject of this paper is the tiling of Euclidean spaces. By a tiling of $\mathbf{R}^n$ we mean a representation of $\mathbf{R}^n$ as the union of 'tiles' – think of an n-dimensional jigsaw puzzle filling all of $\mathbf{R}^n$ – where:

(a) there is a fixed finite set S of 'prototiles', which are pairwise noncongruent homeomorphs of the closed n-ball;
(b) each tile is an isometric 'copy' of some prototile, that is, it is the image of a prototile by a symmetry operation (translation and/or rotation);
(c) the interiors of the tiles do not overlap;
(d) given a tile, its boundary can be covered by other tiles, with no overlapping, in only a finite number of ways.

(Condition (d) forbids the following pair of 3-dimensional prototiles: a circular cylinder of diameter 1 and height L (i.e. $\{(x, y, z): x^2 + y^2 \leqslant \frac{1}{2}, 0 \leqslant z \leqslant L\}$), and a cube of side 2 with a semicircular groove of diameter 1 (i.e. $\{(x, y, z) \in [0, 2] \times [-1, 1] \times [0, 2]: x^2 + y^2 \geqslant \frac{1}{2}\}$.) These are not allowed because one tile can be slid along the other. In fact if the height L of the cylindrical tile is irrational, this tile set would otherwise be a counterexample to the theorem below.)

Tilings of Euclidean spaces are of interest for a myriad of reasons; we are motivated by certain questions concerning sets of prototiles (copies of) which can tile space but only in special ways. This line of development is due to Hao Wang, who (because of certain questions in logic [5], [14]–[17]) first studied sets of prototiles which could tile the plane but *only* with tilings which were *nonperiodic*. (A finite set of nonoverlapping tiles is called a 'patch', and a tiling is called periodic if it consists of a lattice of translates of one of its patches.) A notable example is the set of prototiles discovered by Roger Penrose which can tile the plane but only nonperiodically [5], [11].

The work of Wang eventually led to a succession of efforts (culminating in

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[6]) to find sets of prototiles for which the tilings were of necessity more and more 'disordered' – a notion which is inherently vague; see however [11]. We consider a property that a tiling might have, called 'local isomorphism'.

DEFINITION. A tiling enjoys the *local isomorphism* property if for every patch $\mathscr{P}$ in the tiling there is some distance $d(\mathscr{P})$ such that every sphere of diameter $d(\mathscr{P})$ in the tiling contains an isometric copy of the patch.

It is one of the frequently noted features of the Penrose set that all of its tilings satisfy the local isomorphism property (and with a rather small value of $d(\mathscr{P})$) [5]. Various authors have used the local isomorphism property as a hallmark of the 'regularity' or 'order' of certain tilings ([5], [7], [13]) or similar arrays or patterns ([1], [2], [3]), sometimes using terms such as 'weak periodicity' or 'recurrence' in place of 'local isomorphism'. There are two goals to this paper. The first is the following technical result and its corollary.

THEOREM. *If a set of prototiles admits a tiling of space, it must admit a tiling satisfying the local isomorphism property.*

This shows that local isomorphism is not at all an unusual property of a set of prototiles; in fact it is universal! (This answers an open problem, Exercise 11.2.4 in [5].) To further justify this adjective it is useful to introduce the concept of 'unique ergodicity'. This is a nondegeneracy condition that a set S of prototiles usually satisfies in practice, namely that the set $V(S)$ of all its tilings is nonempty and supports one and only one G-invariant Borel probability measure (where G is the symmetry group of the Euclidean space). (All our proofs go through if G is instead merely a subgroup of the full symmetry group of the Euclidean space. See [11] for a discussion of unique ergodicity as a form of nondegeneracy.) For nondegenerate prototile sets we prove the following corollary.

COROLLARY. *If $V(S)$ is uniquely ergodic with unique measure ρ, then all tilings in $V(S)$ have the local isomorphism property except for a subset of ρ-measure zero. Furthermore, the diameter function $d(\mathscr{P})$ is independent of the tiling.*

The second goal of this paper is to emphasize the usefulness of ergodic theory as a tool in tiling theory: indeed, the main step in our proof of the above results is to interpret the subject of tiling as part of dynamical systems.

2. Proof of the results

Our first step is to put a topology on the space $V(S)$ (assumed nonempty) of all tilings of some given set S of prototiles. The motivating idea is that tilings should be close if they differ only slightly inside some large bounded region. For this we use the Hausdorff distance of a pair of compact subsets A, B of $\mathbf{R}^n$, defined as $\mu[A, B] = \max\{h(A, B), h(B, A)\}$, where $h(A, B) = \sup_{a\in A} \inf_{b\in B} \|a-b\|$. We define a countable base for the topology on $V(S)$ using some countable dense subset G' of the topological group G of symmetries, as follows. Given some $n \geqslant 1$, a set of positive rationals $\{r_j\}_1^n$, and a patch of tiles $\{g'_j(P'_j)\}_1^n$ (where $g'_j \in G'$, $P'_j \in S$), we define the open set: the set of all tilings which contain a patch $\{g_j(P_j)\}_1^n$ such that $\mu[g_j(P_j), g'_j(P'_j)] < r_j$ for all $j \leqslant n$.

A simple consequence of the definitions, particularly condition (d), is the following.

LEMMA 1. *If $\mathscr{P} = \{g_j(P_j)\}_{j=1}^n$ is a patch and if $G_0 \subseteq G$ is a neighborhood of the identity, then there exists an $\varepsilon > 0$ such that: if $0 < r_1, \ldots, r_n < \varepsilon$ and if T is a tiling in the open set defined by $\mathscr{P}$ and $r_1, \ldots, r_n$, then T contains a patch of the form $h(\mathscr{P})$ for some $h \in G_0$.*

Sketch of the proof of Lemma 1. Fix a tile $\tau \in \mathscr{P}$. For any ball containing τ, it follows from condition (d) that there are only finitely many possible ways to extend τ to a patch covering this ball. So, if for the moment we consider that τ is fixed as a tile in T, it is clear that we can restrict the tiles near τ by some ε to force T to contain $\mathscr{P}$. But, in fact, we can choose ε to force T to contain a small shift of τ; hence we can force T to contain a small shift of $\mathscr{P}$. □

An important feature of the topology on $V(S)$ is the following.

LEMMA 2. *The space $V(S)$ of tilings from a given prototile set S is compact.*

Proof of Lemma 2. (Note that this proof does not require condition (d), and therefore holds even for very general tilings.) It is enough to prove the sequential compactness of the metrizable space $V(S)$ [12]. Choose a sequence $\{l_i: i \geqslant 1\}$ of points in $\mathbf{R}^n$ such that any tile must contain some l_i. (For example, one could choose a sufficiently fine lattice, numbered to spiral outward from the origin.) Now suppose we are given a sequence $\{T_i: i \geqslant 1\}$ of tilings in $V(S)$. The idea is to force the tilings to agree more and more closely on larger and larger patches. The first part of our proof will be an induction, where the induction hypothesis on $m \geqslant 1$ has the following three parts.

(1) There is a subsequence $\{T_i^m: i \geqslant 1\}$ of $\{T_i: i \geqslant 1\}$, and for each i a patch of m distinct tiles $\{\tau_{i,j}^m: 1 \leqslant j \leqslant m\}$ in T_i^m, such that:
   - (i) $\{T_i^m: i \geqslant 1\}$ is a subsequence of $\{T_i^{m-1}: i \geqslant 1\}$, where we define $T_i^0 = T_i$;
   - (ii) for $1 \leqslant j \leqslant m$, the sequence $\{\tau_{i,j}^m: i \geqslant 1\}$ converges, in the Hausdorff metric, to a tile (independent of m) which we will call τ_j.

2. The interior of the tile τ_m is disjoint from the interiors of all 'previous' τ_j (that is, τ_j for $1 \leqslant j < m$).
3. τ_m contains the point $l_{p(m)}$, where $l_{p(m)}$ is defined to be the first l_i which is not contained in any previous τ_j.

The induction hypothesis holds for $m = 1$ by the following theorem.

SELECTION THEOREM ([5]). *Given an infinite sequence of tiles τ_j all with some point in common and all congruent to a tile τ', then some subsequence of the τ_j converges to a tile congruent to τ'.*

(In fact our proof uses some of the technique of that of the Selection Theorem.) Now suppose $m \geqslant 2$ and that the hypothesis is true for $m-1$. For $i \geqslant 1$ pick some tile $\hat{\tau}_i^m$ in T_i^{m-1} which contains the point $l_{p(m)}$, and consider the sequence $\{\hat{\tau}_i^m : i \geqslant 1\}$. (It is not hard to see that there exists $I(m)$ such that if $i \geqslant I(m)$ then $\hat{\tau}_i^m \neq \tau_{i,j}^{m-1}$, for any $1 \leqslant j < m$.) By the Selection Theorem and the fact that S is finite, the sequence $\{\hat{\tau}_i^m : i \geqslant 1\}$ contains a convergent subsequence, $\{\hat{\tau}_{s(i)}^m : i \geqslant 1\}$; we also require that $s(m) > I(m)$ for all m. For all $i \geqslant 1$ and $1 \leqslant j < m$ define: $T_i^m = T_{s(i)}^{m-1}$, $\tau_{i,j}^m = \tau_{s(i),j}^{m-1}$, and $\tau_{i,m}^m = \hat{\tau}_{s(i)}^m$. Finally, let $\tau_m = \lim_i \hat{\tau}_{s(i)}^m$. It is easy to check that the induction holds for this choice of τ_m and T_i^m. Namely, for all $i \geqslant I(m)$ the interior of $\tau_{i,m}^m$ is disjoint from those of any $\tau_{i,j}^m$ for $1 \leqslant j < m$, so when we take the limit in i the interior of τ_m is disjoint from those of all previous τ_j. Clearly $l_{p(m)} \in \tau_m$. And finally, the required convergence property for the $\tau_{i,j}^m$ is preserved when we take a subsequence.

So the induction holds for all $m \geqslant 1$. To continue the proof of the lemma we now diagonalize: define $T_i' = T_i^i$, and let T' be the tiling consisting of the $\{\tau_j\}$. (That the $\{\tau_j\}$ constitute a tiling is easy to see since they can be approximated by patches of tilings.) We claim that $T_i' \to T'$. To see this, consider any open set O defined by a patch $\{\tilde{\tau}_j : 1 \leqslant j \leqslant k\}$ and positive numbers $\{r_j : 1 \leqslant j \leqslant k\}$ and containing T'. For $1 \leqslant j \leqslant k$ we know $\lim_i \tau_{i,j}^k = \tau_j$, so for large enough i it follows that T_i' is in O. This proves that $T_i' \to T'$, and since $\{T_i'\}$ is a subsequence of $\{T_i\}$ the lemma is proven. □

Getting back to the proof of the theorem, we note that the compact space $V(S)$, together with the action on it of the group G of symmetries of $\mathbf{R}^n$, constitute a dynamical system. We say the nonempty closed invariant subset X of $V(S)$ is 'minimal invariant' if there is no closed subset of X, other than X itself or the empty set, which is invariant under G. It is a simple application of the Axiom of Choice to prove that such a subset of $V(S)$ must exist [4]; so fix such an X. Given any nonempty relatively open subset O of X, since $\bigcup_{g \in G} g^{-1}(O)$ is a nonempty invariant relatively open subset of X, it follows that it must equal X. From the compactness of $V(S)$ and therefore of X it follows that there exists a finite subset $\{g_i\}$ of G such that $\bigcup_i g_i^{-1}(O) = X$.

Using this, we will show that any tiling $T \in X$ has the local isomorphism property: so suppose we are given a patch $\mathscr{P}$ in T. It is a simple consequence of Lemma 1 that there is an open ball B of diameter $2\,\mathrm{diam}(\mathscr{P})+1$ in $\mathbf{R}^n$ containing $\mathscr{P}$, and a basic open set U containing T, such that all the tilings in U contain a copy of $\mathscr{P}$ inside B. (Note that $\mathrm{diam}(\mathscr{P})+1$ may not be large enough: a set of diameter d may not fit inside any ball of diameter d, but will always fit inside some ball of diameter $2d$.) Let $\{g_i : i \in I\}$ be the finite set of symmetries such that $\bigcup_{i\in I} g_i^{-1}(U)$ contains X, as guaranteed above, and therefore contains the orbit of T under G. Then any ball in T of diameter $2\sup\{\|g_i(b)-b'\| : i \in I, b \in B, b' \in B\}$ will contain a copy of $\mathscr{P}$, which proves the theorem. As for the corollary, it follows from the fact that the support of a uniquely ergodic measure is minimal invariant [4]. □

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