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Particle Lattice Models and the Dynamical Instability of Many-Body Systems*

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Abstract. We consider the notion of dynamical instability of many-body systems wherein states, which are arbitrarily close initially, are not close at some other fixed time. In controlling the dynamics of interacting systems of identical Fermions moving on a lattice, we isolate a basic mechanism which causes instability.

1. Introduction

There are few calculations which establish qualitative control over the dynamics of interacting, many-body, nonrelativistic systems *as a function of particle number*. We are particularly interested here in controlling the following "instability" phenomenon: A many-body system, which we assume to be qualitatively independent of its (large but finite) particle number n and volume V_n, can admit a naturally related family of initial states f_n with the property that for each n there is a well defined evolution of f_n for all time, and yet for which there is an *instability in the finite particle system* which is made manifest when the initial state $f_\infty = \lim_{n\to\infty} f_n$ only has a well defined evolution for finite time. For example, one can easily set up a classical mechanical system of n point particles, $p^1, \ldots, p^n$, and an initial state f_n such that at time $t = t_j = \sum_{k=1}^{j} 2^{-k}$ particle p^j hits the "target particle" p^1 imparting a unit of momentum in a fixed direction. It is clear that for large but finite n, something unusual occurs just before $t = 1$ (p^1 attains arbitrarily high momentum) which causes the breakdown of the evolution of f_∞ at $t = 1$. We emphasize that the "catastrophic" feature of the infinite particle system is only a manifestation of a real instability of the n-particle system and not just an anomaly of infinite particle systems. (We use the term "instability" because even though f_n and f_m are arbitrarily close, being close to f_∞, their evolved states $f_n(t)$ and $f_m(t)$ cannot be close at $t = 1$ or else they would define a state, $f_\infty(1)$, which we know does not exist.) A parallel with the phenomenon of phase transitions is clear — models of finite particle systems do not exhibit phase transitions in the sense of actual discontinuity or nondifferentiability of thermodynamic functions of temperature, but the inherent instability of the finite particle system is made manifest by these features of the corresponding infinite particle system.

Instability seems to be a key obstacle in the dynamical control of continuous quantum mechanical nonrelativistic many-body systems, though we are unaware of any convincing proof that instabilities or catastrophies, of the type described above, can actually occur in quantum systems; see [1—4]. The main goal of this

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paper is to attack the contrary problem, of whether the given mechanism (of arbitrarily large momentum transfer by individual particles) is in some sense the only possible mechanism for causing dynamical instability. We give evidence in this direction by calculations of models which are otherwise quite general, but in which this particular mechanism is removed in a natural manner: by replacing the physical space $\boldsymbol{R}^3$ by $\boldsymbol{Z}^3$, i.e. by working with n-body Schrödinger *difference* equations. In this way the momentum of a particle is represented by a *bounded* difference operator, thus naturally preventing any single particle from accepting or transmitting a large transfer of momentum. To simplify our situation further we only consider identical Fermions, so that we do not have to consider initial states which could be in a sense "unstable after zero time" (e.g. infinite density states).

2. Notation

Let $\boldsymbol{Z}^3$ be the infinite three dimensional lattice and e_j, $j=1,2,3$, the basis vector $(\delta_1(j), \delta_2(j), \delta_3(j))$ where $\delta_k(\cdot)$ is the Kronecker delta function. Let $H^{(n)}$, $n=1,2,\ldots$, be the Hilbert space of complex functions f on the Cartesian product, $\times_{k=1}^{n} \boldsymbol{Z}_k^3$, which are antisymmetric in the particle label k. We define $H^{(0)}=\boldsymbol{C}$ and $H=\bigoplus_{n=0}^{\infty} H^{(n)}$. For each x in $\boldsymbol{Z}^3$ we define the Fermi field operator $a(x)$ on a general vector f in the Fock space H, with components $f_m(x_1, \ldots, x_m)$ in $H^{(m)}$, by

$$(a(x)f)_n(x_1, \ldots, x_n) = (n+1)^{\frac{1}{2}} f_{n+1}(x, x_1, \ldots, x_n).$$

We note that $a(x)$ is bounded, with operator norm $\|a(x)\|=1$. We define F as the family of nonempty finite subsets of $\boldsymbol{Z}^3$ partially ordered by inclusion, and for each W in F, $P(W)$ denotes the (finite dimensional) operator algebra of polynomials in $a(x)$ and the adjoint $a^*(y)$, x, y in W. Finally we define the operator algebra $P=\bigcup_{W\in F} P(W)$, and its norm closure, $\mathfrak{A}$, (the CAR algebra). For general orientation we refer to the texts [5—7].

3. Particle Lattice Models and Their Stability

Schrödinger mechanics for n identical spinless Fermions in the "box" V in F, interacting via the two-body potential $\Phi(x, y)$, is given by

$$\frac{\partial f_n}{\partial t}(x_1, \ldots, x_n) = iH_V^S(x_1, \ldots, x_n) f_n(x_1, \ldots, x_n)$$

where $H_V^S = H_V + S_V$, $\Phi(x, y) = \varphi(x-y)$ for some real function φ in $l_1(\boldsymbol{Z}^3)$, and

$$H_V = -\tfrac{1}{4}\sum_{x\in V}\sum_{j=1}^{3} a^*(x)[a(x+2e_j)+a(x-2e_j)-2a(x)]$$
$$\quad +\tfrac{1}{2}\sum_{x,y\in V}\varphi(x-y)a^*(y)a^*(x)a(x)a(y)$$
$$S_V = \tfrac{1}{4}\sum_{j=1}^{3}\Big[\sum_{\substack{x\in V\\ x+2e_j\notin V}} a^*(x)a(x+2e_j) + \sum_{\substack{x\in V\\ x-2e_j\notin V}} a^*(x)a(x-2e_j)\Big].$$

If we define the function E on $F \cup \emptyset$ by

$$E(W) = 0, \text{ when the cardinality, } |W|, \text{ of } W \text{ is 0 or } \geqq 3,$$
$$E(\{x\}) = \tfrac{3}{2} a^*(x) a(x),$$
$$E(\{x, y\}) = -\tfrac{1}{4} \textstyle\sum_{j=1}^{3} [\delta_y(x+2e_j) + \delta_x(y+2e_j)][a^*(x)a(y) + a^*(y)a(x)]$$
$$+ \tfrac{1}{2} [\varphi(x-y) + \varphi(y-x)] a^*(x) a^*(y) a(y) a(x), \qquad x \neq y$$

then we may write $H_V^S = \sum_{W \in V} E(W)$. It is clear that the restriction of E to F is a function from F to P with the properties:

i) if A is in $P(V_A)$ and $V \cap V_A = \emptyset$, then $[E(V), A] = 0$

ii) $\sum_{k=0}^{\infty} \sup_{y \in \mathbf{Z}^3} \sum_{\substack{W \in F \\ y \in W \\ |W| = k+1}} \exp(k) \|E(W)\| \equiv \|E\| < \infty$.

Therefore a simple variation of the proof of [5; 77.6.1] gives a norm bound on the n^{th} order commutator $[H_V^S, A]^{(n)}$:

$$\|[H_V^S, A]^{(n)}\| \leqq \|A\| \exp(|V_A|) n! (2\|E\|)^n, \qquad \text{for any } A \text{ in } \quad P(V_A),$$

where here $\|E\| \leqq 10 + 3 \sum_{y \in \mathbf{Z}^3} |\varphi(y)| < \infty$. Therefore given A in P and real t, $|t| < 1/(2\|E\|) \equiv t_E$, there is a unique element $\alpha_V^t(A)$ in $\mathfrak{A}$ such that

$$\|\alpha_V^t(A) - \textstyle\sum_{n=0}^{N} [H_V^S, A]^{(n)} \dfrac{(it)^n}{n!} \| \xrightarrow[N \to \infty]{} 0, \tag{1}$$

uniformly in $|t| \leqq T < t_E$ and uniformly in $V \in F$.
Clearly,

$$\alpha_V^t(A) = \exp(itH_V^S) A \exp(-itH_V^S) \tag{2}$$

which we use to extend the domain of the map α_V^t from P to $\mathfrak{A}$, and to all real t. Since $[H_V^S, A]^{(n)}$ converges in norm, for fixed n, as $V \to \mathbf{Z}^3$, and the convergence in (1) is uniform in V, we may define, for each A in P, and $|t| < t_E$, the element $\beta^t(A)$ of $\mathfrak{A}$ by the properties:

$$\|\beta^t(A) - \alpha_V^t(A)\| \xrightarrow[V \to \mathbf{Z}^3]{} 0, \quad \text{uniformly in} \quad |t| \leqq T < t_E.$$

Clearly α_V^t and β^t are linear maps from P into $\mathfrak{A}$. Furthermore for A in P, $\|\alpha_V^t(A)\| = \|A\|$ for all V, so $\|\beta^t(A)\| = \|A\|$, and we may extend β^t to a linear isometry on $\mathfrak{A}$. Next we show that in fact, $\alpha_V^t(A)$ converges in norm for all A in $\mathfrak{A}$ and all real t. Given A in $\mathfrak{A}$, $|t| \leqq T < t_E$ and $\varepsilon > 0$, choose a fixed A' in P such that $\|A - A'\| < \varepsilon/3$. Then

$$\|\alpha_V^t(A) - \beta^t(A)\| \leqq \|\alpha_V^t(A - A')\| + \|(\alpha_V^t - \beta^t)(A')\| + \|\beta^t(A' - A)\|,$$

which shows that $\alpha_V^t(A)$ converges [to $\beta^t(A)$] as $V \to \mathbf{Z}^3$, for all A in $\mathfrak{A}$, uniformly in $t \leqq T < t_E$. Then

$$\|\alpha_V^t[\alpha_V^t(A)] - \beta^t[\beta^t(A)]\| \leqq \|\alpha_V^t\{[\alpha_V^t - \beta^t](A)\}\| + \|[\alpha_V^t - \beta^t][\beta^t(A)]\|$$

implies that $\alpha_V^{2t}(A) = \alpha_V^t[\alpha_V^t(A)]$ converges in norm (to $\beta^t[\beta^t(A)]$) as $V \to \mathbf{Z}^3$, defining an element $\alpha^{2t}(A)$ of $\mathfrak{A}$. By iterating this argument one sees that $\alpha_V^t(A)$ is Cauchy in norm for all A in $\mathfrak{A}$ and uniformly for t in compact sets, defining $\alpha^t(A)$. Furthermore, it follows very easily from the corresponding properties of $\{\alpha_V^t | t \in \boldsymbol{R}\}$ that $\{\alpha^t | t \in \boldsymbol{R}\}$ is a strongly continuous one parameter group of *-automorphisms of $\mathfrak{A}$. Finally we note that $\alpha^t(A)$ is independent of the specific surface term S_V used in defining H_V^S. We summarize our argument in the

Theorem. *The family* $\{\alpha_V^t | V \in F\}$ *of *-automorphisms of the CAR algebra defined in* (2) *is strongly Cauchy in V, uniformly for t in compact sets. The limits,* $\{\alpha^t | t \in \boldsymbol{R}\}$ *form a strongly continuous one parameter group of *-automorphisms of* $\mathfrak{A}$.

(*Note:* It is easy to see that, at the expense of doubling our notation, we could establish the same results for identical Fermions with spin – in other words these "particle lattice models" are just a generalization of the spin lattice models of Streater [8] and Robinson [9] obtained by allowing the spins to move on the lattice, i.e. to possess and transfer momentum.)

Although for different purposes one may wish to use different C^*-subalgebras of $\mathfrak{A}$ as an algebra of observables, one would expect any such choice to remain invariant under each α_V^t and thus under α^t. (This is clearly the case for the "even" and "gauge-invariant" subalgebras, for example.) Therefore we have the

Corollary. *No instability occurs for these models, in the sense of the Introduction.*

4. Summary

We have considered "particle lattice models" of n identical Fermions moving on a finite subset V of the lattice, $\boldsymbol{Z}^3$, with rather general interactions, and have shown that the dynamics of such systems stabilizes as n and V are made large. For reasons detailed in the Introduction, we feel that this should be of value in controlling the dynamics of similar systems, where $\boldsymbol{Z}^3$ is replaced by $\boldsymbol{R}^3$. As a by-product, it is hoped that, as these models allow a transfer of momentum (unlike the spin lattice models), they will provide more realistic examples in which to investigate the qualitative dynamical (and nondynamical) features of nonrelativistic many-body systems.

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