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The pinwheel tilings of the plane

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Introduction

The subject of forced tilings (also called aperiodic tilings) was created by the philosopher Hao Wang in 1961 [9]-[12] as a tool in the study of certain decidability problems in the propositional calculus. It is concerned with the patterns generated by "tiles" as they are used to tile space; the formalism is the following. One defines a fixed, finite number of shapes (henceforth called "prototiles") in Euclidean n-space, $\mathcal{E}^n$, for $n \geq 2$. The prototiles are usually required to be rather nice topologically, at least homeomorphs of the closed unit ball. One then makes arbitrarily many congruent copies, called "tiles," of these prototiles, and considers all ways (called "tilings") that such tiles may provide a simultaneous covering and packing of $\mathcal{E}^n$; a tiling is thus an unordered collection of tiles for which the union is all of $\mathcal{E}^n$, but such that the interiors of each pair of tiles do not intersect.

Wang's original problem was to determine if it was possible to have a finite set of prototiles, with associated tilings of space, for which *all* the tilings were *nonperiodic*. (A tiling is "nonperiodic" if it is not invariant under any single simultaneous translation of its tiles.) This was settled in the affirmative in the thesis of Wang's student Robert Berger [1].

Slowly over the years, due to the efforts of Raphael Robinson [8], Roger Penrose [2], Shahar Mozes [4] and many others (see [3], [6], [7] for other references), Wang's problem has been generalized to ask whether it is possible for a fixed, finite set of prototiles to generate tilings of space *but only very complicated tilings*, where "very complicated" is interpreted as appropriate, but always implying nonperiodicity. In some interpretations, particularly the first ones associated with physical models of quasicrystals, "very complicated" means "without crystallographic symmetry." In later work, "very complicated" often means "disordered" in the probabilistic sense used to study

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patterns in nonlinear dynamics. Within the field of logic, there was a brief line of development in which "very complicated" meant "nonrecursive."

All published examples, of finite sets of prototiles which can only tile the plane nonperiodically, have the feature that in every tiling each prototile only appears in finitely many orientations. Therefore, for these examples one could add the requirement that copies of a prototile be not just congruent to the prototile, but congruent by a translation; to recover the tilings of these previous examples one might then need to increase the sets of prototiles to larger but still finite sets, effectively declaring certain rotated or reflected prototiles as new, distinct prototiles. In other words, the use of congruence in the production of copies of the prototiles was effectively replaceable by translation in all of these examples, at the expense of increasing the number of prototiles to a larger finite number. This is not true of the example of this paper. In this example there is a finite set of prototiles, and in *every* associated tiling of the plane, tiles appear in *infinitely* many orientations; all the connected parts of the Euclidean group are needed to analyze the tilings by this set of prototiles, not just the translation subgroup of the Euclidean group. This constitutes a major advance in the field, which began with models only requiring discrete translations (using "Wang dominoes"), and expanded to continuous translations with models such as those of Penrose; this introduction of rotations adds a distinctly new element to the subject, of particular interest to certain applications such as the physics of quasicrystals. There are other features of this example which are novel, but we postpone discussion of them to the end of the paper.

1. The substitution tesselation of the plane; the tesselation and its Levels

We begin with a certain hierarchical structure, a tesselation of the plane which motivated our tiling example; the tesselation is due to John H. Conway (unpublished).

We define the "unmarked prototile" as the right triangle with the following vertices in the Cartesian plane: (0,0), (2,0), (2,1). (To be precise, it is the closed convex hull of these points.) The interior angles at the vertices of this triangle (and any similar triangle) are of three sizes, small, medium and large, so the vertices are denoted S, M and L. Similarly the edges of this triangle (and any similar triangle) are of three sizes and are denoted S, M and L. See Figure 1 below. We now define the (substitution) tesselation Θ based on this unmarked prototile. It consists of isometric copies of the unmarked prototile, obtained by the following iterative procedure. Define "the first type C Triangle of Level 0" to be the unmarked prototile. Consider the map which takes

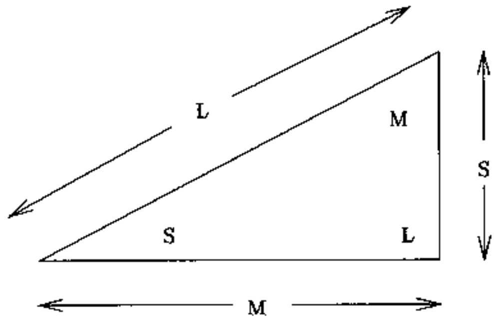

FIGURE 1

this Triangle into a similar Triangle (which we call "the first type C Triangle of Level 1"), with vertices $(-2, 1)$, $(2, -1)$ and $(3, 1)$, and *composed of* five isometric copies of the first type C Triangle of Level 0, with each of the five labeled with a "type" A, B, C, D, or E, as follows: a Triangle of type A with vertices $(-2, 1)$, $(0, 1)$ and $(0, 0)$; a Triangle of type B with vertices $(0, 1)$, $(0, 0)$ and $(2, 1)$; a Triangle of type C with vertices $(0, 0)$, $(2, 0)$ and $(2, 1)$ (coinciding with the first type C Triangle of Level 0); a Triangle of type D with vertices $(0, 0)$, $(2, 0)$ and $(2, 1)$; and a Triangle of type E with vertices $(2, 1)$, $(3, 1)$ and $(2, -1)$. In summary, the first type C Triangle of Level 1 is a certain collection of five "component" Triangles. See Figure 2.

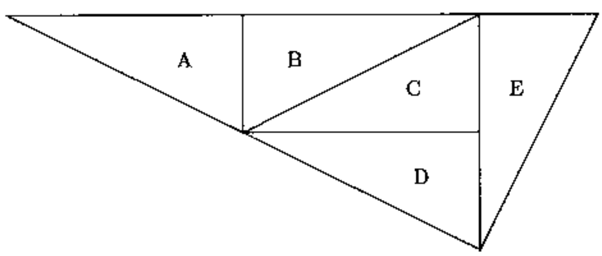

FIGURE 2

Next we repeat this process, mapping the first type C Triangle of Level 1 into the first type C Triangle of Level 2, with vertices $(-5, 5)$, $(1, -3)$ and $(5, 0)$; it is similar to the former and composed of five Triangles (each composed of five subtriangles) isometric to the former, which are in the same geometric relation to the former as are the five created at the first step, with the first type C Triangle of Level 1 as the C in this Level 2 Triangle. See Figure 3. Continuing this process leads to the desired tesselation Θ of the plane. See Figure 4. We define Triangles of Level n as those Triangles,

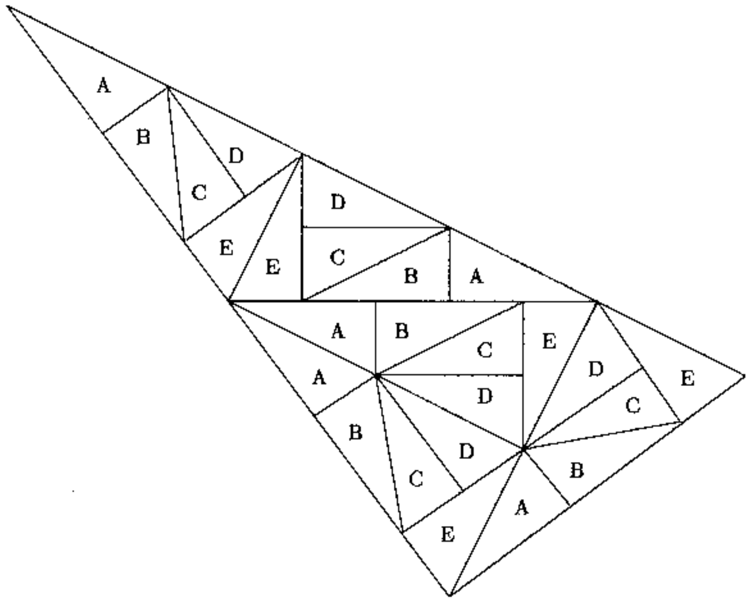

FIGURE 3

created above, which are isometric to the first type C Triangle of Level n; each has a "type" (A $\cdots$ E) defined by its relative position in the unique Level $n+1$ Triangle containing it. We also define a "class" (A1, B1,$\cdots$, E1, A2, B2,$\cdots$, E2) for each Triangle; the first component in the symbol represents the type, and the second component, either 1 or 2, distinguishes between Triangles which are reflections of one another. See Figure 5. (This pattern of Triangles of all Levels is the key feature of the tesselation Θ on which we will focus our attention. Also, we emphasize that by definition the above Triangles are each fixed sets in the plane, and not movable in any sense.) It is of prime importance to this paper that the Triangles of each Level appear at infinitely many orientations in Θ, as results from the following simple lemma (Theorem 6.15 in [5]) applied to the small angle $S = \tan^{-1}(1/2)$.

LEMMA. *For any rational number r other than 0 and ± 1, $\tan^{-1}(r)$ is irrational with respect to π.*

2. Marks in the tesselation Θ

Our main objective is to define a finite set of prototiles which can tile the plane but only with tilings that have some of the geometric relationships of Conway's tesselation Θ: in particular, the hierarchical relationships between consecutive Levels of Triangles. This will be done in Section 3. As preparation,

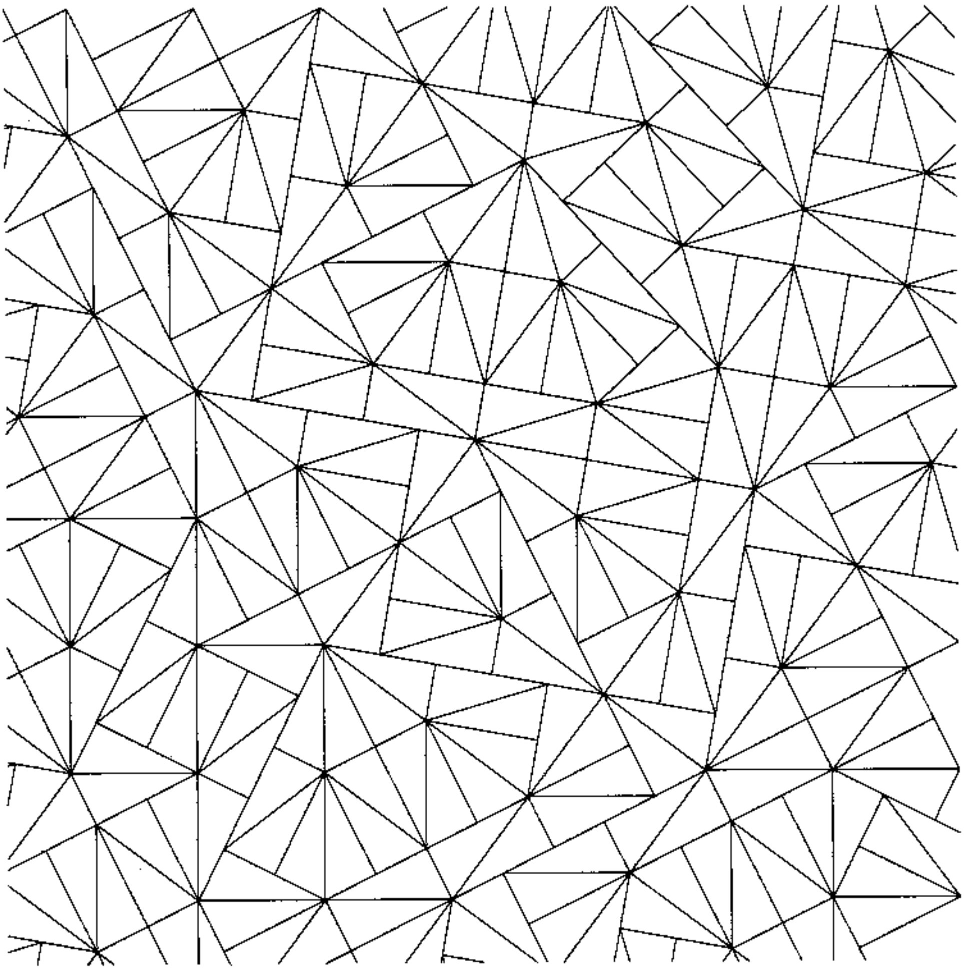

FIGURE 4

we begin by adding "marks" to the (smallest) Triangles in Θ. Each Triangle of Level 0 in the tesselation Θ will be called a *Tile*, and will have a mark associated with (but not necessarily located at) each of its vertices, μ_S, μ_M, μ_L, where the subscript refers to the angle of the vertex. (We will do what we can to explain the significance of the various marks, though this may only be fully clarified by our proofs below, and the remarks afterward. The basic idea will be to record enough information in the marks of the Tiles so that similarly defined but movable "tiles" will only have tilings with a hierarchical structure analogous to that of the Triangles. There will be a brief summary of the marks at the end of this section.)

Each of the three vertex marks of a Tile will contain a variety of information about the edges of those Triangles in the tesselation Θ which are related

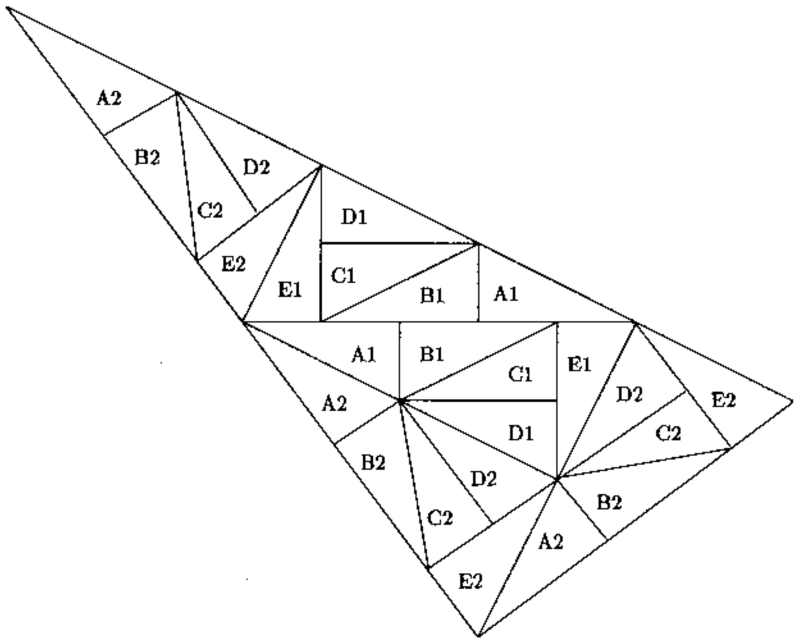

FIGURE 5

in some sense to the vertex. In particular, with each vertex V of each Tile T there will be recorded information concerning each edge of each Tile which either has a coinciding vertex, or which has an edge with V at its center. Since (by definition) abutting Tiles have edges which coincide in an interval, the lines representing Tile edges will be separated into double lines, as follows: At vertex V of Tile T, the information at V is given in linear order. There are two edges of T meeting at V. Measuring angles as positive when referring to clockwise rotation, we call that edge of T at V the "first" one which is a positive angle from the other, the other being called the "last" edge of T at V. We will record the angle between edges at V. The first angle will be that of Tile T at V, and we use the abbreviations S, M and L noted above. There must be some Tile T′ abutting T at the first edge of T at V. Since this edge of T′ at V and the first edge of T at V coincide in an interval, the angle between them is zero, and does not need to be recorded. Therefore, we include in the information at V: the angle of T (called A_1), then some information about the first edge of T (called E_1^1), then information about the abutting edge of T′ (called E_1^2), then the angle of T′ (called A_2), information about the other edge of T′ (called E_2^1), ..., and finally, information about the last edge of T (called E_e^2; e is used to denote the last two edges). So the information at V is a sequence of triples, each beginning with an angle, and then information about two edges which coincide in an interval; the k^{th} such triple is referred to as A_k, E^1_k and E^2_k. Edge-marks E^j_k will be said to "refer to" or "lie on" the appropriate edges of a Tile or Triangle (and are denoted by thick intervals in some figures). Now we need to specify what the information is in each E^j_k, where $j = 1, 2$ and $1 \leq k \leq e$.

We are concerned with the edges not just of Tiles, but of Triangles of all Levels. We call a "complete" edge only the following: the small edges of Triangles of type A, B, C, and D; the medium edges of Triangles of type C, D and E; and the large edges of Triangles of type B and C. Note that an edge of a Triangle is complete if and only if it is not part or all of an edge of a higher Level Triangle. For each of the three edges of each Tile, we will record information about the complete edge of highest level of which it is a part; we record the "size" of that edge (S, M or L), what type Triangle (A $\cdots$ E) it is an edge of, and with what type Triangle (A $\cdots$ E) this last Triangle is "associated"; within each E^j_k we use variable names J, N and P for the three pieces of information just described—$J \in \{\mathrm{S}, \mathrm{M}, \mathrm{L}\}$ for the size of the complete edge; $N \in \{\mathrm{A}, \ldots, \mathrm{E}\}$ for the type of Triangle of which the complete edge is a part; and $P \in \{\mathrm{A}, \ldots, \mathrm{E}\}$ for the type of Triangle of which that of N is one of the five components.

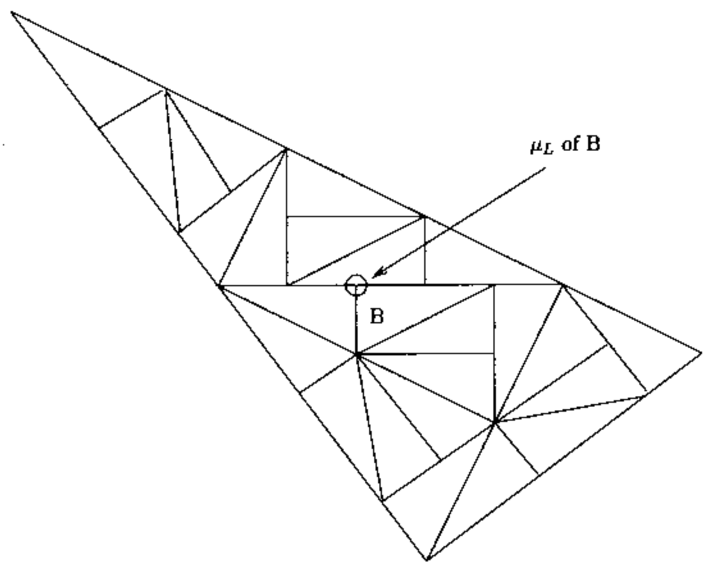

FIGURE 6

As an example, the vertex mark μ_{L} of the Tile of type B in Figure 6 has the following structure: $\mu_{\mathrm{L}} = \{A_1, E^1_1, E^2_1, A_2, E^1_2, E^2_2, A_e, E^1_e, E^2_e\}$, where

$E_k^j = (J, N, P, \cdots)$ and so:

$$\mu_{\mathrm{L}} = \{\mathrm{L}, (\mathrm{S}, \mathrm{B}, \mathrm{C}, \cdots), (\mathrm{S}, \mathrm{A}, \mathrm{C}, \cdots), \mathrm{L}, (\mathrm{L}, \mathrm{C}, \cdots), (\mathrm{L}, \mathrm{B}, \cdots), \\ \pi, (\mathrm{L}, \mathrm{B}, \cdots), (\mathrm{L}, \mathrm{C}, \cdots)\}$$

There are three other pieces of information recorded in each E_k^j, described by variables F, G and H. Using the notation $E_k^j[\alpha]$ for the edge-mark E_m^j at angle α beyond that of E_k^j, we assign $F \in \{\mathrm{S}, \mathrm{M}, \mathrm{L}, \mathrm{Z}, \mathrm{R}\}$ the value Z in E_k^1 (resp. E_k^2) if there is an edge $E_k^2[+\pi]$ (resp. $E_k^1[+\pi]$) at V, referring to a different Tile than does E_k^1 (resp. E_k^2), which is part of the same complete edge as that of E_k^1 (resp. E_k^2). We let $F = \mathrm{S}$ (resp. M, resp. L) if the complete edge of E_k^j ends at V at an angle of S (resp. M, resp. L) with the rest of its Triangle. Now $F = \mathrm{R}$ in E_k^j if a different value has not been specified above. (As one sees in the above example, it sometimes happens that in a vertex of a Tile T, information must be given about an abutting Tile T′ which does not share this vertex, as illustrated in Figure 6; in such a case we use the value R for F.) To expand on the above example then, we have: $E_k^j = (J, N, P, F, \cdots)$ and

$$\mu_{\mathrm{L}} = \{\mathrm{L}, (\mathrm{S}, \mathrm{B}, \mathrm{C}, \mathrm{L}, \cdots), (\mathrm{S}, \mathrm{A}, \mathrm{C}, \mathrm{L}, \cdots), \mathrm{L}, (\mathrm{L}, \mathrm{C}, \cdot, \mathrm{Z}, \cdots), \\ (\mathrm{L}, \mathrm{B}, \cdot, \mathrm{R}, \cdots), \pi, (\mathrm{L}, \mathrm{B}, \cdot, \mathrm{R}, \cdots), (\mathrm{L}, \mathrm{C}, \cdot, \mathrm{Z}, \cdots)\}.$$

In order to discuss variables G and H we need the following convention. For each Triangle T of Level $n \geq 1$ we define three Tiles in the Triangle, $\mathrm{T_S}$, $\mathrm{T_M}$ and $\mathrm{T_L}$, by: $\mathrm{T_S}$ shares its small vertex $\mathrm{V_{SS}} \equiv \mathrm{V_S}$ with that of the Triangle, another vertex $\mathrm{V_{SM}}$ of $\mathrm{T_S}$ lies on the large edge of the Triangle, and the third vertex $\mathrm{V_{SL}}$ of $\mathrm{T_S}$ lies on the medium edge of the Triangle; $\mathrm{T_M}$ shares its medium vertex $\mathrm{V_{MM}} \equiv \mathrm{V_M}$ with that of the Triangle, another vertex $\mathrm{V_{MS}}$ of $\mathrm{T_M}$ lies on the large edge of the Triangle, and the third vertex $\mathrm{V_{ML}}$ of $\mathrm{T_M}$ lies on the small edge of the Triangle; $\mathrm{T_L}$ shares its small vertex $\mathrm{V_{LL}} \equiv \mathrm{V_S}$ with the large vertex of the Triangle, and another vertex $\mathrm{V_{LM}}$ of $\mathrm{T_L}$ lies on the small edge of the Triangle. For Level 0 Triangles, which are Tiles, we define $\mathrm{T_S}$, $\mathrm{T_M}$ and $\mathrm{T_L}$ as the Tile itself, with $\mathrm{V_{SS}}$ as $\mathrm{V_S}$, $\mathrm{V_{MM}}$ as $\mathrm{V_M}$, and $\mathrm{V_{LL}}$ as $\mathrm{V_L}$. Note that this generalizes the concept of $\mathrm{V_S}$, $\mathrm{V_M}$ and $\mathrm{V_L}$ of Tiles to $\mathrm{V_{SS}}$, $\mathrm{V_{MM}}$ and $\mathrm{V_{LL}}$ of Triangles.

The variable G has the following structure:

$$G = (G_{\mathrm{A}}^{\alpha 1\mathrm{M}}, G_{\mathrm{E}}^{\alpha 1\mathrm{S}}, G_{\mathrm{B}}^{\alpha 1\mathrm{S}}, G_{\mathrm{D}}^{\alpha 1\mathrm{S}}, G_{\mathrm{D}}^{\alpha 1\mathrm{M}}, G_{\mathrm{A}}^{\alpha 2\mathrm{M}}, G_{\mathrm{E}}^{\alpha 2\mathrm{S}}, G_{\mathrm{B}}^{\alpha 2\mathrm{S}}, G_{\mathrm{D}}^{\alpha 2\mathrm{S}}, G_{\mathrm{D}}^{\alpha 2\mathrm{M}}, G_{\mathrm{A}}^{\epsilon 1\mathrm{M}}, \\ G_{\mathrm{E}}^{\epsilon 1\mathrm{S}}, G_{\mathrm{B}}^{\epsilon 1\mathrm{S}}, G_{\mathrm{D}}^{\epsilon 1\mathrm{S}}, G_{\mathrm{D}}^{\epsilon 1\mathrm{M}}, G_{\mathrm{A}}^{\epsilon 2\mathrm{M}}, G_{\mathrm{E}}^{\epsilon 2\mathrm{S}}, G_{\mathrm{B}}^{\epsilon 2\mathrm{S}}, G_{\mathrm{D}}^{\epsilon 2\mathrm{S}}, G_{\mathrm{D}}^{\epsilon 2\mathrm{M}}),$$

where each G_m^{jkn} takes values in $\{+1, -1\}$. We will now define these variables on each complete edge, assigning a value at the edge-marks at each end of the edge and prescribing an algorithm for assigning values at the other edge-marks on the edge. (The geometric significance of variables G and H is less simple than that of the previous marks. This will be discussed at the end of this section.)

(M1) $[G_X^{jk\mathrm{M}}]$. Let X = A or D. Consider any Tile which is part of a Triangle T, with vertices V_1 and V_2 of the Tile lying on an edge of T, and E_{I}^1 (resp. E_e^2) of V_2 and E_e^2 (resp. E_{I}^1) of V_1 referring to that edge of T. If $F = \mathrm{Z}$ in E_{I}^1 (resp. E_e^2) of V_2, then $G_X^{jk\mathrm{M}}$ has the same value in E_e^2 (resp. E_{I}^1) of V_1 as it has in $E_{\mathrm{I}}^2[-\pi]$ (resp. $E_e^1[+\pi]$) of V_2, unless V_2 has an edge-mark $E_{\mathrm{I}}^2[-\pi/2]$ (resp. $E_e^1[+\pi/2]$) which contains the values (J, N, F, P) = (S, A, L, A) or (S, A, L, D), in which case the two G values have opposite sign. (In Figure 7,

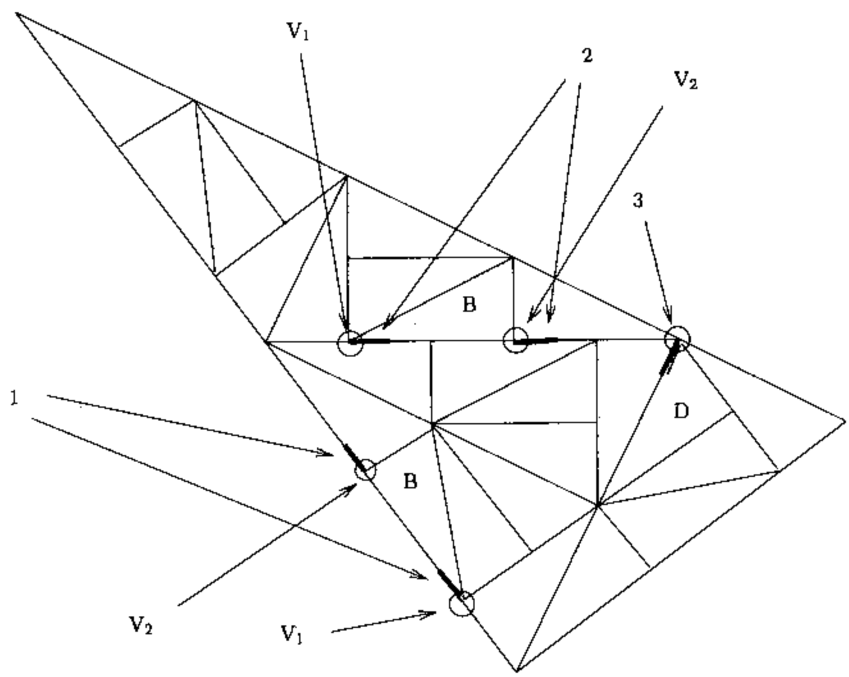

FIGURE 7

numbers 1 and 2 each label a pair of edge-marks; for each pair there is the $E_e^2[-\pi]$ of $\mathrm{V_L}$ and the E_e^2 of $\mathrm{V_S}$ of a Tile of type B. For pair 1, G_{D}^{jkn} has different values in the two edge-marks, and for pair 2, G_{D}^{jkn} has the same values in the two edge-marks.) If $F \neq \mathrm{Z}$ in E_{I}^1 (resp. E_e^2) of V_2, and we do not have $k = 2$ (resp. $k = 1$) and (J, N, F) = (S, X, L) in E_{I}^1 (resp. E_e^2) of V_2, then $G_X^{\epsilon k\mathrm{M}} = +1$ in that edge-mark; $k = 1$ (resp. $k = 2$) and (J, N, F) = (S, A, L) or (S, D, L) in E_{I}^1 (resp. E_e^2) of V_2, then $G_X^{\alpha k\mathrm{M}} = +1$ in that edge-mark. (In

Figure 7, label 3 indicates an edge-mark E_1^1 of vertex $\mathrm{V_M}$ of a Tile of type D, in which $G_{\mathrm{X}}^{\alpha k\mathrm{M}} = G_{\mathrm{X}}^{\epsilon k\mathrm{M}} = +1$ from above.)

(M2) $[G_X^{\epsilon k\mathrm{M}}$ in $X]$. Let T be a Triangle of class $X1 =$ A1 or D1 (resp. $X2 =$ A2 or D2). In edge-mark E_e^2 (resp. E_1^1) of vertex $\mathrm{V_{ML}}$, $G_X^{\epsilon 2\mathrm{M}}$ (resp. $G_X^{\epsilon 1\mathrm{M}}$) has the same value as that of $G_X^{\alpha 2\mathrm{M}}$ (resp. $G_X^{\alpha 1\mathrm{M}}$) in the edge-mark E_1^1 (resp. E_e^2) of vertex $\mathrm{V_{MS}}$. See Figure 8A.

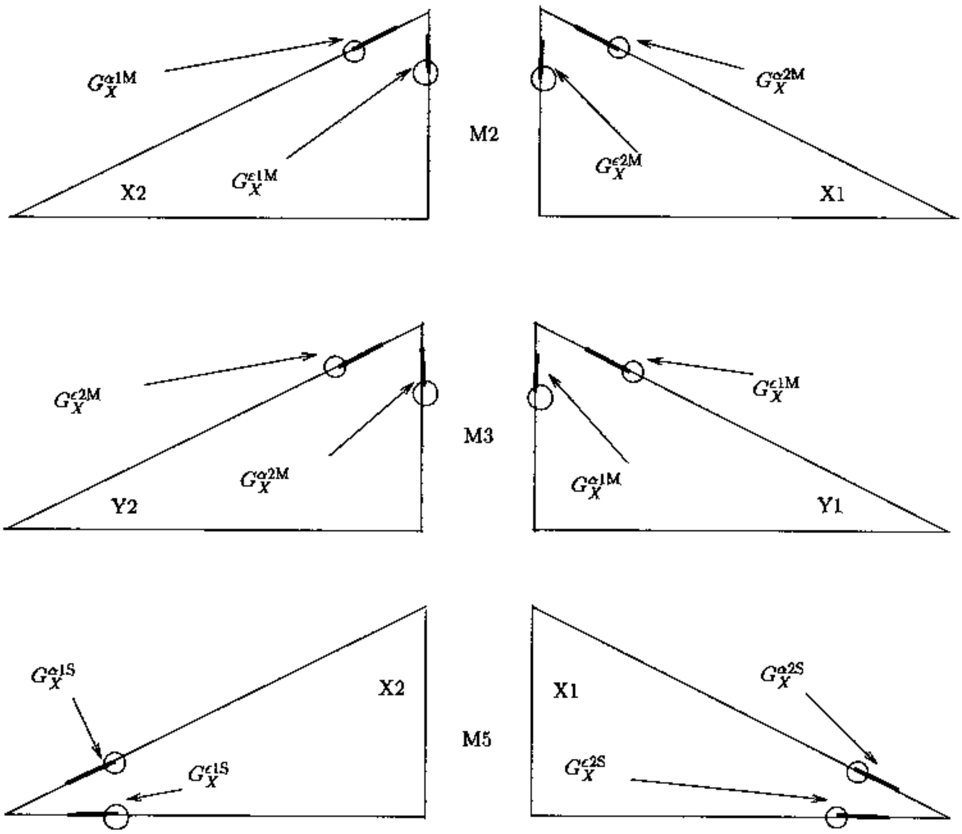

FIGURE 8A

(M3) $[G_X^{\alpha k\mathrm{M}}$ in $Y]$. Let T be a Triangle of class $Y1 =$ A1 or D1 (resp. $Y2 =$ A2 or D2), and let $X =$ A or D. In edge-mark E_e^2 (resp. E_1^1) of vertex $\mathrm{V_{ML}}$, $G_X^{\alpha 1\mathrm{M}}$ (resp. $G_X^{\alpha 2\mathrm{M}}$) has the negative of the value of $G_X^{\epsilon 1\mathrm{M}}$ (resp. $G_X^{\epsilon 2\mathrm{M}}$) in the edge-mark E_1^1 (resp. E_e^2) of vertex $\mathrm{V_{MS}}$. See Figure 8A.

(M4) $[G_X^{jk\mathrm{S}}]$. Let $X =$ D or E. Consider any Tile which is part of a Triangle T, with vertices $\mathrm{V_1}$ and $\mathrm{V_2}$ of the Tile lying on an edge of T, and E_1^1 (resp. E_e^2) of $\mathrm{V_2}$ and E_e^2 (resp. E_1^1) of $\mathrm{V_1}$ referring to the edge of T. If $F = \mathrm{Z}$ in E_1^1 (resp. E_e^2) of $\mathrm{V_2}$, then $G_X^{jk\mathrm{S}}$ has the same value in E_e^2 (resp. E_1^1)

of V_1 as it has in $E_1^2[-\pi]$ (resp. $E_e^1[+\pi]$) of V_2, unless (i) V_2 has an edge-mark $E_1^2[-\pi/2]$ (resp. $E_e^1[+\pi/2]$) which contains the values $(J,N,F,P) = (\mathrm{S,A,L,E})$ or $(\mathrm{S,A,L,D})$; or (ii) V_2 has an edge-mark E_e^2 (resp. E_1^1) which contains the values $(J,N,F,P) = (\mathrm{S,A,M,B})$, in which case the two G values have opposite sign. If $F \neq \mathrm{Z}$ in E_1^1 (resp. E_e^2) of V_2, and we do not have (i) $k=2$ (resp. $k=1$) and $(J,N,F) = (\mathrm{M,X,L})$ in E_1^1 (resp. E_e^2) of V_2, then $G_X^{\epsilon k\mathrm{S}} = +1$ in that edge-mark; (ii) $k=1$ (resp. $k=2$) and $(J,N,F) = (\mathrm{M,E,L})$ or $(\mathrm{M,D,L})$ or $(\mathrm{L,B,M})$ in E_1^1 (resp. E_e^2) of V_2, then $G_X^{\alpha k\mathrm{S}} = +1$ in that edge-mark.

(M5) $[G_X^{\epsilon k\mathrm{S}}$ in $X]$. Let T be a Triangle of class $X1 = \mathrm{E1}$ or D1 (resp. $X2 = \mathrm{E2}$ or D2). In edge-mark E_1^1 (resp. E_e^2) of vertex V_{SL}, $G_X^{\epsilon 2\mathrm{S}}$ (resp. $G_X^{\epsilon 1\mathrm{S}}$) has the same value as that of $G_X^{\alpha 2\mathrm{S}}$ (resp. $G_X^{\alpha 1\mathrm{S}}$) in the edge-mark E_e^2 (resp. E_1^1) of vertex V_{SM}. See Figure 8A.

(M6) $[G_X^{\alpha k\mathrm{S}}$ in $Y]$. Let T be a Triangle of class $Y1 = \mathrm{E1}$ or D1 (resp. $Y2 = \mathrm{E2}$ or D2), and let $X = \mathrm{E}$ or D. In edge-mark E_1^1 (resp. E_e^2) of vertex V_{SL}, $G_X^{\alpha 1\mathrm{S}}$ (resp. $G_X^{\alpha 2\mathrm{S}}$) has the negative of the value of $G_X^{\epsilon 1\mathrm{S}}$ (resp. $G_X^{\epsilon 2\mathrm{S}}$) in the edge-mark E_e^2 (resp. E_1^1) of vertex V_{SM}. See Figure 8B.

(M7) $[G_X^{\alpha k\mathrm{S}}$ in B]. Let T be a Triangle of class B1 (resp. B2) and let $X = \mathrm{E}$ or D. In edge-mark E_1^1 (resp. E_e^2) of vertex V_{SM}, $G_X^{\alpha 1\mathrm{S}}$ (resp. $G_X^{\alpha 2\mathrm{S}}$) has the negative of the value of $G_X^{\epsilon 1\mathrm{S}}$ (resp. $G_X^{\epsilon 2\mathrm{S}}$) in edge-mark E_e^2 (resp. E_1^1) of vertex V_{SL}. See Figure 8B.

(M8) $[G_{\mathrm{B}}^{jk\mathrm{S}}]$. Consider any Tile which is part of a Triangle T, with vertices V_1 and V_2 of the Tile lying on an edge of T, and E_1^1 (resp. E_e^2) of V_2 and E_e^2 (resp. E_1^1) of V_1 referring to the edge of T. If $F = \mathrm{Z}$ in E_1^1 (resp. E_e^2) of V_2, then $G_{\mathrm{B}}^{jk\mathrm{S}}$ has the same value in E_e^2 (resp. E_1^1) of V_1 as it has in $E_1^2[-\pi]$ (resp. $E_e^1[+\pi]$) of V_2, unless (i) V_2 has an edge-mark $E_1^2[+\pi/2]$ (resp. $E_e^1[-\pi/2]$) which contains the values $(J,N,F,P) = (\mathrm{S,A,L,E})$ or $(\mathrm{S,A,L,D})$; or (ii) V_2 has an edge-mark E_e^2 (resp. E_1^1) which contains the values $(J,N,F,P) = (\mathrm{S,A,M,B})$, in which case the two G values have opposite sign. If $F \neq \mathrm{Z}$ in E_1^1 (resp. E_e^2) of V_2, and we do not have (i) $k=2$ (resp. $k=1$) and $(J,N,F) = (\mathrm{M,E,L})$ or $(\mathrm{M,D,L})$ or $(\mathrm{L,B,M})$ in E_1^1 (resp. E_e^2) of V_2, then $G_X^{\epsilon k\mathrm{S}} = +1$ in E_1^1 (resp. E_e^2) of V_2; or (ii) $k=1$ (resp. $k=2$) and $(J,N,F) = (\mathrm{L,B,M})$ in E_1^1 (resp. E_e^2) of V_2, then $G_X^{\alpha k\mathrm{S}} = +1$ in E_1^1 (resp. E_e^2) of V_2.

(M9) $[G_{\mathrm{B}}^{\epsilon k\mathrm{S}}$ in $Y]$. Let T be a Triangle of class $Y1 = \mathrm{E1}$ or D1 (resp. $Y2 = \mathrm{E2}$ or D2). In edge-mark E_1^1 (resp. E_e^2) of vertex V_{SL}, $G_{\mathrm{B}}^{\epsilon 2\mathrm{S}}$ (resp. $G_{\mathrm{B}}^{\epsilon 1\mathrm{S}}$) has the negative of the value of $G_{\mathrm{B}}^{\alpha 2\mathrm{S}}$ (resp. $G_{\mathrm{B}}^{\alpha 1\mathrm{S}}$) in the edge-mark E_e^2 (resp. E_1^1) of vertex V_{SM}. See Figure 8B.

(M10) $[G_{\mathrm{B}}^{\epsilon k\mathrm{S}}$ in B]. Let T be a Triangle of class B1 (resp. B2). In edge-mark E_1^1 (resp. E_e^2) of vertex V_{SM}, $G_{\mathrm{B}}^{\epsilon 2\mathrm{S}}$ (resp. $G_{\mathrm{B}}^{\epsilon 1\mathrm{S}}$) has the negative of the

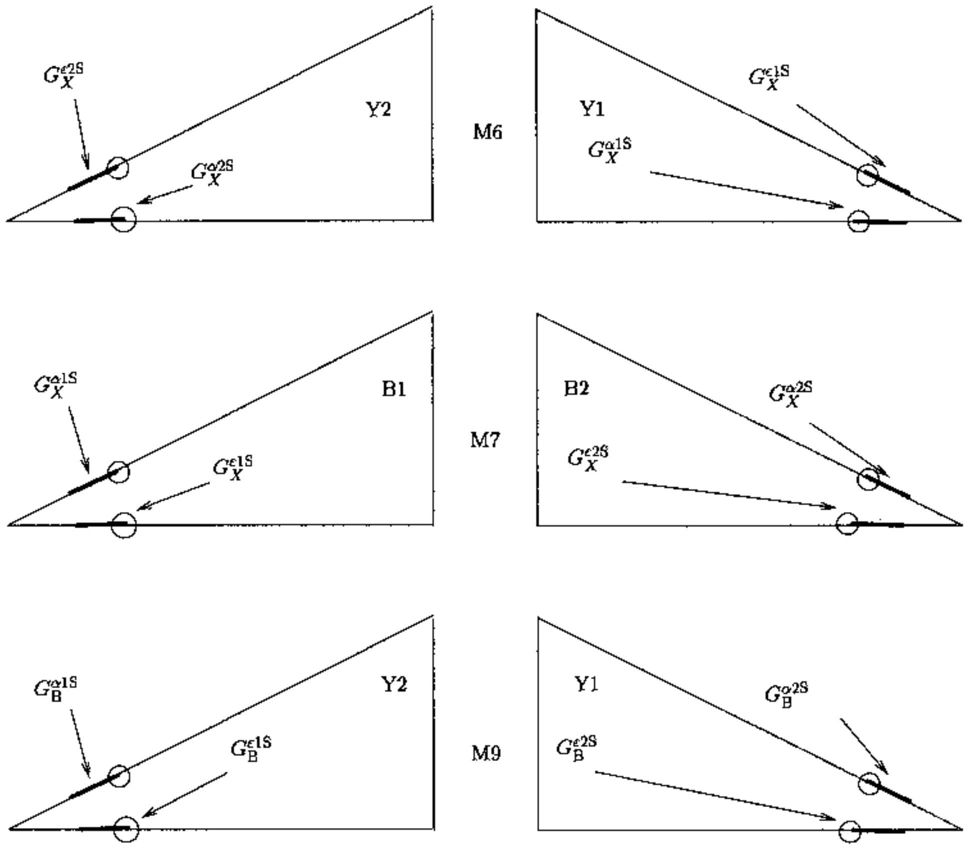

FIGURE 8B

value of $G_B^{\alpha 2S}$ (resp. $G_B^{\alpha 1S}$) in the edge-mark E_e^2 (resp. E_1^1) of vertex V_{SL}. See Figure 8C.

(M11) [$G_B^{\alpha kS}$ in B]. Let T be a Triangle of class B1 (resp. B2). In edge-mark E_1^1 (resp. E_e^2) of vertex V_{SM}, $G_B^{\alpha 1S}$ (resp. $G_B^{\alpha 2S}$) has the same value as that of $G_B^{\epsilon 1S}$ (resp. $G_B^{\epsilon 2S}$) in the edge-mark E_e^2 (resp. E_1^1) of vertex V_{SL}. See Figure 8C.

(Note: The above values of G are complicated but well-defined. To see that they are well defined just note that even though one value is sometimes defined in terms of another, this process always ends since it always refers to a variable on a higher Level edge; in the tesselation Θ every Triangle is contained in infinitely many Triangles of type C, on the edges of which all G variables are explicitly defined, and the process cannot take us outside any Triangle of type C.)

$H \in \{+1, -1\}$ basically maintains a "parity" between two specific points in each Triangle of Level $n \geq 1$, marked V1 and V5 in Figure 12A, and V6

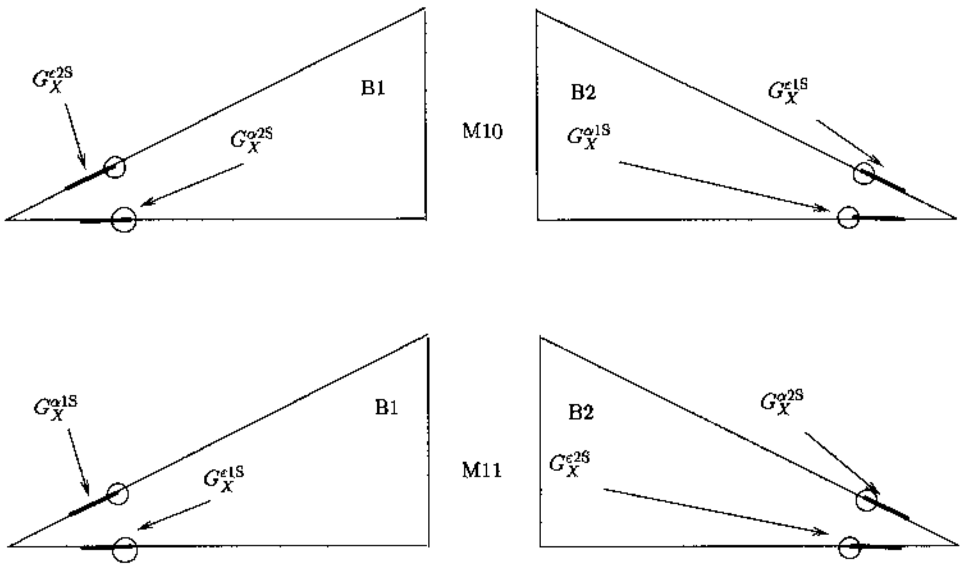

FIGURE 8C

and V10 in Figure 12B. (See below.) The usefulness of this parity must remain obscure until the proof of Theorem 5.4.

The variable H has the following structure:

$$H = (H_{\mathrm{A}}^{1\mathrm{M}}, H_{\mathrm{E}}^{1\mathrm{S}}, H_{\mathrm{B}}^{1\mathrm{S}}, H_{\mathrm{D}}^{1\mathrm{S}}, H_{\mathrm{D}}^{1\mathrm{M}}, H_{\mathrm{A}}^{2\mathrm{M}}, H_{\mathrm{E}}^{2\mathrm{S}}, H_{\mathrm{B}}^{2\mathrm{S}}, H_{\mathrm{D}}^{2\mathrm{S}}, H_{\mathrm{D}}^{2\mathrm{M}}),$$

where each H_m^{kn} takes values in $\{+1, -1\}$, as follows.

(M12) Consider any Tile with E_1^1 (resp. E_e^2) of vertex V_2 and E_e^2 (resp. E_1^1) of vertex V_1 referring to the same edge of the Tile, and assume $F = \mathrm{Z}$ in E_1^1 (resp. E_e^2) of V_2. Then the ratio of H_m^{kn} and $G_m^{\varepsilon kn}$ has the same value in $E_1^2[+\pi]$ (resp. $E_e^1[+\pi]$) of V_2 as in E_e^2 (resp. E_1^1) of V_1. Also, $H_m^{kn} = G_m^{\varepsilon kn}$ in all edge-marks except as prescribed below, in M13, M14 or M15, for certain edge-marks in small edges of Triangles of type A and D, and medium edges of type D.

(M13) Consider any vertex V of a prototile with edge-mark E_1^1 (resp. E_e^2) in which $(J, N, F) = (\mathrm{S}, \mathrm{D}, \mathrm{M})$; in particular, the edge-mark is lying in the small edge of a Triangle of class D1 (resp. D2). See Figure 9. Then the ratio of $H_m^{1\mathrm{M}}$ and $G_m^{\varepsilon 1\mathrm{M}}$ (resp. $H_m^{2\mathrm{M}}$ and $G_m^{\varepsilon 2\mathrm{M}}$) in E_1^1 (resp. E_e^2) equals the value of $G_m^{\varepsilon 1\mathrm{M}}$ (resp. $G_m^{\varepsilon 2\mathrm{M}}$) in E_2^1 (resp. E_{e-1}^2). Also, the ratio of $H_m^{1\mathrm{S}}$ and $G_m^{\varepsilon 1\mathrm{S}}$ (resp. $H_m^{2\mathrm{S}}$ and $G_m^{\varepsilon 2\mathrm{S}}$) in E_1^1 (resp. E_e^2) equals the value of $G_m^{\varepsilon 1\mathrm{S}}$ (resp. $G_m^{\varepsilon 2\mathrm{S}}$) in E_e^2 (resp. E_1^1).

(M14) Consider any Tile with E_1^1 (resp. E_e^2) of vertex V_2 and E_e^2 (resp. E_1^1) of vertex V_1 referring to the same edge of the Tile, and assume $(J, N, F) =$

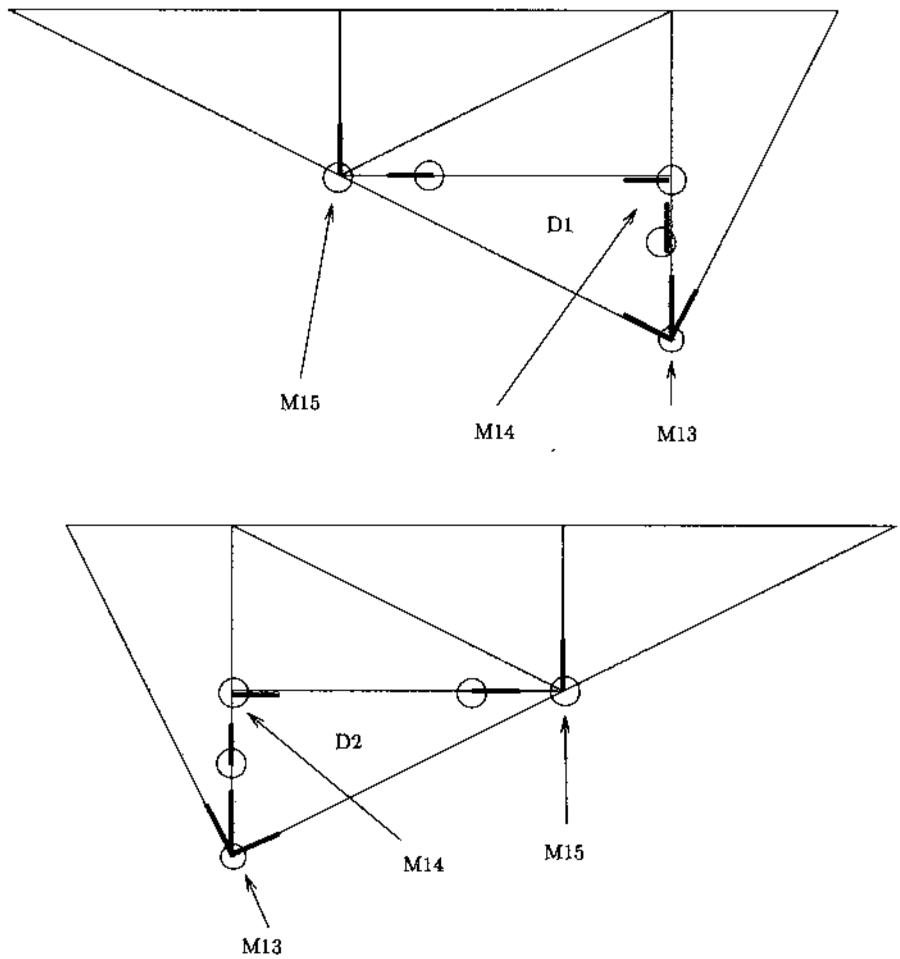

FIGURE 9

(M, D, L) in E_1^1 (resp. E_e^2) of V_2. See Figure 9. Then the ratio of H_m^{2n} and $G_m^{\varepsilon 2n}$ (resp. H_m^{1n} and $G_m^{\varepsilon 1n}$) is equal in E_e^2 (resp. E_1^1) of V_1 to its value in $E_1^2[-\pi/2]$ (resp. $E_e^1[+\pi/2]$) of V_2.

(M15) Consider any Tile with E_1^1 (resp. E_e^2) of vertex V_2 and E_e^2 (resp. E_1^1) of vertex V_1 referring to the same edge of the Tile, and assume $(J, N, F) =$ (M, D, S) in E_1^1 (resp. E_e^2) of V_2. See Figure 9. Then the ratio of H_m^{2n} and $G_m^{\varepsilon 2n}$ (resp. H_m^{1n} and $G_m^{\varepsilon 1n}$) is equal in E_e^2 (resp. E_1^1) of V_1 to its value in $E_1^2[+\pi/2]$ (resp. $E_e^1[-\pi/2]$) of V_2.

We now summarize the above addition of marks to the Tiles in the tesselation Θ. The marks all concern the hierarchical structure of the Triangles in Θ, and more specifically they record certain facts about their complete edges.

At each vertex in Θ it is recorded, on all the Tiles sharing that vertex, which types of complete edges are associated with that vertex, what type Triangle each such edge belongs to, and what type Triangle is the parent of the latter Triangle. We also record at what angles these edges meet, and whether or not the edge ends at that vertex. This is all implicit in the variables A, J, N, P and F, and is rather easy to understand visually. The variables G and H refer to less obvious features of the tesselation Θ. Both these (families of) variables take values ± 1, and do not have a natural absolute meaning; they should be thought of as phases, and convey information through the agreement or disagreement of the values of variables on pairs of edges meeting at a vertex. In Θ the relative values of these variables are fixed at certain such intersections, as illustrated in Figures 8A,B,C and 9, and the values then oscillate as one moves past vertices on a given complete edge, with certain specified exceptions, as illustrated in Figure 7; these exceptions each have simple geometric meaning in terms of edges of certain Triangles of lower Level appearing at the vertex. The reason for the definitions is that certain pairs of variables meet in phase at a vertex in Θ only under a unique circumstance; for example variables $G_{\mathrm{A}}^{\alpha 1\mathrm{M}}$ and $G_{\mathrm{A}}^{\epsilon 1\mathrm{M}}$ meet in phase in the geometric relation shown in Figure 8A if and only if they meet at the medium vertex of a Triangle of class A1. This is shown in the proof of Theorem 5.1 below. To repeat then, the actual value of any of the variables in the G or H families has no geometric meaning, it is only the relative values of pairs of variables meeting at a vertex which may have a (simple geometric) meaning in terms of the hierarchy of Triangles.

3. Tiling the plane; the prototiles and their matching rules

We now consider the problem of trying to reproduce the structure of the substitution tesselation Θ by the tilings of a finite set of prototiles. Before we begin, we need to introduce a simple technique used almost universally in forced tiling.

As stated above, tiling consists of simultaneous covering and packing; the packing condition intuitively requires (given the covering condition) that the tiles must fit together like a jig-saw puzzle, the boundary of one tile nestled into the boundaries of its neighbors. Consider for example the system in the left half of Figure 10 of two prototiles in the plane which are both roughly square with edges aligned (but with jagged edges). (They can only tile the plane in checkerboard fashion.)

Now consider in the right half of Figure 10 the set of two perfect squares, with edges aligned, which have "colors" $(1, 2, 3, 4, 1', 2', 3', 4')$ assigned to their edges; the colors come in "complementary" pairs: j and j' are complements. If we consider this a pair of prototiles, and allow translated copies (called

FIGURE 10

tiles) to be made, preserving the assignment of colors to edges, and add the "matching rule" that in a tiling tiles must abut full edge to full edge and with complementary colors meeting, then it is clear that we will reproduce, in any reasonable sense, the previous example. In other words, one can often simplify an example of prototiles by using simpler shapes, but with added "colors" and "matching rules" to make up for the simplified boundary. Reversing this process, if one is given a set of prototiles with colors and matching rules, it is straightforward to replace it by a set of prototiles with more complicated boundaries and no colors or matching rules, and this is what we will do.

We sketch here a justification for this claim. Assume there is a finite set of polygonal prototiles, the set of all edges of which is called Q, and assume rules governing, for each edge, which edges it may abut in a tiling, this information being summarized in the symmetric set $\mathrm{K} \subset \mathrm{Q} \times \mathrm{Q}$. We assume that edges may only abut along their full length in a tiling. In some examples not satisfying this condition, noted below, one can add "vertices" on the original edges to produce smaller edges which then satisfy the condition. We now show that the effect of such rules is that K can be reproduced by suitably modifying the edges of the polygons, and requiring that the new tiles tile the plane in the usual geometric sense, if and only if a certain mild condition C is satisfied by K, namely that if (a, b), (a, c) and (d, b) are in K then so is (d, c). The necessity of this condition for the desired conclusion is obvious. We sketch the sufficiency, using [4]. Define two edges a and b to be equivalent if there exists an edge c such that (a, c) and (b, c) are both in K. It follows from condition C that this is an equivalence relation. Next we define for each equivalence class a unique complementary class as follows. For class E the complement is that class E' such that (a, b) is in K for some a in E and b in E'. It follows from condition C that this defines a unique class E' independent of choice of representatives. This is sufficient to define a unique family of zig-zag curves to modify the edges, one for each pair of complementary classes, so that only edges from complementary classes will fit together; one could take the same number of bumps for all edges—all congruent for a given curve—and vary the height of the bumps from class to class to guarantee the uniqueness.

We will have in fact only one basic shape (and its reflection), the unmarked prototile of the tesselation Θ, but we will define a set of colors, which we will call "marks," with a notion of complementary pairs, and a set of "matching rules," and we will analyze those tilings of the plane by tiles which satisfy the matching rules, the object being to show that these tilings all exhibit the basic structure of Triangles of all Levels which we see in Θ. We begin by introducing the marks.

The marks were already introduced for the Tiles of Θ. Intuitively, we would like to define our prototile set as all the distinct versions of the unmarked prototile and its reflection (reflecting about its small edge, say) obtained by adding to them those marks which appear on the Tiles which are obtained from these two unmarked prototiles by orientation-preserving isometries. This could be done, but it is somewhat unsatisfying in that it would then be difficult to determine precisely which of the conceivable mark combinations are used. So we instead define our prototiles as those obtained by using all possible marks, but satisfying an explicit list of "restrictions."

A "(marked) prototile" consists of the unmarked prototile, or its reflection about its small edge (which we will now call another unmarked prototile), together with three (vertex) marks $\mu_{\mathrm{S}}, \mu_{\mathrm{M}}, \mu_{\mathrm{L}}$, each of the three of the form $(A_1, E_1^1, E_1^2, A_2, E_2^1, E_2^2, \ldots, A_e, E_e^1, E_e^2)$, where e varies between 3 and 8 (e stands for "end") and where:

(a) $A_j \in \{\mathrm{S}, \mathrm{M}, \mathrm{L}, \pi\}$,

(b) each E_j^k is of the form (J, N, P, F, G, H), where:

(c) $J \in \{\mathrm{S}, \mathrm{M}, \mathrm{L}\}$,

(d) $N \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}\}$,

(e) $P \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}\}$,

(f) $F \in \{\mathrm{S}, \mathrm{M}, \mathrm{L}, \mathrm{R}, \mathrm{Z}\}$,

(g) $G = (G_{\mathrm{A}}^{\alpha 1\mathrm{M}}, G_{\mathrm{E}}^{\alpha 1\mathrm{S}}, G_{\mathrm{B}}^{\alpha 1\mathrm{S}}, G_{\mathrm{D}}^{\alpha 1\mathrm{S}}, G_{\mathrm{D}}^{\alpha 1\mathrm{M}}, G_{\mathrm{A}}^{\alpha 2\mathrm{M}}, G_{\mathrm{E}}^{\alpha 2\mathrm{S}}, G_{\mathrm{B}}^{\alpha 2\mathrm{S}}, G_{\mathrm{D}}^{\alpha 2\mathrm{S}}, G_{\mathrm{D}}^{\alpha 2\mathrm{M}}, G_{\mathrm{A}}^{\epsilon 1\mathrm{M}}, G_{\mathrm{E}}^{\epsilon 1\mathrm{S}}, G_{\mathrm{B}}^{\epsilon 1\mathrm{S}}, G_{\mathrm{D}}^{\epsilon 1\mathrm{S}}, G_{\mathrm{D}}^{\epsilon 1\mathrm{M}}, G_{\mathrm{A}}^{\epsilon 2\mathrm{M}}, G_{\mathrm{E}}^{\epsilon 2\mathrm{S}}, G_{\mathrm{B}}^{\epsilon 2\mathrm{S}}, G_{\mathrm{D}}^{\epsilon 2\mathrm{S}}, G_{\mathrm{D}}^{\epsilon 2\mathrm{M}})$, where each G_m^{jkn} takes values in $\{+1, -1\}$.

(h) $H = (H_{\mathrm{A}}^{1\mathrm{M}}, H_{\mathrm{E}}^{1\mathrm{S}}, H_{\mathrm{B}}^{1\mathrm{S}}, H_{\mathrm{D}}^{1\mathrm{S}}, H_{\mathrm{D}}^{1\mathrm{M}}, H_{\mathrm{A}}^{2\mathrm{M}}, H_{\mathrm{E}}^{2\mathrm{S}}, H_{\mathrm{B}}^{2\mathrm{S}}, H_{\mathrm{D}}^{2\mathrm{S}}, H_{\mathrm{D}}^{2\mathrm{M}})$, where each H_m^{kn} takes values in $\{+1, -1\}$.

(We note that although these are called "vertex marks," they could easily be implemented by encoding their information in bumps and dents in the edges of the tiles, away from the vertices.) As mentioned above, we do not allow our prototiles to have all possible values of the above marks; we allow precisely those combinations satisfying the following set of restrictions.

Every prototile must contain marks of one of the following ten "classes," A1 $\cdots$ E2 (and thus five "types," A $\cdots$ E, where the type of class Xj is denoted X), the "original" ones, with vertices at (0,0), (2,0), and (2,1), satisfying one of A2, B1, C1, D2, E2, and the "reflected" ones, with vertices at (2,0), (4,0) and (2,1), satisfying one of A1, B2, C2, D1, E1, as shown in Figures 11A,B.

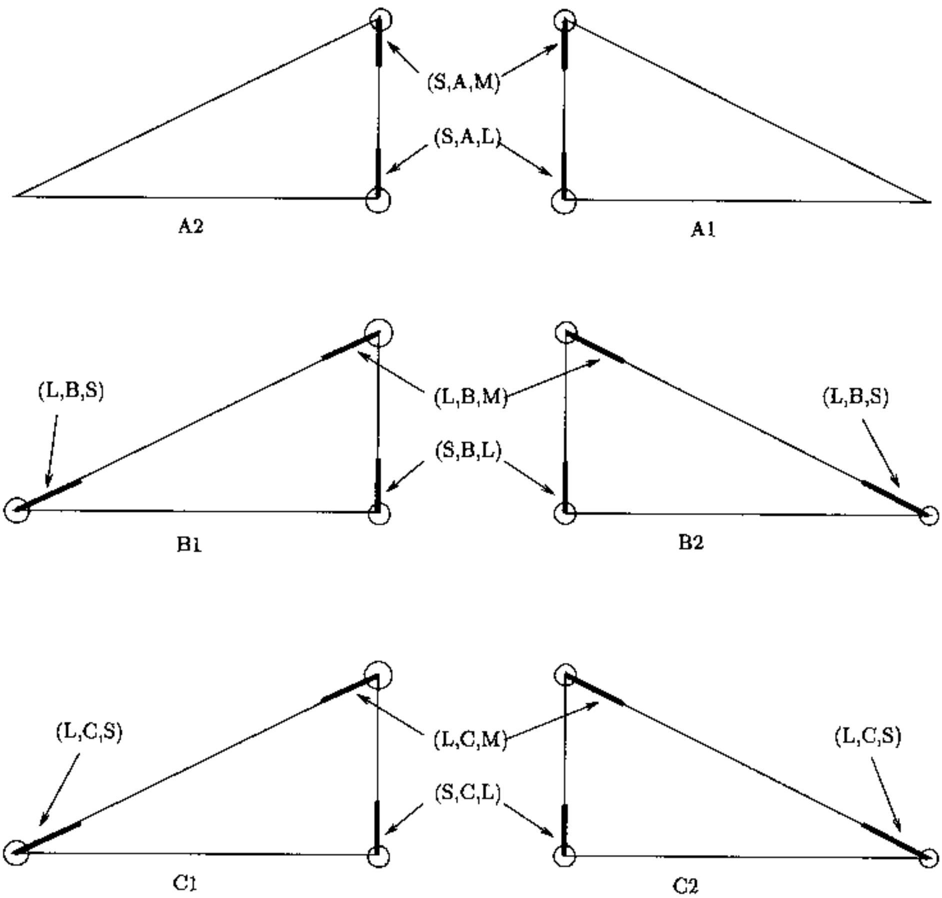

FIGURE 11A

(It will follow from further restrictions, namely V5 and V10, that these ten classes are mutually exclusive.)

(A1) In V_L, E_e^2 contains $(J, N, F) = (\mathrm{S}, \mathrm{A}, \mathrm{L})$, and in V_M, E_1^1 contains $(J, N, F) = (\mathrm{S}, \mathrm{A}, \mathrm{M})$.

(B1) In V_L, E_1^1 contains $(J, N, F) = (\mathrm{S}, \mathrm{B}, \mathrm{L})$; in V_M, E_1^1 contains $(J, N, F) = (\mathrm{L}, \mathrm{B}, \mathrm{M})$; and in V_S, E_e^2 contains $(J, N, F) = (\mathrm{L}, \mathrm{B}, \mathrm{S})$.

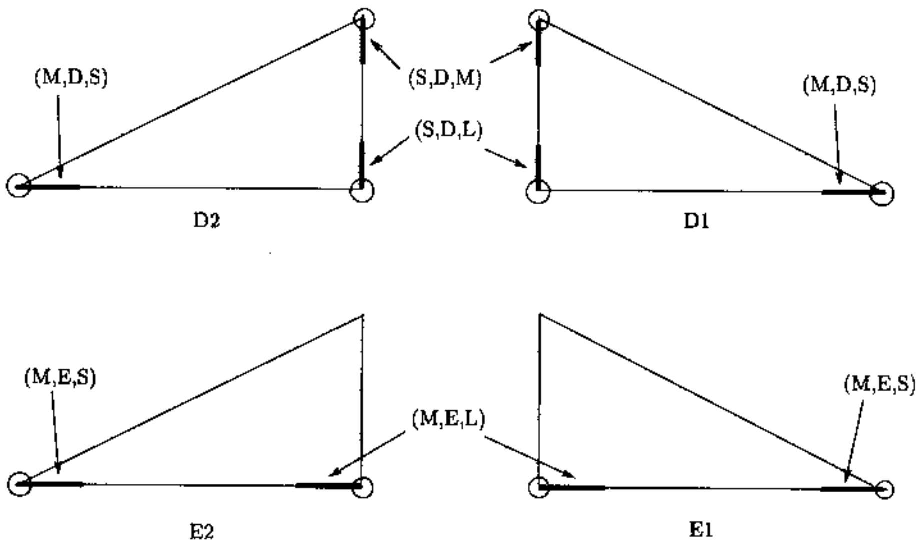

FIGURE 11B

(C1) In $\mathrm{V_S}$, E_e^2 contains $(J,N,F) = (\mathrm{M,C,S})$; in $\mathrm{V_M}$, E_1^1 contains $(J,N,F) = (\mathrm{L,C,M})$; and in $\mathrm{V_L}$, E_1^1 contains $(J,N,F) = (\mathrm{S,C,L})$.

(D1) In $\mathrm{V_S}$, E_e^2 contains $(J,N,F) = (\mathrm{M,D,S})$; in $\mathrm{V_L}$, E_e^2 contains $(J,N,F) = (\mathrm{M,D,L})$; and in $\mathrm{V_M}$, E_1^1 contains $(J,N,F) = (\mathrm{S,D,M})$.

(E1) In $\mathrm{V_S}$, E_e^2 contains $(J,N,F) = (\mathrm{M,E,S})$, and in $\mathrm{V_L}$, E_1^1 contains $(J,N,F) = (\mathrm{M,E,L})$.

(A2) In $\mathrm{V_L}$, E_1^1 contains $(J,N,F) = (\mathrm{S,A,L})$, and in $\mathrm{V_M}$, E_e^2 contains $(J,N,F) = (\mathrm{S,A,M})$.

(B2) In $\mathrm{V_L}$, E_e^2 contains $(J,N,F) = (\mathrm{S,B,L})$; in $\mathrm{V_M}$, E_e^2 contains $(J,N,F) = (\mathrm{S,B,M})$; and in $\mathrm{V_S}$, E_1^1 contains $(J,N,F) = (\mathrm{L,B,S})$.

(C2) In $\mathrm{V_S}$, E_1^1 contains $(J,N,F) = (\mathrm{L,C,S})$; in $\mathrm{V_M}$, E_e^2 contains $(J,N,F) = (\mathrm{S,C,M})$; and in $\mathrm{V_L}$, E_e^2 contains $(J,N,F) = (\mathrm{M,C,L})$.

(D2) In $\mathrm{V_S}$, E_1^1 contains $(J,N,F) = (\mathrm{M,D,S})$; in $\mathrm{V_L}$, E_1^1 contains $(J,N,F) = (\mathrm{S,D,L})$; and in $\mathrm{V_M}$, E_e^2 contains $(J,N,F) = (\mathrm{S,D,M})$.

(E2) In $\mathrm{V_L}$, E_e^2 contains $(J,N,F) = (\mathrm{M,E,L})$, and in $\mathrm{V_S}$, E_1^1 contains $(J,N,F) = (\mathrm{M,E,S})$.

The following restrictions can be understood from Figures 12A-12F; we record in the restrictions V1 $\cdots$ V10 almost everything that can be easily deduced as occurring in the tesselation Θ at such vertices. The references to variables H and G will need special analysis.

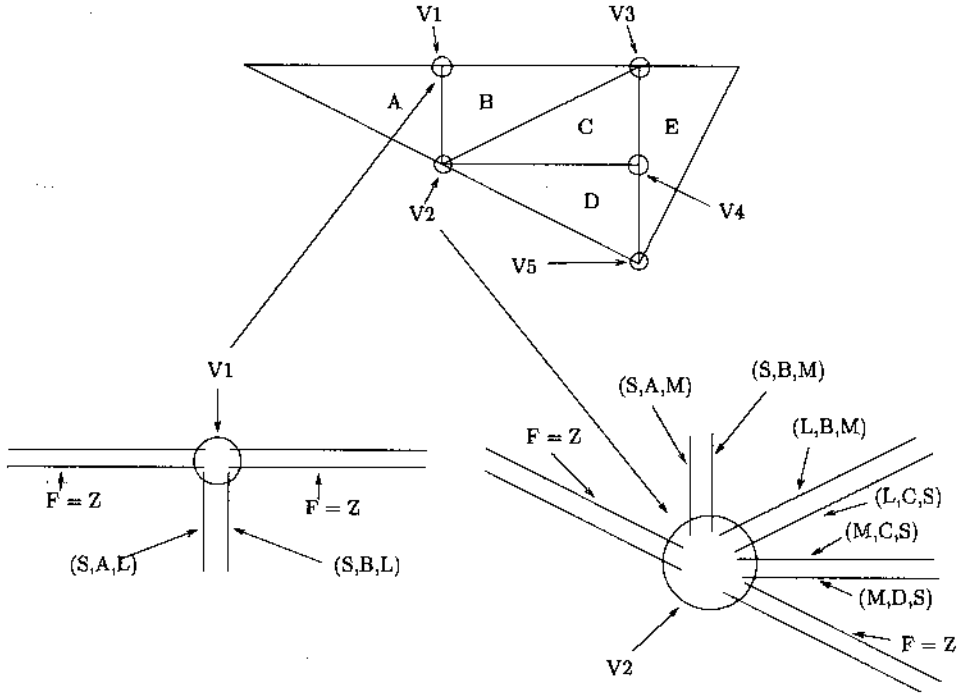

FIGURE 12A

(V1) Consider a prototile T, with vertices V_1, V_2 and V_3, such that E_1^1 of V_2 and E_e^2 of V_1 refer to the same edge of T. (For vertices V_1, V_2 and V_3, of a tile—as opposed to a Tile—the phrase "E_1^1 of V_2 and E_e^2 of V_1 refer to the same edge" means by definition that V_1, V_2 and V_3 are in clockwise order in the plane.) If in V_1 either of the following conditions is satisfied: E_e^2 contains $(J,N,F) = (\mathrm{S},\mathrm{A},\mathrm{L})$, or E_e^1 contains $(J,N,F) = (\mathrm{S},\mathrm{B},\mathrm{L})$, then so is the other, and also: $F = \mathrm{Z}$ in $E_e^1[+\pi/2]$ and $E_e^2[-\pi/2]$; P has the same value in E_e^1 and E_e^2; the quotient of $H_m^{1\mathrm{M}}$ by $G_m^{\epsilon 1\mathrm{M}}$ in E_1^1 of V_2 equals $G_m^{\alpha 1\mathrm{M}}$ in $E_e^2[-\pi/2]$ of V_1 if $P = m$ in E_e^2 of V_1; and the quotient of $H_m^{1\mathrm{S}}$ by $G_m^{\epsilon 1\mathrm{S}}$ in E_1^1 of V_2 equals $G_m^{\alpha 1\mathrm{S}}$ in $E_e^1[+\pi/2]$ of V_1 if $P = m$ in E_e^2. Also if in V_1 we have, in E_e^2, that P has value B, then $(J,N) = (\mathrm{L},\mathrm{B})$ in $E_e^1[+\pi/2]$, and $(J,N) = (\mathrm{L},\mathrm{B})$ in $E_e^2[-\pi/2]$; if P has value C, then $(J,N) = (\mathrm{L},\mathrm{C})$ in $E_e^1[+\pi/2]$ and in $E_e^2[-\pi/2]$.

(V2) If a vertex of a prototile satisfies any of the following conditions: E_k^1 contains $(J,N,F) = (\mathrm{S},\mathrm{A},\mathrm{M})$; E_k^2 contains $(J,N,F) = (\mathrm{S},\mathrm{B},\mathrm{M})$; E_{k+1}^1 contains $(J,N,F) = (\mathrm{L},\mathrm{B},\mathrm{M})$; E_{k+1}^2 contains $(J,N,F) = (\mathrm{L},\mathrm{C},\mathrm{S})$; E_{k+2}^1 contains $(J,N,F) = (\mathrm{M},\mathrm{C},\mathrm{S})$; E_{k+2}^2 contains $(J,N,F) = (\mathrm{M},\mathrm{D},\mathrm{S})$, then it satisfies the others, and also: $F = \mathrm{Z}$ in E_{k-1}^2 and in E_{k+3}^1; P has the same value in E_k^1, E_k^2, E_{k+1}^1, E_{k+1}^2, E_{k+2}^1 and E_{k+2}^1.

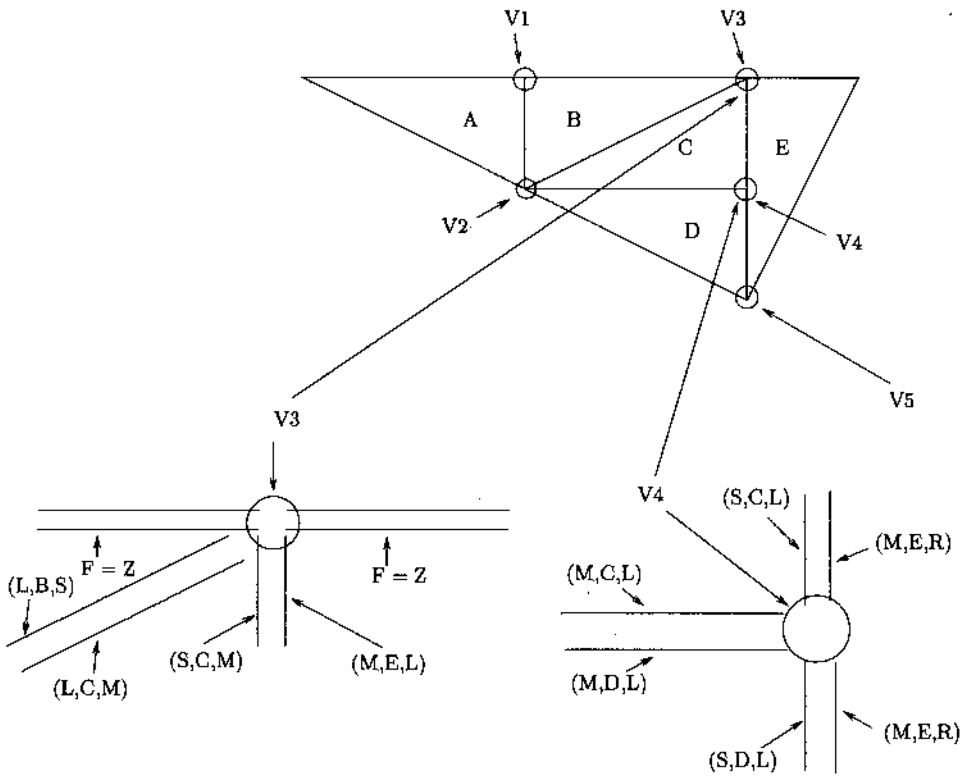

FIGURE 12B

(V3) If a vertex of a prototile satisfies any of the following conditions: E_k^2 contains $(J, N, F) = (\mathrm{L}, \mathrm{B}, \mathrm{S})$; E_k^1 contains $(J, N, F) = (\mathrm{L}, \mathrm{C}, \mathrm{M})$; E_{k-1}^2 contains $(J, N, F) = (\mathrm{S}, \mathrm{C}, \mathrm{M})$; E_{k-1}^1 contains $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{L})$, then it satisfies the others, and also: $F = \mathrm{Z}$ in $E_{k-1}^2[-\pi/2]$ and in E_{k+1}^1 ; P has the same value in E_k^2, E_k^1, E_{k-1}^2 and E_{k-1}^1.

(V4) If a vertex of a prototile satisfies any of the following conditions: E_k^1 contains $(J, N, F) = (\mathrm{S}, \mathrm{C}, \mathrm{L})$; $E_k^2[-\pi/2]$ contains $(J, N, F) = (\mathrm{M}, \mathrm{C}, \mathrm{L})$; $E_k^1[-\pi/2]$ contains $(J, N, F) = (\mathrm{M}, \mathrm{D}, \mathrm{L})$; $E_k^2[-\pi]$ contains $(J, N, F) = (\mathrm{S}, \mathrm{D}, \mathrm{L})$, then it satisfies the others, and also: E_k^2 contains $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{R})$; $E_k^1[-\pi]$ contains $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{R})$; P has the same value in E_k^2, E_k^1, $E_k^2[-\pi/2]$, $E_k^1[-\pi/2]$, $E_k^2[-\pi]$ and $E_k^1[-\pi]$.

(V5) If a vertex of a prototile satisfies either of the following conditions: E_1^1 contains $(J, N, F) = (\mathrm{S}, \mathrm{D}, \mathrm{M})$, or E_1^2 contains $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{S})$, then it satisfies the other and: P has the same value in E_1^1 and E_1^2; the quotient of $H_m^{1\mathrm{M}}$ by $G_m^{\varepsilon 1\mathrm{M}}$ in E_1^1 equals $G_m^{\varepsilon 1\mathrm{M}}$ in E_2^1 if $P = m$ in E_1^1; the quotient of $H_m^{1\mathrm{S}}$ by $G_m^{\varepsilon 1\mathrm{S}}$ in E_1^1 equals $G_m^{\varepsilon 1\mathrm{S}}$ in E_e^2 if $P = m$ in E_1^1. Also, if, in E_1^1, P has value B, then $F = \mathrm{Z}$ in E_e^2, and $(J, N, F) = (\mathrm{S}, \mathrm{B}, \mathrm{L})$ in E_2^1; and if P has

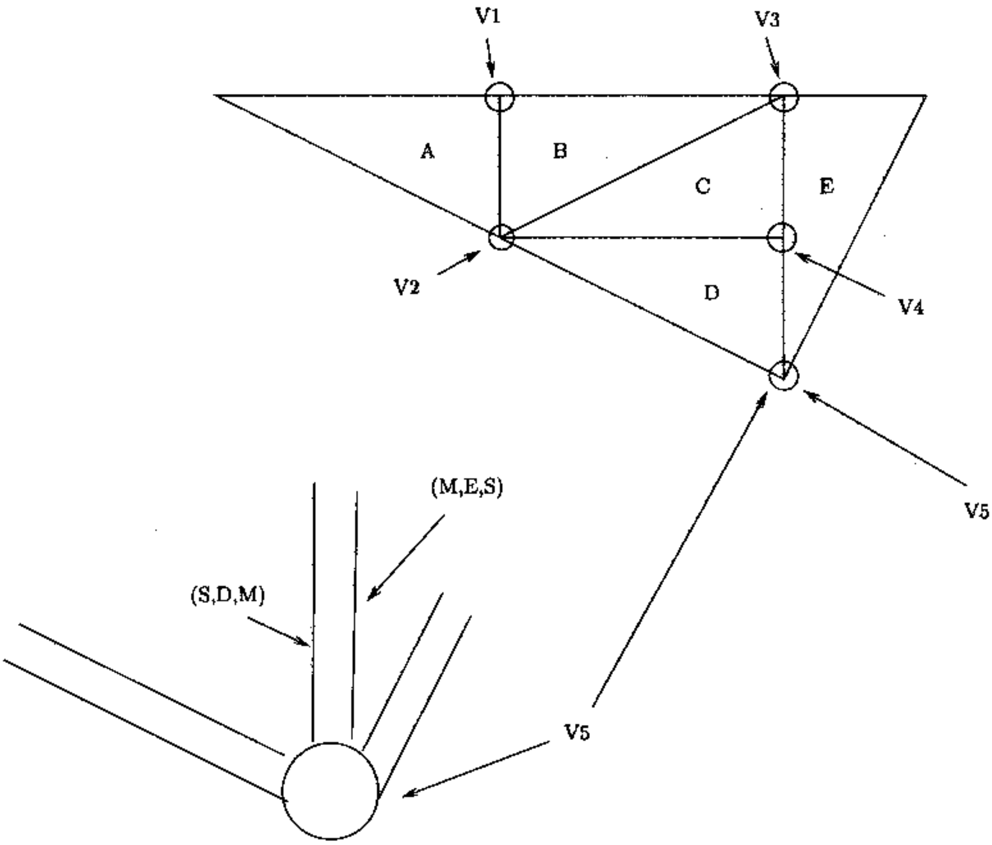

FIGURE 12C

value C, then $(J, N, F) = (\mathrm{M}, \mathrm{C}, \mathrm{L})$ in E_e^2 and $(J, N, F) = (\mathrm{S}, \mathrm{C}, \mathrm{L})$ in E_2^1; if P has value A, then $(J, N, F) = (\mathrm{S}, \mathrm{A}, \mathrm{L})$ in E_2^1, and $F = \mathrm{Z}$ in E_e^2; D, then $(J, N, F) = (\mathrm{M}, \mathrm{D}, \mathrm{L})$ in E_e^2, and $(J, N, F) = (\mathrm{S}, \mathrm{D}, \mathrm{L})$ in E_2^1; if P has value E, then $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{L})$ in E_e^2, and $F = \mathrm{Z}$ in E_2^1 and in $E_e^2[+\pi]$.

(V6) Consider a prototile T, with vertices V_1, V_2 and V_3, such that E_1^1 of V_1 and E_e^2 of V_2 refer to the same edge of T. If in V_1 either of the following conditions is satisfied: E_1^1 contains $(J, N, F) = (\mathrm{S}, \mathrm{A}, \mathrm{L})$, or E_1^2 contains $(J, N, F) = (\mathrm{S}, \mathrm{B}, \mathrm{L})$, then so is the other, and also: $F = \mathrm{Z}$ in $E_1^1[+\pi/2]$ and $E_1^2[-\pi/2]$; P has the same value in E_1^1 and E_1^2; the quotient of $H_m^{\alpha 2\mathrm{M}}$ by $G_m^{\alpha 2\mathrm{M}}$ in E_e^2 of V_2 equals $G_m^{\alpha 2\mathrm{M}}$ in $E_1^1[+\pi/2]$ of V_1 if $P = m$ in E_1^1 of V_1; and the quotient of $H_m^{\alpha 2\mathrm{S}}$ by $G_m^{\alpha 2\mathrm{S}}$ in E_e^2 of V_2 equals $G_m^{\alpha 2\mathrm{S}}$ in $E_1^2[-\pi/2]$ of V_1 if $P = m$ in E_1^1. And if in V_1 we have, in E_1^1, that P has value B, then $(J, N) = (\mathrm{L}, \mathrm{B})$ in $E_1^2[-\pi/2]$ and $E_1^1[+\pi/2]$; if P has value C, then $(J, N) = (\mathrm{L}, \mathrm{C})$ in $E_1^2[-\pi/2]$ and $E_1^1[+\pi/2]$.

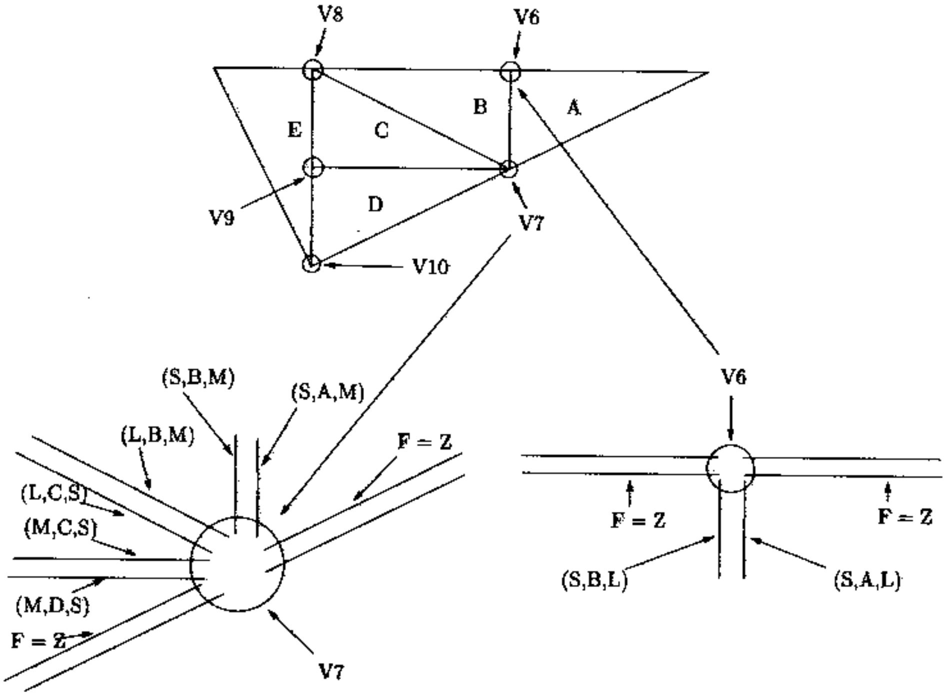

FIGURE 12D

(V7) If a vertex of a prototile satisfies any of the following conditions: E_k^2 contains $(J, N, F) = (\mathrm{S}, \mathrm{A}, \mathrm{M})$; E_k^1 contains $(J, N, F) = (\mathrm{S}, \mathrm{B}, \mathrm{M})$; E_{k-1}^2 contains $(J, N, F) = (\mathrm{L}, \mathrm{B}, \mathrm{M})$; E_{k-1}^1 contains $(J, N, F) = (\mathrm{L}, \mathrm{C}, \mathrm{S})$; E_{k-2}^2 contains $(J, N, F) = (\mathrm{M}, \mathrm{C}, \mathrm{S})$; E_{k-2}^1 contains $(J, N, F) = (\mathrm{M}, \mathrm{D}, \mathrm{S})$, then it satisfies the others, and also: $F = \mathrm{Z}$ in E_{k+1}^1 and E_{k-3}^2; P has the same value in E_k^1, E_k^2, E_{k+1}^1, E_{k+1}^2, E_{k+2}^1 and E_{k+2}^1.

(V8) If a vertex of a prototile satisfies any of the following conditions: E_k^1 contains $(J, N, F) = (\mathrm{L}, \mathrm{B}, \mathrm{S})$; E_k^2 contains $(J, N, F) = (\mathrm{L}, \mathrm{C}, \mathrm{M})$; E_{k+1}^1 contains $(J, N, F) = (\mathrm{S}, \mathrm{C}, \mathrm{M})$; E_{k+1}^2 contains $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{L})$, then it satisfies the others, and also: $F = \mathrm{Z}$ in $E_{k+1}^1[+\pi/2]$ and E_{k-1}^2; P has the same value in E_k^1, E_k^2, E_{k+1}^1 and E_{k+1}^2.

(V9) If a vertex of a prototile satisfies any of the following conditions: E_k^2 contains $(J, N, F) = (\mathrm{S}, \mathrm{C}, \mathrm{L})$; $E_k^1[+\pi/2]$ contains $(J, N, F) = (\mathrm{M}, \mathrm{C}, \mathrm{L})$; $E_k^2[+\pi/2]$ contains $(J, N, F) = (\mathrm{M}, \mathrm{D}, \mathrm{L})$; $E_k^1[+\pi]$ contains $(J, N, F) = (\mathrm{S}, \mathrm{D}, \mathrm{L})$, then it satisfies the others, and also: E_k^1 contains $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{R})$; $E_k^2[+\pi]$ contains $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{R})$; P has the same value in E_k^2, E_k^1, $E_k^2[+\pi/2]$, $E_k^1[+\pi/2]$, $E_k^2[+\pi]$ and $E_k^1[+\pi]$.

(V10) If a vertex of a prototile satisfies either of the following conditions: E_e^2 contains $(J, N, F) = (\mathrm{S}, \mathrm{D}, \mathrm{M})$, or E_e^1 contains $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{S})$; then

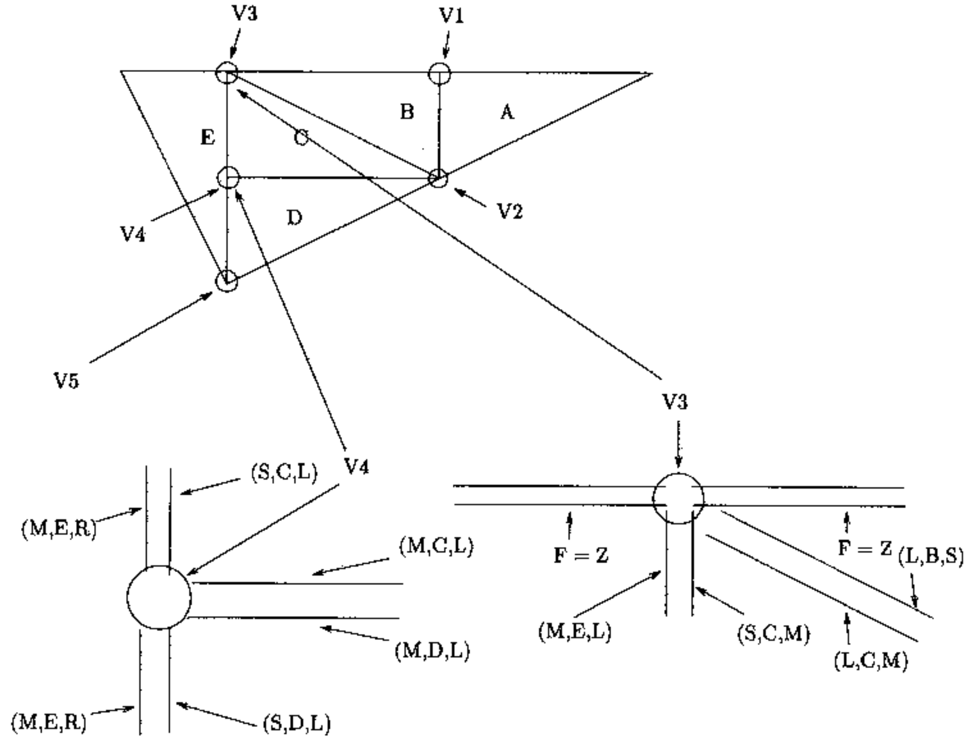

FIGURE 12E

it satisfies the other, and: P has the same value in E_e^2 and E_e^1; the quotient of $H_m^{2\mathrm{M}}$ by $G_m^{\varepsilon 2\mathrm{M}}$ in E_e^2 equals $G_m^{\varepsilon 2\mathrm{M}}$ in E_{e-1}^2 if $P = m$ in E_e^2; and the quotient of $H_m^{2\mathrm{S}}$ by $G_m^{\varepsilon 2\mathrm{S}}$ in E_e^2 equals $G_m^{\varepsilon 2\mathrm{S}}$ in E_1^1 if $P = m$ in E_e^2. Also if, in E_e^2, P has value: A, then $(J, N, F) = (\mathrm{S}, \mathrm{A}, \mathrm{L})$ in E_{e-1}^2, and $F = \mathrm{Z}$ in E_1^1; D, then $(J, N, F) = (\mathrm{M}, \mathrm{D}, \mathrm{L})$ in E_1^1 and $(J, N, F) = (\mathrm{S}, \mathrm{D}, \mathrm{L})$ in E_{e-1}^2; B, then $(J, N, F) = (\mathrm{S}, \mathrm{B}, \mathrm{L})$ in E_{e-1}^2, and $F = \mathrm{Z}$ in E_1^1; C, then $(J, N, F) = (\mathrm{M}, \mathrm{C}, \mathrm{L})$ in E_1^1 and $(J, N, F) = (\mathrm{S}, \mathrm{C}, \mathrm{L})$ in E_{e-1}^2; E, then $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{L})$ in E_1^1, and $F = \mathrm{Z}$ in E_{e-1}^2 and in $E_{e-1}^1[+\pi]$.

Finally we make some restrictions which follow easily from examining Figures 3, 13, 14 and 15; R9 and R10 are less obvious, and we will analyze them carefully.

(R1) In any E_k^j, $(J, N) \neq (\mathrm{M}, \mathrm{A}), (\mathrm{L}, \mathrm{A}), (\mathrm{M}, \mathrm{B}), (\mathrm{L}, \mathrm{D}), (\mathrm{S}, \mathrm{E}), (\mathrm{L}, \mathrm{E})$, and $F \neq J$.

(R2) In any V_k, $\sum_{j=1}^{e} Size(A_j) = 2\pi$.

(R3) A_1 must have the value of the vertex of the μ component to which it belongs.

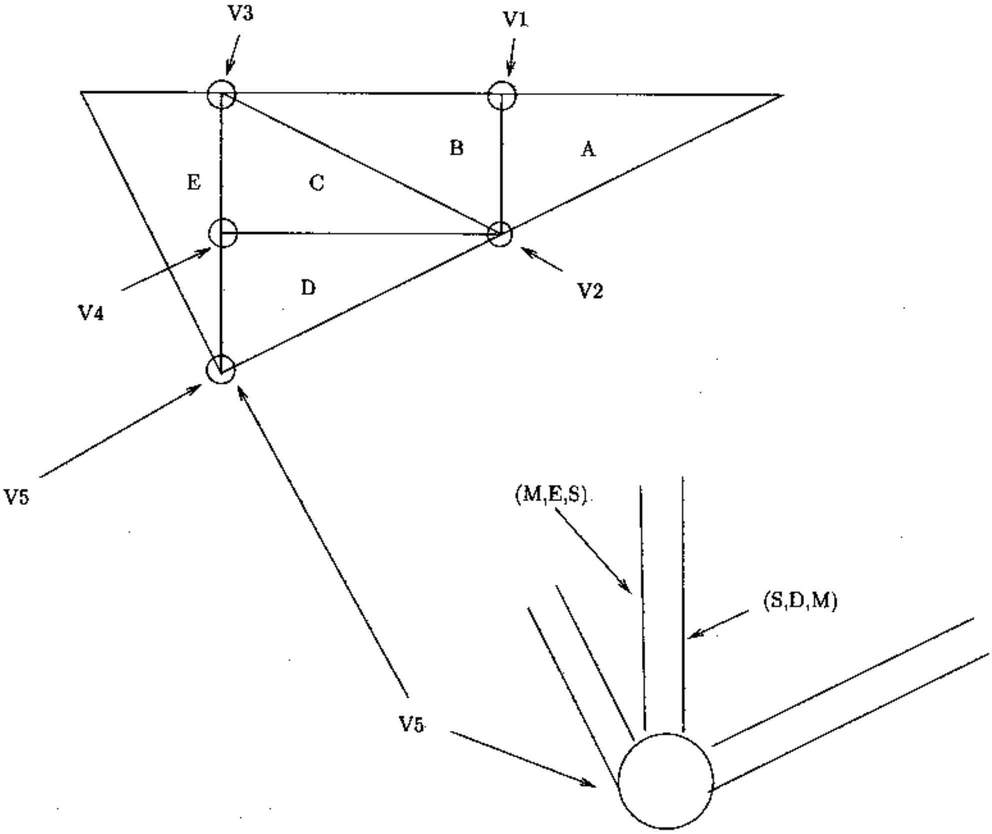

FIGURE 12F

(R4) If E_n^m and E_q^p belong to two different vertices and refer to the same edge of a prototile, then J (resp. N, resp. P) must have the same value in E_n^m as in E_q^p.

(R5) In any vertex of a prototile, if one of a pair E_k^1, E_k^2 of edge-marks has (J, N) values: (i) (S, A), then for the other $(J, N) = (\mathrm{S}, \mathrm{B})$, and then one has $F = \mathrm{L}$ (resp. $F = \mathrm{Z}$ or R , resp. $F = \mathrm{M}$) if and only if the other has $F = \mathrm{L}$ (resp. $F = \mathrm{Z}$ or R, resp. $F = \mathrm{M}$); (ii) (S, B), then for the other $(J, N) = (\mathrm{S}, \mathrm{A})$; (iii) (L, B), then for the other $(J, N) = (\mathrm{L}, \mathrm{C})$, and then one has $F = \mathrm{S}$ (resp. $F = \mathrm{Z}$ or R) if and only if the other has $F = \mathrm{M}$ (resp. $F = \mathrm{Z}$ or R); (iv) (L, C), then for the other $(J, N) = (\mathrm{L}, \mathrm{B})$; (v) (M, C), then for the other $(J, N) = (\mathrm{M}, \mathrm{D})$, and then one has $F = \mathrm{L}$ (resp. $F = \mathrm{S}$, resp. $F = \mathrm{Z}$ or R) if and only if the other has $F = \mathrm{L}$ (resp. $F = \mathrm{S}$, resp. $F = \mathrm{Z}$ or R); (vi) (M, D), then for the other $(J, N) = (\mathrm{M}, \mathrm{C})$; (vii) (S, C), then for the other $(J, N) = (\mathrm{M}, \mathrm{E})$, and then the former has $F = \mathrm{M}$ if and only if the other has $F = \mathrm{L}$, and if the former has $F = \mathrm{L}$ (resp. $F = \mathrm{Z}$ or R) then the other has $F = \mathrm{Z}$ or R (resp. $F = \mathrm{Z}$ or R); (viii) (S, D), then for the other

$(J,N) = (\mathrm{M},\mathrm{E})$, and then the former has $F = \mathrm{M}$ if and only if the other has $F = \mathrm{S}$, and if the former has $F = \mathrm{L}$ (resp. $F = \mathrm{Z}$ or R) then the other has $F = \mathrm{Z}$ or R (resp. $F = \mathrm{Z}$ or R); (ix) (M,E), then for the other $(J,N) = (\mathrm{S},\mathrm{C})$ or (S,D), and if the former has $F = \mathrm{Z}$ or R then the other has $F = \mathrm{Z}$ or R or L.

(R6) If $F = \mathrm{S}$ (resp. M) in E_k^1 then $A_k = \mathrm{S}$ (resp. M). If $F = \mathrm{S}$ (resp. M) in E_k^2 then $A_{k+1} = \mathrm{S}$ (resp. M).

(R7) Assume for an edge-mark E_k^1 (resp. E_k^2) of any V_m that $F = \mathrm{R}$. Then there is an edge-mark $E_k^2[+\pi]$ (resp. $E_k^1[+\pi]$) of V_m, and $F = \mathrm{R}$ in that edge too. The value $F = \mathrm{R}$ may not appear in any edge-mark numbered E_1^1 or E_e^2. Also, the values of J are the same in E_k^1 and $E_k^2[+\pi]$ (resp. E_k^2 and $E_k^1[+\pi]$), the values of N are the same in E_k^1 and $E_k^2[+\pi]$ (resp. E_k^2 and $E_k^1[+\pi]$), and the values of P are the same in E_k^1 and $E_k^2[+\pi]$ (resp. E_k^2 and $E_k^1[+\pi]$). Finally, it cannot happen that $F = \mathrm{R}$ for another edge-mark E_q^p of V_m.

(R8) Assume for an edge-mark E_k^1 (resp. E_k^2) of any V_m that $F = \mathrm{Z}$. Then there is an edge-mark $E_k^2[+\pi]$ (resp. $E_k^1[+\pi]$) of V_m, and $F = \mathrm{Z}$ in that edge too. Also, the values of J are the same in E_k^1 and $E_k^2[+\pi]$ (resp. E_k^2 and $E_k^1[+\pi]$), the values of N are the same in E_k^1 and $E_k^2[+\pi]$ (resp. E_k^2 and $E_k^1[+\pi]$), and the values of P are the same in E_k^1 and $E_k^2[+\pi]$ (resp. E_k^2 and $E_k^1[+\pi]$). Finally, it can only happen that $F = \mathrm{Z}$ for another edge-mark E_q^p of V_m if E_q^p has angle zero with one of the above pair.

(R9) Let V_1, V_2 and V_3 be the three vertices of a prototile, with E_e^2 (resp. E_1^1) of V_1 and E_1^1 (resp. E_e^2) of V_2 both referring to the same tile edge, and $A_1 = \mathrm{S}$ in μ_2. See Figure 13. If $G_m^{\epsilon 1\mathrm{S}}$ in E_e^2 (resp. $G_m^{\alpha 2\mathrm{S}}$ in E_1^1) of V_1 equals $G_m^{\alpha 1\mathrm{S}}$ in E_1^1 (resp. $G_m^{\epsilon 2\mathrm{S}}$ in E_e^2) of V_3, then in V_2: (i) if $m = \mathrm{D}$, and $(J,N) = (\mathrm{M},\mathrm{D})$ in E_1^1 (resp. E_e^2), then $F = \mathrm{S}$ in E_1^1 (resp. E_e^2); (ii) if $m = \mathrm{E}$, and $(J,N) = (\mathrm{M},\mathrm{E})$ in E_1^1 (resp. E_e^2), then $F = \mathrm{S}$ in E_1^1 (resp. E_e^2); (iii) if $m = \mathrm{B}$, and $(J,N) = (\mathrm{L},\mathrm{B})$ in E_e^2 (resp. E_1^1), then $F = \mathrm{S}$ in E_e^2 (resp. E_1^1).

(R10) Let V_1, V_2 and V_3 be the three vertices of a prototile, with E_e^2 (resp. E_1^1) of V_1 and E_1^1 (resp. E_e^2) of V_2 both referring to the same physical edge, and $A_1 = \mathrm{M}$ in μ_2. See Figure 13. If $G_m^{\alpha 1\mathrm{M}}$ in E_e^2 (resp. $G_m^{\alpha 2\mathrm{M}}$ in E_1^1) of V_1, equals $G_m^{\epsilon 1\mathrm{M}}$ in E_1^1 (resp. $G_m^{\epsilon 2\mathrm{M}}$ in E_e^2) of V_3, then in V_2: (i) if $m = \mathrm{D}$, and $(J,N) = (\mathrm{S},\mathrm{D})$ in E_e^2 (resp. E_e^2), then $F = \mathrm{M}$ in E_e^2 (resp. E_e^2); (ii) if $m = \mathrm{A}$, and $(J,N) = (\mathrm{S},\mathrm{A})$ in E_e^2 (resp. E_e^2), then $F = \mathrm{M}$ in E_e^2 (resp. E_e^2).

(R11) Let V_1, V_2 and V_3 be the three vertices of a prototile, with E_e^2 (resp. E_1^1) of V_1 and E_1^1 (resp. E_e^2) of V_2 both referring to the same physical edge, and $A_1 = \mathrm{M}$ in μ_2. See Figure 14. If $(J,N) = (\mathrm{L},\mathrm{B})$ in E_e^2 (resp. E_1^1)

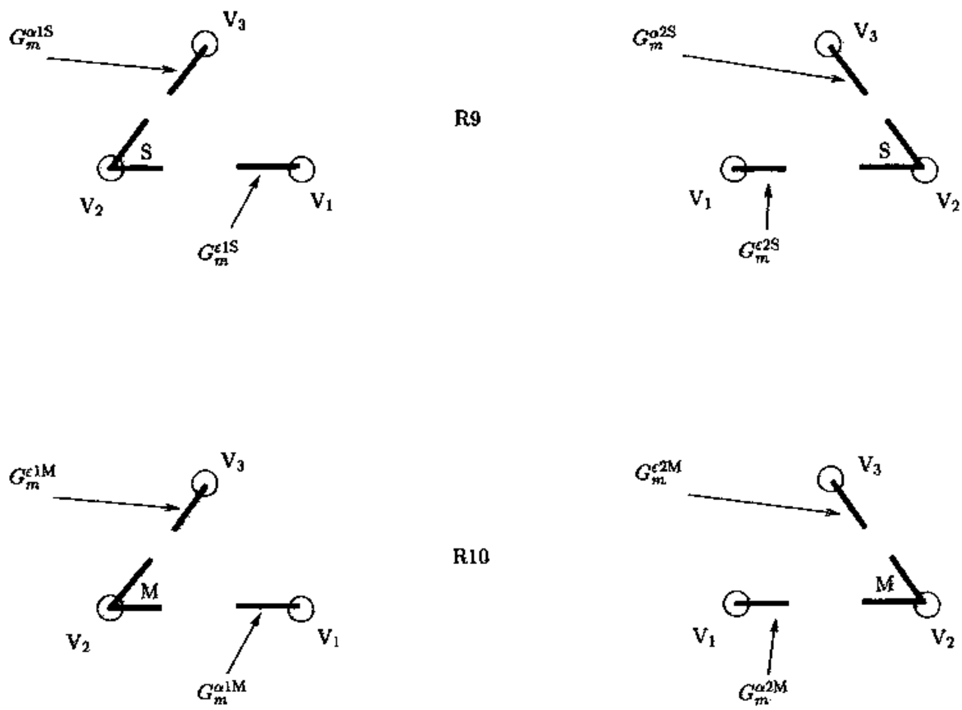

FIGURE 13

of V_1, and $(J,N) = (S,B)$ in E_1^1 (resp. E_e^2) of V_3, then in V_2 we have $F = M$ in both E_1^1 and E_e^2.

(R12) Let V_1, V_2 and V_3 be the three vertices of a prototile, with E_e^2 (resp. E_1^1) of V_1 and E_1^1 (resp. E_e^2) of V_2 both referring to the same physical edge, and $A_1 = M$ in μ_2. See Figure 14. If $(J,N) = (L,C)$ in E_e^2 (resp. E_1^1) of V_1, and $(J,N) = (S,C)$ in E_1^1 (resp. E_e^2) of V_3, then in V_2 we have $F = M$ in both E_1^1 and E_e^2.

(R13) Let V_1, V_2 and V_3 be the three vertices of a prototile, with E_e^2 (resp. E_1^1) of V_1 and E_1^1 (resp. E_e^2) of V_2 both referring to the same physical edge, and $A_1 = M$ in μ_2. See Figure 14. If $(J,N) = (L,C)$ in E_e^2 (resp. E_1^1) of V_1, and $(J,N) = (M,C)$ in E_1^1 (resp. E_e^2) of V_3, then in V_2 we have $F = S$ in both E_1^1 and E_e^2.

(R14) If in two of the vertices of a prototile there are F values indicating that the edge between the vertices ends at the vertices, then the corresponding N value must agree with the class of prototile that it is.

(R15) Let X = A or D. Assume we have a prototile T, with vertices V_1, V_2 and V_3, and that E_1^1 (resp. E_e^2) of V_2 and E_e^2 (resp. E_1^1) of V_1 refer to the same edge of T. Then if $F = Z$ in E_1^1 (resp. E_e^2) of V_2, G_X^{jkM} has

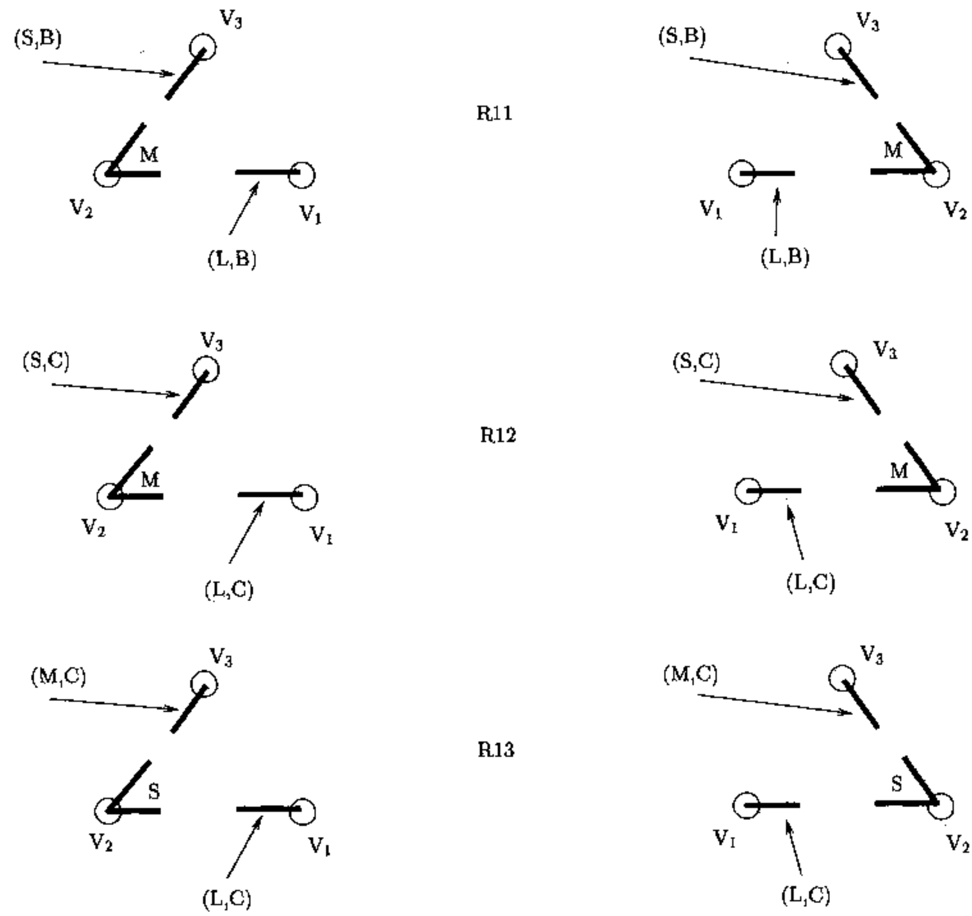

FIGURE 14

the same value in E_e^2 (resp. E_1^1) of V$_1$ as it has in $E_1^2[-\pi]$ (resp. $E_e^1[+\pi]$) of V$_2$, unless V$_2$ has an edge-mark $E_1^2[-\pi/2]$ (resp. $E_e^1[+\pi/2]$) which contains the values $(J, N, F, P) = (\mathrm{S}, \mathrm{A}, \mathrm{L}, \mathrm{A})$ or $(\mathrm{S}, \mathrm{A}, \mathrm{L}, \mathrm{D})$, in which case the two G values have opposite sign. If $F \neq \mathrm{Z}$ in E_1^1 (resp. E_e^2) of V$_2$, and we do not have: (i) $k = 2$ (resp. $k = 1$) and $(J, N, F) = (\mathrm{S}, \mathrm{X}, \mathrm{L})$ in E_1^1 (resp. E_e^2) of V$_2$, then $G_X^{\epsilon k \mathrm{M}} = +1$ in that edge-mark; (ii) $k = 1$ (resp. $k = 2$) and $(J, N, F) = (\mathrm{S}, \mathrm{A}, \mathrm{L})$ or (S, D, L) in E_1^1 (resp. E_e^2) of V$_2$, then $G_X^{\alpha k \mathrm{M}} = +1$ in that edge-mark. See M1.

(R16) Let $X = \mathrm{D}$ or E. Assume we have a prototile T, with vertices V$_1$, V$_2$ and V$_3$, and that E_1^1 (resp. E_e^2) of V$_2$ and E_e^2 (resp. E_1^1) of V$_1$ refer to the same edge of T. If $F = \mathrm{Z}$ in E_1^1 (resp. E_e^2) of V$_2$, then $G_X^{jk\mathrm{S}}$ has the same value in E_e^2 (resp. E_1^1) of V$_1$ as it has in $E_1^2[-\pi]$ (resp. $E_e^1[+\pi]$) of V$_2$, unless: (i) V$_2$ has an edge-mark $E_1^2[+\pi/2]$ (resp. $E_e^1[-\pi/2]$) which contains the values $(J, N, F, P) = (\mathrm{S}, \mathrm{A}, \mathrm{L}, \mathrm{E})$ or $(\mathrm{S}, \mathrm{A}, \mathrm{L}, \mathrm{D})$; or (ii) V_2 has an edge-mark E_e^2 (resp. E_1^1) which contains the values $(J, N, F, P) = (\mathrm{S}, \mathrm{A}, \mathrm{M}, \mathrm{B})$, in which case the two G values have opposite sign. If $F \neq \mathrm{Z}$ in E_1^1 (resp. E_e^2) of V_2, and we do not have: (i) $k = 2$ (resp. $k = 1$) and $(J, N, F) = (\mathrm{M}, \mathrm{X}, \mathrm{L})$ in E_1^1 (resp. E_e^2) of V_2, then $G_X^{\epsilon k\mathrm{S}} = +1$ in that edge-mark; (ii) $k = 1$ (resp. $k = 2$) and $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{L})$ or $(\mathrm{M}, \mathrm{D}, \mathrm{L})$ in E_1^1 (resp. E_e^2) of V_2, then $G_X^{\alpha k\mathrm{S}} = +1$ in that edge-mark. If $F = \mathrm{Z}$ in E_1^1 (resp. E_e^2) of V_2 and we do not have $k = 1$ (resp. $k = 2$) and $(J, N, F) = (\mathrm{L}, \mathrm{B}, \mathrm{S})$ in E_e^2 (resp. E_1^1) of V_1, then $G_X^{\alpha k\mathrm{S}} = +1$ in E_1^1 (resp. E_e^2) of V_2. See M4.

(R17) Assume we have a prototile T, with vertices V_1, V_2 and V_3, and that E_1^1 (resp. E_e^2) of V_2 and E_e^2 (resp. E_1^1) of V_1 refer to the same edge of T. If $F = \mathrm{Z}$ in E_1^1 (resp. E_e^2) of V_2, then $G_B^{jk\mathrm{S}}$ has the same value in E_e^2 (resp. E_1^1) of V_1 as it has in $E_1^2[-\pi]$ (resp. $E_e^1[+\pi]$) of V_2, unless: (i) V_2 has an edge-mark $E_1^2[-\pi/2]$ (resp. $E_e^1[+\pi/2]$) which contains the values $(J, N, F, P) = (\mathrm{S}, \mathrm{A}, \mathrm{L}, \mathrm{E})$ or $(\mathrm{S}, \mathrm{A}, \mathrm{L}, \mathrm{D})$; or (ii) V_2 has an edge-mark E_e^2 (resp. E_1^1) which contains the values $(J, N, F, P) = (\mathrm{S}, \mathrm{A}, \mathrm{M}, \mathrm{B})$, in which case the two G values have opposite sign. If $k = 2$ (resp. $k = 1$) and we do not have $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{L})$ or $(\mathrm{M}, \mathrm{D}, \mathrm{L})$ in E_1^1 (resp. E_e^2) of V_2, then $G_B^{\alpha k\mathrm{S}} = +1$ in that edge-mark. If $k = 1$ (resp. $k = 2$) and we do not have $(J, N, F) = (\mathrm{L}, \mathrm{B}, \mathrm{S})$ in E_e^2 (resp. E_1^1) of V_1, then $G_B^{\alpha k\mathrm{S}} = G_B^{\epsilon k\mathrm{S}} = +1$ in E_1^1 (resp. E_e^2) of V_2. See M8.

(R18) $H_m^{kn} = G_m^{\epsilon kn}$ in an edge-mark if in that edge-mark $(J, N) \neq (\mathrm{S}, \mathrm{A}), (\mathrm{S}, \mathrm{D}), (\mathrm{M}, \mathrm{D})$, or if $(J, N, F) = (\mathrm{S}, \mathrm{A}, \mathrm{L}), (\mathrm{S}, \mathrm{D}, \mathrm{L})$ or $(\mathrm{M}, \mathrm{D}, \mathrm{S})$.

(R19) Assume we have a prototile T, with vertices V_1, V_2 and V_3, and that E_1^1 (resp. E_e^2) of V_2 and E_e^2 (resp. E_1^1) of V_1 refer to the same edge of T. If $F = \mathrm{Z}$ in E_1^1 (resp. E_e^2) of V_2, then the ratio of H_m^{kn} and $G_m^{\epsilon kn}$ is the same in $E_1^2[+\pi]$ (resp. $E_e^1[+\pi]$) of V_2 as it is E_e^2 (resp. E_1^1) of V_1. See Figure 15.

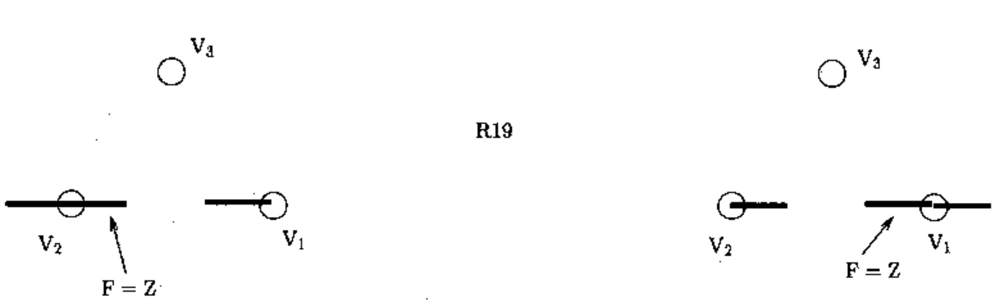

FIGURE 15

(R20) Consider any prototile with E_1^1 (resp. E_e^2) of V_2 and E_e^2 (resp. E_1^1) of V_1 referring to the same edge. If $F \neq \mathrm{Z}$ in E_1^1 (resp. E_e^2) of V_2, and it is not the case that: (i) $k = 2$ (resp. $k = 1$) and $(J, N, F) = (\mathrm{S}, \mathrm{X}, \mathrm{L})$ in E_1^1 (resp. E_e^2) of V_2, then $G_X^{\epsilon k\mathrm{M}} = +1$ in that edge-mark; (ii) $k = 1$ (resp. $k = 2$) and $(J, N, F) = (\mathrm{S}, \mathrm{A}, \mathrm{L})$ or (S, D, L) in E_1^1 (resp. E_e^2) of V_2, then $G_X^{\alpha k\mathrm{M}} = +1$ in that edge-mark (see M1); (iii) $k = 2$ (resp. $k = 1$) and $(J, N, F) = (\mathrm{M}, \mathrm{X}, \mathrm{L})$ in E_1^1 (resp. E_e^2) of V_2, then $G_X^{\epsilon k\mathrm{S}} = +1$ in that edge-mark; (iv) $k = 1$ (resp. $k = 2$) and $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{L})$ or $(\mathrm{M}, \mathrm{D}, \mathrm{L})$ or $(\mathrm{L}, \mathrm{B}, \mathrm{M})$ in E_1^1 (resp. E_e^2) of V_2, then $G_X^{\alpha k\mathrm{S}} = +1$ in that edge-mark (see M4); (v) $k = 2$ (resp. $k = 1$) and $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{L})$ or $(\mathrm{M}, \mathrm{D}, \mathrm{L})$ or $(\mathrm{L}, \mathrm{B}, \mathrm{M})$ in E_1^1 (resp. E_e^2) of V_2, then $G_X^{\epsilon k\mathrm{S}} = +1$ in E_1^1 (resp. E_e^2) of V_2; (vi) $k = 1$ (resp. $k = 2$) and $(J, N, F) = (\mathrm{L}, \mathrm{B}, \mathrm{M})$ in E_1^1 (resp. E_e^2) of V_2, then $G_X^{\alpha k\mathrm{S}} = +1$ in E_1^1 (resp. E_e^2) of V_2 (see M8).

(R21) Assume we have a prototile T, with vertices V_1, V_2 and V_3, and that E_1^1 (resp. E_e^2) of V_2 and E_e^2 (resp. E_1^1) of V_1 refer to the same edge of T. If $(J, N, F) = (\mathrm{M}, \mathrm{D}, \mathrm{L})$ in E_e^2 (resp. E_1^1) of V_1, then the ratio of H_m^{1n} and $G_m^{\epsilon 1n}$ in E_1^1 (resp. the ratio of H_m^{2n} and $G_m^{\epsilon 2n}$ in E_e^2) of V_2 equals the ratio of H_m^{1n} and $G_m^{\epsilon 1n}$ in $E_e^2[+\pi/2]$ (resp. the ratio of H_m^{2n} and $G_m^{\epsilon 2n}$ in $E_1^1[-\pi/2]$) of V_1.

(R22) Assume we have a prototile T, with vertices V_1, V_2 and V_3, and that E_1^1 (resp. E_e^2) of V_2 and E_e^2 (resp. E_1^1) of V_1 refer to the same edge of T. If $(J, N, F) = (\mathrm{M}, \mathrm{D}, \mathrm{S})$ in E_e^2 (resp. E_1^1) of V_1, then the ratio of H_m^{1n} and $G_m^{\epsilon 1n}$ in E_1^1 (resp. the ratio of H_m^{2n} and $G_m^{\epsilon 2n}$ in E_e^2) of V_2 equals the ratio of H_m^{1n} and $G_m^{\epsilon 1n}$ in $E_e^1[-\pi/2]$ (resp. the ratio of H_m^{2n} and $G_m^{\epsilon 2n}$ in $E_1^2[+\pi/2]$) of V_1. We must now define the tiles of our system and their "matching rules."

We define a "tile" as an (orientation-preserving) isometric image of one of the two unmarked prototiles, together with its marks. (We could define the tiles of class X2 to be reflections—with appropriate changes of marks—of those of class X1 if it was desired to minimize the number of prototiles.) In order to define "matching rules" for these tiles, we first define what it means for two tiles to be "neighbors." Two tiles are neighbors if their intersection consists of an edge of each. Also, a tile of type E will be called the neighbor of a tile of type C if they intersect precisely in the small edge of the C and half the medium edge of the E, with the medium vertex of the C and the large vertex of the E coinciding. A tile of type E is a neighbor of a tile of type D if they intersect precisely in the small edge of the D and half the medium edge of the E, with the medium vertex of the D and the small vertex of the E coinciding. A tile of type E is a neighbor of a tile of type A if they intersect precisely in the small edge of the E and half the medium edge of the A, with the small vertex of the A and the medium vertex of the E coinciding. A tile of type A is a neighbor of a tile of type B if they intersect precisely in half the medium edges of each, with the small vertex of the B in the middle of the edge of the A, and the large vertex of the A in the middle of the edge of the B. And finally, a tile of type B is a neighbor of a tile of type B if they intersect precisely in half the medium edges of each, with the large vertex of each B in the middle of the edge of the other B. See Figure 16. By the data above,

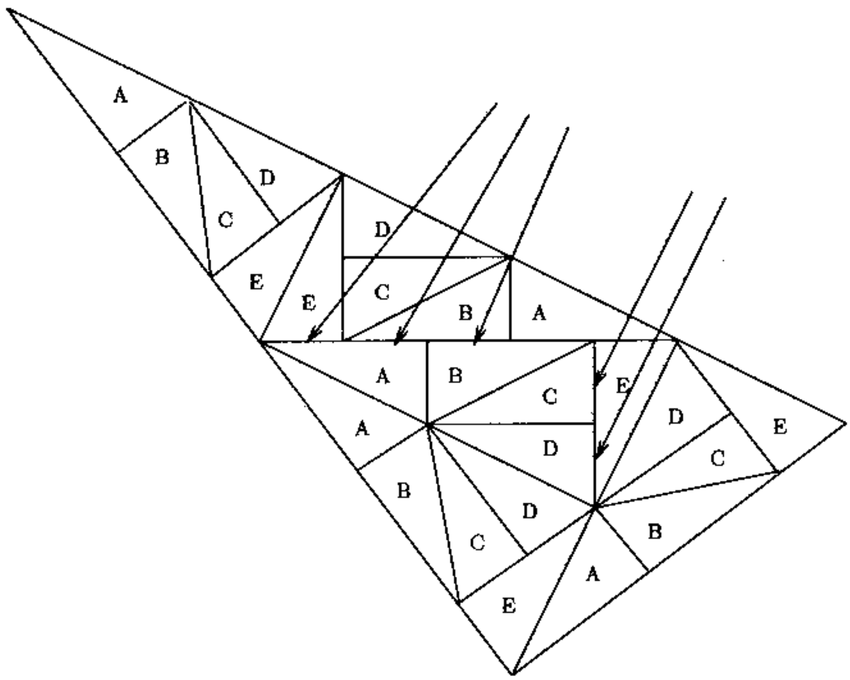

FIGURE 16

the matching rules for tiles are: In a tiling, two tiles T_m and T_n may abut if they are neighbors, and whenever a vertex V_m of T_m coincides with a vertex V_n of T_n, and T_n is in the positive direction from T_m about the common vertex, then E_k^j of V_n equals E_{k+1}^j of V_m, and A_k of V_n equals A_{k+1} of V_m, where $e+1 \equiv 1$.

4. Hierarchical structure in the tilings

We define a triangle of level 0 (note the distinction from "Triangle" and "Level") as a tile; its class and type are those of the tile. We inductively define a triangle of level $n \geq 1$ as an "appropriate collection" (defined below) of five triangles of level $n-1$ together with certain conditions on values of F, N and J as noted below.

An appropriate collection of five triangles of level n is the image, by a single isometry of the plane, of: (i) a triangle of type A with vertices $5^{n/2}(-2,1)$,

$5^{n/2}(0,1)$ and $(0,0)$; (ii) a triangle of type B with vertices $5^{n/2}(0,1)$, $(0,0)$ and $5^{n/2}(2,1)$; (iii) a triangle of type C with vertices $(0,0)$, $5^{n/2}(2,0)$ and $5^{n/2}(2,1)$; (iv) a triangle of type D with vertices $(0,0)$, $5^{n/2}(2,0)$ and $5^{n/2}(2,1)$; and (v) a triangle of type E with vertices $5^{n/2}(2,1)$, $5^{n/2}(3,1)$ and $5^{n/2}(2,-1)$. For any such collection we carry over the definition we used for Triangles of tiles T_S, T_M and T_L, and vertices V_{SS}, V_{SM} etc.

To be a triangle of level $n \geq 1$ we require, moreover, $F = Z$ in all edge-marks referring to tile edges which coincide in an interval with one of the three edges of the collection, except possibly for edge-marks in those vertices coinciding with the three vertices of the collection, which however must satisfy the following. The large triangle must belong to one of these "classes" (see Figures 17A,B), which from R8 are mutually exclusive:

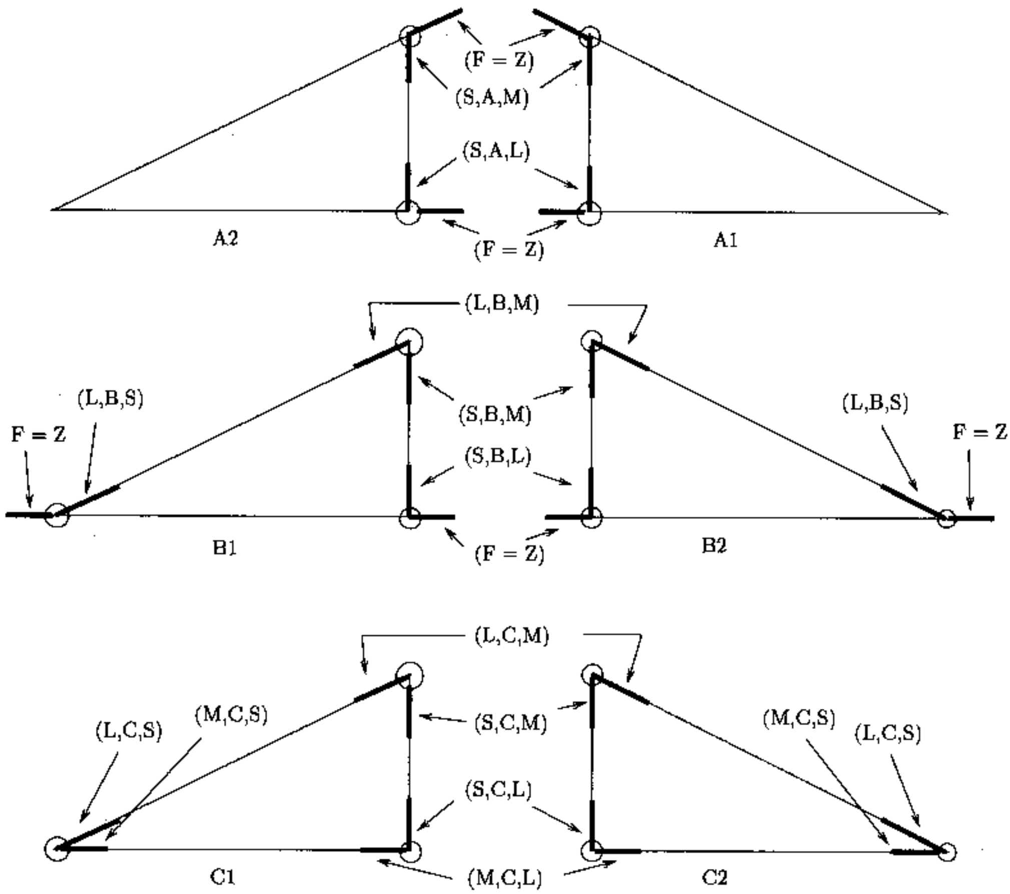

FIGURE 17A

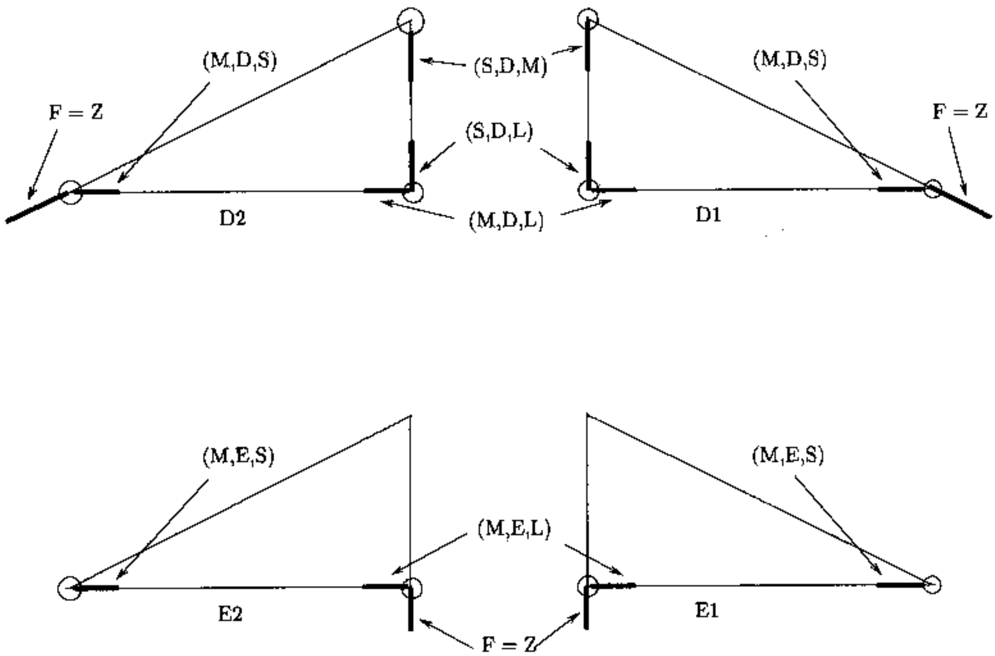

FIGURE 17B

(A1) (resp. A2) In V_{MM}, E_1^1 (resp. E_e^2) contains $(J, N, F) = (\mathrm{S}, \mathrm{A}, \mathrm{M})$, and E_e^2 (resp. E_1^1) contains $F = \mathrm{Z}$, and in V_{SS}, E_e^2 (resp. E_1^1) contains $(J, N, F) = (\mathrm{S}, \mathrm{A}, \mathrm{L})$, and E_2^1 (resp. E_{e-1}^1) contains $F = \mathrm{Z}$.

(B1) (resp. B2) In V_{MM}, E_e^2 (resp. E_1^1) contains $(J, N, F) = (\mathrm{S}, \mathrm{B}, \mathrm{M})$, and E_1^1 (resp. E_e^2) contains $(J, N, F) = (\mathrm{L}, \mathrm{B}, \mathrm{M})$, and in V_{SS}, E_1^1 (resp. E_e^2) contains $(J, N, F) = (\mathrm{S}, \mathrm{B}, \mathrm{L})$, and E_{e-1}^2 (resp. E_2^1) contains $F = \mathrm{Z}$, and in V_{SS}, E_e^2 (resp. E_1^1) contains $(J, N, F) = (\mathrm{L}, \mathrm{B}, \mathrm{S})$, and E_1^1 (resp. E_e^2) contains $F = \mathrm{Z}$.

(C1) (resp. C2) In V_{MM}, E_e^2 (resp. E_1^1) contains $(J, N, F) = (\mathrm{S}, \mathrm{C}, \mathrm{M})$, and E_1^1 (resp. E_e^2) contains $(J, N, F) = (\mathrm{L}, \mathrm{C}, \mathrm{M})$, and in V_{SS}, E_1^1 (resp. E_e^2) contains $(J, N, F) = (\mathrm{S}, \mathrm{C}, \mathrm{L})$, and E_{e-1}^2 (resp. E_2^1) contains $(J, N, F) = (\mathrm{M}, \mathrm{C}, \mathrm{L})$, and in V_{SS}, E_e^2 (resp. E_1^1) contains $(J, N, F) = (\mathrm{L}, \mathrm{C}, \mathrm{S})$, and E_1^1 (resp. E_e^2) contains $(J, N, F) = (\mathrm{M}, \mathrm{C}, \mathrm{S})$.

(D1) (resp. D2) In V_{MM}, E_1^1 (resp. E_e^2) contains $(J, N, F) = (\mathrm{S}, \mathrm{D}, \mathrm{M})$, and in V_{SS}, E_e^2 (resp. E_1^1) contains $(J, N, F) = (\mathrm{S}, \mathrm{D}, \mathrm{L})$, and E_2^1 (resp. E_{e-1}^2) contains $(J, N, F) = (\mathrm{M}, \mathrm{D}, \mathrm{L})$, and in V_{SS}, E_1^1 (resp. E_e^2) contains $F = \mathrm{Z}$, and E_e^2 (resp. E_1^1) contains $(J, N, F) = (\mathrm{M}, \mathrm{D}, \mathrm{S})$.

(E1) (resp. E2) In V_{SS}, E_e^2 (resp. E_1^1) contains $F = \mathrm{Z}$, and E_2^1 (resp. E_{e-1}^2) contains $(J, N, F) = (\mathrm{M}, \mathrm{E}, \mathrm{L})$, and in V_{SS}, E_e^2 (resp. E_1^1) contains $(J, N, F) =$

(M, E, S). A triangle of class Xj and level $n \geq 1$, defined above, is said to be of type X.

In summary, a triangle of level $n \geq 1$ is defined inductively as a collection of five triangles of level $n - 1$ in one of two possible specified geometric relationships, satisfying one of ten (mutually exclusive) sets of conditions on values of F, N and J in those edge-marks referring to the edges of the collection; $F = \mathrm{Z}$ at all interior vertices, and (F, N, J) exhibit one of ten specified, mutually exclusive, patterns at the three vertices of the collection. Note (using V1 $\cdots$ V10) that triangles of level 0 (namely tiles) satisfy all the properties of triangles of level n, for $n = 0$.

5. Results

THEOREM 5.1. *If in the substitution tesselation Θ of the plane the Tiles are considered as tiles, there is a tiling of the plane with all matching rules satisfied.*

Proof. We need to prove that the marks on the Tiles satisfy the restrictions for marks of tiles, A1 $\cdots$ E2, V1 $\cdots$ V10, and R1 $\cdots$ R22, and that the neighbor rules are satisfied.

For the most part, the restrictions on tiles and the definitions of "neighbor" are constructed so as to conform obviously with the tesselation Θ. We first show that abutting Tiles occur in Θ only in ways allowed by the neighbor rules.

Our claim is obviously true for Level 1 Triangles. New types of neighbor combinations may appear as a result of some Triangle of a certain Level n being divided in five parts according to our procedure; two abutting Tiles in a Level n Triangle, when subdivided, may create along their common edge something new at Level $n + 1$. Consequently, we just have to verify that any two Tiles, neighbors according to the rules, give rise only to such neighboring Tiles after being divided. We now examine four cases.

If the Tiles match (full) edge-to-edge, with equal angles at coinciding vertices, then they produce only edge-to-edge matchings after the subdivision.

If the Tiles match edge-to-edge, with unequal angles at coinciding vertices (this happens in the tesselation Θ only for L edges, but this is irrelevant to our argument), then they produce edge-to-edge matchings unless they border along L edges. In this case they produce the A-to-A and A-to-B matchings on half the medium edge and the A-to-E small edge to half medium edge matching (see Figure 3).

All matchings of an S edge to half an M edge give rise to edge-to-edge matchings after a subdivision (see Figure 3).

All half medium edge to half medium edge matchings give rise to edge-to-edge matchings after a subdivision (see Figure 3). This completes our argument concerning the neighbor rules.

The only restrictions we consider are R9, R10, and those parts of V1 and V6 which refer to the H variables.

Consider a typical case of R9: Assume $(J, N) = (\mathrm{M}, \mathrm{D})$ in E_1^1 of V_2, and $G_{\mathrm{D}}^{\epsilon 1\mathrm{S}}$ in E_e^2 of V_1 equals $G_{\mathrm{D}}^{\alpha 1\mathrm{S}}$ in E_1^1 of V_3. We will show that this only occurs if the Tile is as shown in Figure 18, namely the Tile $\mathrm{T_S}$ of a Triangle of class

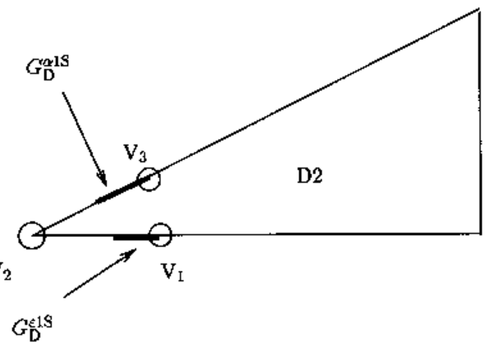

FIGURE 18

D2 (which is, of course, consistent with the conclusion of R9). We know from the hypothesis that there is a Triangle T of type D, with the medium edge of T containing vertices V_1 and V_2 of the Tile. Our proof will be by contradiction.

Now assume that $F \neq \mathrm{S}$ in E_1^1 of V_2. It follows that V_2 and V_3 lie on a complete edge which ends at V_2, and from the geometry this edge must be either: the medium edge of a type D or E Triangle, or the large edge of a type B Triangle. The three cases are similar, and we illustrate the argument with the case of the medium edge of a type E Triangle, T'. The geometry must then be as in Figure 19. But this cannot be our situation because of M6, with $Y1 = \mathrm{E1}$ and $X = \mathrm{D}$, which completes our analysis of R9.

We now consider R10. More specifically assume $G_{\mathrm{A}}^{\alpha 1\mathrm{M}}$ in E_e^2 of V_1 equals $G_{\mathrm{A}}^{\epsilon 1\mathrm{M}}$ in E_1^1 of V_3, and that $(J, N) = (\mathrm{S}, \mathrm{A})$ in E_e^2 of V_2; we must show that $F = \mathrm{M}$ in E_e^2 of V_2 as in Figure 20. We know from the hypothesis there is a Triangle T of type A, with the short edge of T containing vertices V_3 and V_2 of the Tile. Our proof will be by contradiction. So assume $F \neq \mathrm{M}$ in E_e^2 of V_2. It follows that V_2 and V_1 lie on a complete edge which ends at V_2, and from the geometry this edge must be the short edge of a type D or A Triangle. The two cases are similar, and we illustrate the argument with the case of a type D Triangle, T'. The geometry must then be as in Figure 21. But this cannot be our situation because of M3 with $Y1 = \mathrm{D1}$ and $X = \mathrm{A}$; this completes our discussion of R10.

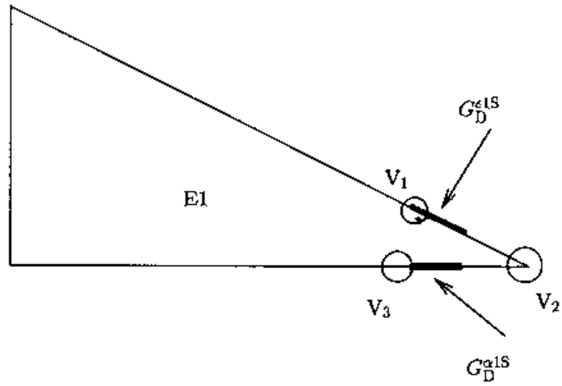

FIGURE 19

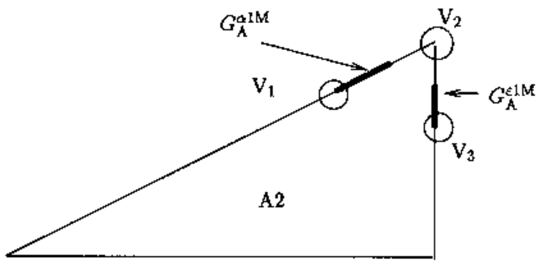

FIGURE 20

For V1 and V6 we will need two lemmas.

LEMMA 5.2. a) *Let* V_1 *and* V_2 *be vertices of a Tile, both lying on the medium edge (not necessarily a complete edge) of a Triangle of Level* $n \geq 1$, *with* V_1 *at one end of the edge and* E_1^1 *(resp.* E_e^2*) of* V_2 *and* E_e^2 *(resp.* E_1^1*) of* V_1 *referring to the edge. Assume* V_4 *is a vertex of a Tile lying at the other end of this medium edge, with* E_1^1 *(resp.* E_e^2*) of* V_4 *referring to the edge. Then*

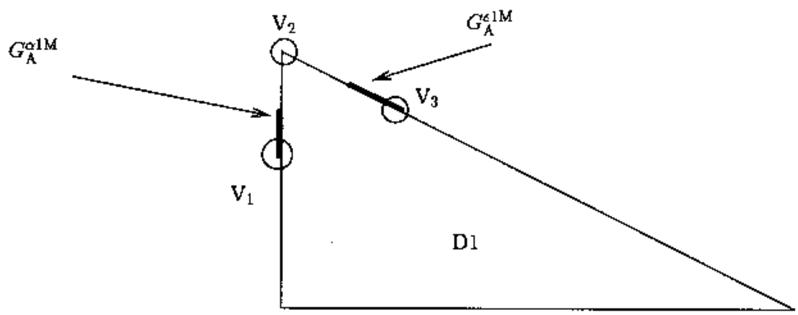

FIGURE 21

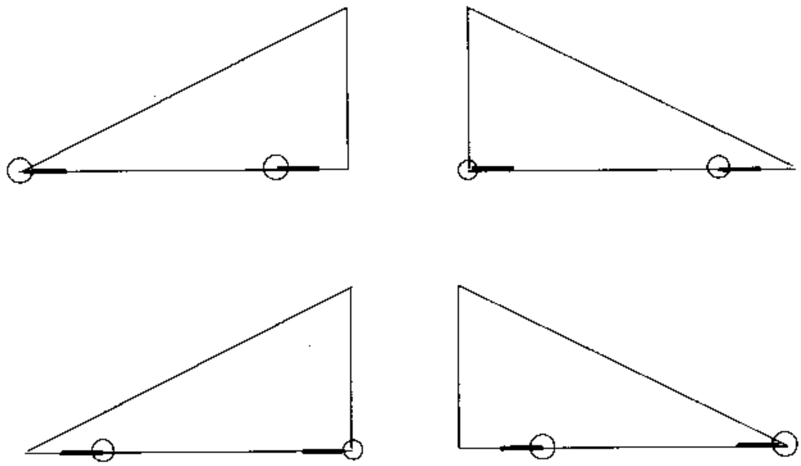

FIGURE 22A

G_n^{jkM} *has the same value in* E_1^1 *(resp.* E_e^2*) of* V_2 *as it has in* E_1^1 *(resp.* E_e^2*) of* V_4. *See Figure* 22A.

b) *Let* V_1 *and* V_2 *be a vertices of a Tile, both lying on the medium edge (not necessarily a complete edge) of a Triangle of Level* $n \geq 1$, *with* V_1 *at the end of the edge meeting the small angle, and* E_1^1 *(resp.* E_e^2*) of* V_2 *and* E_e^2 *(resp.* E_1^1*) of* V_1 *referring to the edge. Assume* V_4 *is a vertex of a Tile lying at the other end of this medium edge, with* E_1^1 *(resp.* E_e^2*) of* V_4 *referring to the edge. Then* G_n^{jkS} *has the same value in* E_1^1 *(resp.* E_e^2*) of* V_2 *as it has in* E_1^1 *(resp.* E_e^2*) of* V_4. *See Figure* 22B.

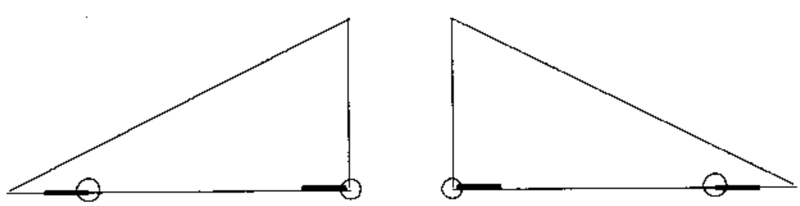

FIGURE 22B

Proof of Lemma 5.2. a) The case of Level $n = 1$ follows from M1. The higher Level cases then follow from the A-D symmetry of M1, since the medium edge is composed of the large edges of an A and a D Triangle, and any changes G_n^{jkM} undergoes in the A is reproduced in the D.

b) The case of Levels $n = 1, 2$ follows from M4 and M8. And now the same argument used in part a) can be applied, which completes our proof of the lemma. □

LEMMA 5.3. *Consider a Triangle of type E and Level $n \geq 0$, with E_1^1 (resp. E_e^2) of* V_{MM} *and E_e^2 (resp. E_1^1) of* V_{MS} *referring to the large edge. Then G_n^{jkM} has the same value in E_e^2 (resp. E_1^1) of* V_{SM} *as in E_e^2 (resp. E_1^1) of* V_{MS} *if and only if it has the same value in E_1^1 (resp. E_e^2) of* V_{LL} *as in E_1^1 (resp. E_e^2) of* V_{ML}. *See Figure* 22C.

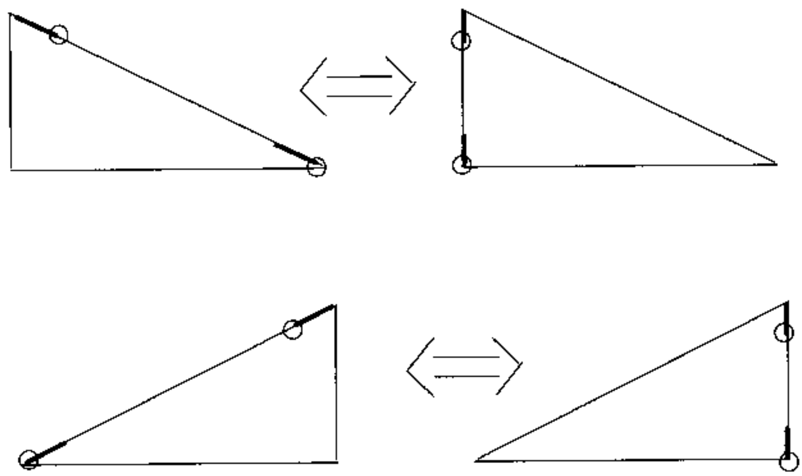

FIGURE 22C

Proof of Lemma 5.3. First note from M1 that cases $n = 0, 1$ hold. The cases $n \geq 2$ then follow by induction, where Lemma 5.2 is used to reduce to the component E Triangle of one lower Level. This ends our proof of the lemma. □

The nonobvious portions of V1 and V6 concern the values of the H_m^{kn}. However the case for variables H_m^{kS} follows immediately from R10 and Lemma 5.2b, and for variables H_m^{kM} from R9, Lemma 5.2a and Lemma 5.3. This completes our proof of Theorem 5.1. □

THEOREM 5.4. *In any tiling of the plane satisfying the matching rules, there is a unique decomposition into triangles of level $n \geq 0$.*

Proof. We begin with an outline of the proof. To prove by induction the existence of unique triangles of all levels one needs to show that each triangle of level N is contained in a unique "appropriate collection" at level $N+1$, which furthermore has all the distinguishing characteristics of some class, so that one can unambiguously form the level $N+2$ "appropriate collection." Some of this is easy to arrange using the "accessible" parts of the components of the collection, in particular the F values at the large vertex of the collection, which is accessible from the E component. (Accessible means known directly from the definition of the class of the component.) The harder step is to arrange for the needed characteristics of the collection at inaccessible parts of the components, in particular the F values at small and medium vertices of the collection. This is accomplished by encoding the information in the G variables, which can then carry it from accessible to inaccessible parts along pathways defined by the F and H variables, where R9 or R10 is applied.

To begin the proof we first note that for $n = 0$ the conclusion is satisfied by the assumption of a tiling. So to continue the induction, assume we have a tiling of the plane (with all matching rules for tiles satisfied of course), and that the tiling is uniquely decomposable into triangles of level n for $0 \leq n \leq N$. We must show that the tiling is uniquely decomposable into triangles of level $N+1$.

It is routine to check by R5 and V1 $\cdots$ V10 that given any triangle of level N in the tiling there is a unique way to extend it to an "appropriate collection" of five such triangles, and that there is a unique value of P in the nine complete edges of the five level N triangles, in particular the small edge of the A and the small and medium edges of the D. It then follows from R4, R8, V1, V5, V6 and V10 that this P value is converted into the N values in all those edges of the collection appropriate for the value of P, where the N values are meant to help define a type, and thus class, for the collection, making it an $N+1$ triangle. It remains to show that the needed F values are automatically correct, thereby justifying these N values.

The needed F values (namely all Z) in edge-marks interior to the edges of the collection are automatically correct by properties of the five component triangles. The only possible difficulty is therefore at the three vertices of the collection. There are ten cases, depending on the ten possible classes, A1 $\cdots$ E2, of the collection. The proof is similar for all cases. Using the P value used to determine the class of the triangle of level $N+1$, we get the needed information as follows, the general method being illustrated in the interests of clarity for the cases of class B1 and A2.

We consider first the case where $P = \mathrm{B}$ in the edge-marks in, say, the small edge of the component triangle of type D of the collection. The needed information in $\mathrm{V_{LL}}$ of the collection follows from V5 when we use this value of P in the small edge of the triangle of type D, which shares its medium vertex with the small vertex of $\mathrm{T_L}$. The needed information in $\mathrm{V_{MM}}$ of the collection follows from R11, applied to tile $\mathrm{T_M}$, again using the above P value, but now using it with R4 and R8.

The needed information in $\mathrm{V_{SS}}$ of the collection follows from R9 applied to tile $\mathrm{T_S}$ as follows. We know from V5 together with R19, R21 and R22 that $G_{\mathrm{B}}^{\alpha 1\mathrm{S}}$ in $E_e^1[+\pi/2]$ of vertex $\mathrm{T_{LL}}$ of the component A triangle equals $G_{\mathrm{B}}^{\varepsilon 1\mathrm{S}}$ in $E_1^2[-\pi/2]$ of vertex $\mathrm{T_{LL}}$ of the collection. To carry this information to vertex $\mathrm{V_{SS}}$ of the collection, in order to use R9 and V3, we use Lemma 5.2, which, like

Lemma 5.3, holds for "appropriate collections" of triangles as well as Triangles since it depends only on the variation of the G and H variables, which are the same for Tiles and tiles.

We now consider another case of the theorem, class A2, and only the needed information at the medium vertex of the collection (the information at the large vertex following as above).

The needed information in $\mathrm{V_{MM}}$ of the collection follows from R10 applied to tile $\mathrm{T_M}$, as follows. We know from V5 together with R19, R21 and R22 that $G_{\mathrm{A}}^{\alpha 1\mathrm{M}}$ in $E_e^1[-\pi/2]$ of vertex $\mathrm{T_{LL}}$ of the component A triangle equals $G_{\mathrm{A}}^{\epsilon 1\mathrm{M}}$ in E_1^1 of vertex $\mathrm{T_{LL}}$ of the collection. To carry this information to vertex $\mathrm{V_{MM}}$ of the collection, in order to use R10, we use V2, Lemmas 5.2 and 5.3.

This completes our proof of Theorem 5.4. □

6. Concluding remarks

In the above we have constructed the first example of a finite set of shapes (prototiles) which can tile the Euclidean plane, but *only* using tiles in infinitely many orientations (and thus of course only nonperiodically.) This requires, essentially for the first time, the use of rotations in the study of forced tilings. We make the following supplementary remarks.

* If our restrictions for tile marks are replaced by the stronger restrictions that the only marks allowed are those which appear on Tiles in the tesselation Θ, then of course Theorems 5.1 and 5.4 still hold. (The reason we did not do this is that it would then be more difficult to determine if a given marked prototile satisfied the restrictions.)

** It is common in the tiling literature to restrict attention to tilings of polygonal tiles which are "edge-to-edge," that is, in which the vertex of a tile can only intersect another tile at a vertex, as distinct from the way our tiles of type E meet tiles of type C, for example. To accommodate this (mistaken) prejudice, we note that our results can be recaptured with this additional property by the doubling of the number of prototiles; each of our prototiles listed above could be decomposed into two by cutting on a straight line from the center of the medium edge to the medium vertex, each half retaining of course the information on the two vertices it inherits; see Figure 23. One would then add matching rules along these fresh edges, forcing only the original pairs' ability to meet along such an edge.

*** It is convenient to insert a distinguished point within each triangle of type C, where the bisector of the large angle meets a straight line perpendicular to the center of the small edge, as in Figure 24. It is easy to check that for

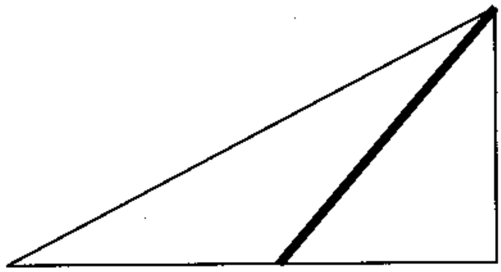

FIGURE 23

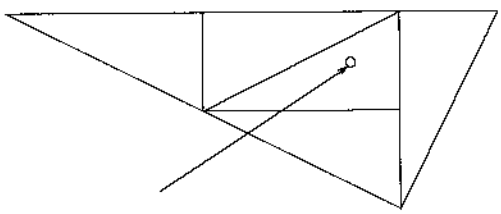

FIGURE 24

a triangle of type C of level $n \geq 1$ this point coincides with that associated with the triangle of type C of one lower level (and of course continuing down to level 0). These points are of interest because the triangles appear rotated about these points, in opposite directions for triangles of the same type but different class. This is the origin of the name we have given to this tiling system, with reference to the pinwheel toy which has blades, attached to a stick, which rotate in the wind.

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