Pointwise ergodic theory on operator algebras

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We extend Birkhoff's pointwise ergodic theorem from classical mechanics to the overlap with quantum mechanics.

1. INTRODUCTION

Ergodic theory was invented to elucidate the dynamical behavior of nonrelativistic many-body systems as described in a classical mechanical formalism. Soon thereafter it also was called upon to perform similar service for the same systems treated with quantum mechanics.

We will demonstrate below what we believe is the first nontrivial analog for quantum systems of Birkhoff's pointwise ergodic theorem[1]—nontrivial in that it does not require any special properties of the dynamics, in particular it does not require discrete spectrum of the Hamiltonian.[2] As there have been many misdirected versions of quantum ergodic theory, we feel it appropriate to state our objectives and results carefully.

In order to describe problems in classical and quantum mechanics in a parallel fashion it is convenient to use the algebraic formalism[3,4] in which the observables of the physical system are represented by the self-adjoint elements of a norm-separable C^*-algebra A with unit I, and the physical states by a subset of the set S of (mathematical) states of A; $f(a)$, for f in S and a in A, then represents the expected value of a when the system is in the state f. The classical mechanics of a system is described with an Abelian A, and the quantum mechanics with a non-Abelian A. The detailed structure of the dynamics is only imperfectly understood at present for interesting physical systems, but we know from simple examples[4–6] that it cannot be grossly misleading to assume that time evolution is represented by a one-parameter group $\{\alpha^t \mid t\in\mathbb{R}\}$ of $*$-automorphisms α^t of A. Thus the expectation value at time zero, $f(a)$, would evolve in time t in the Heisenberg picure to $f(\alpha^t a)$ or, equivalently, in the Schrödinger picture to $\tilde{\alpha}^t f(a)$, where $\tilde{\alpha}^t$ is the dual of α^t.

Unfortunately, this structure for the dynamics, which we will call "C^*-dynamics," has been shown[7,8] to be physically untenable in many imporant cases, while there is evidence to support hope that a certain modification might be generally acceptable.[9,10] This more general form, which we will term W^*-dynamics, requires certain (physically) distinguished states f_β, which are time invariant and such that in each of their GNS representations, π_β, of A the dynamics is represented by one-parameter groups $\{\alpha^t_\beta \mid t\in\mathbb{R}\}$ of $*$-automorphisms of the W^*-algebras $\pi_\beta(A)''$. Therefore, for a state f which is the restriction to A of a state (also denoted f) in the predual of some $\pi_\beta(A)''$, the dynamics is again of the form $f(a)\to f(\alpha^t a)$ or $f(a)\to\tilde{\alpha}^t f(a)$. It is, however, now an important problem to make sense of the evolution "$\tilde{\alpha}^t f$" if f is not of the above type. This problem was considered in Refs. 7 and 11 but only partially solved, in a sense described in later sections.

For simplicity, in the remainder of this section we will ignore the evidence of the last paragraph and assume not only a C^*-dynamics on A but also the appropriateness of a discrete time variable, n, so that the "orbit" of an initial state f would be $\{\tilde{\alpha}^n f \mid n\in\mathbb{Z}\}$. (There are certainly some physically interesting models where this is justified.[4,6])

A common approach to understanding the gross dynamical behavior of physical systems (particularly in statistical mechanics) is through "time average" quantities of the form

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\tilde{\alpha}^n f(a).$$

Of course, before such quantities can be convincingly manipulated, it is necessary to prove their existence. Historically, ergodic theory was invented to solve what we call the

Primary ergodic problem: Prove, for as many states f in S and observables a in A as possible, the existence of

$$\lim_{N\to\infty}\frac{1}{N}\sum_{N=0}^{N-1}\tilde{\alpha}^n f(a).$$

The most significant results in this direction actually correspond to the special form of the

Secondary ergodic problem: Prove, for as many states f in S as possible, the existence of

$$w^*\text{-}\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\tilde{\alpha}^n f. \tag{1}$$

In particular, the results referred to fall in two classes, "mean" and "pointwise" theories. For both one assumes given a state $\bar{f}$ in S which is a fixed point of $\tilde{\alpha}$. Mean ergodic theorems then prove (1) for a relatively small class of states f, mathematically and physically similar to $\bar{f}$, namely for f in L_1, the norm closure of

$$\bigcup_{n\geq 1}\{g\in S \mid g\leq n\bar{f}\};$$

Pointwise (or individual) ergodic theorems prove (1) for "$\bar{f}$-almost every" state f in S.

Mean ergodic theory in the above sense was developed for Abelian A only (i.e., for classical mechanical systems) in Refs. 12 and 13 and in the general non-Abelian (i.e., quantum mechanical) setting in Refs. 14 and 15. The pointwise theory was developed for Abelian A in Ref. 1, and it is the effort to obtain a satisfactory noncommutative generalization of this result (i.e., satisfactory quantum version) which is the subject of this paper. For concreteness it is perhaps convenient to keep in mind the above problems for the threedimensional Heisenberg model on an infinite lattice,[9] a highly nontrivial model where the above considerations are easily formulated and of definite interest.

2. NOTATION AND AN EXTENSION OF LANCE'S THEOREM

Throughout this section, M will denote a W^*-algebra, $\bar{f}$ a faithful state in the predual M_* of M, and A a σ-weakly dense, norm-separable sub-C^*-algebra of M containing the unit I. Further, α will denote a *-automorphism of M such that its dual $\tilde{\alpha}$ has $\bar{f}$ for a fixed point.

In this notation, Lance has proven[16] (see also Ref. 17):

Theorem I: There exists a norm continuous linear projection $T: M \to M$ and, for each finite subset A' of A, *a sequence* $\{P_j | j \in \mathbb{N}\}$ of projections in M (the sequence dependent on A') such that $\lim_{n\to\infty} \bar{f}(P_n) = 1$ and

$$\lim_{N\to\infty} \left\| \left[\frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(a) - Ta \right] P_j \right\| = 0$$

for each a in A' and j in $\mathbb{N}$.

As our first step we prove the following related result.

Theorem II: If we further assume that $\bar{f}$ is tracial, then the above sequence $\{P_j | j \in \mathbb{N}\}$ can be chosen independent of $A' \subseteq A$.

Proof: Let $\{a_k | k \in \mathbb{N}\}$ be a norm dense subset of A. For A' being the singleton $\{a_k\}$, let $\{P_j(k) | j \in \mathbb{N}\}$ be the sequence guaranteed by Theorem I. By choosing a subsequence if necessary for each k, we can assume that

$$\bar{f}[P_j(k)] > 1 - 1/2^{k+j}.$$

Define

$$Q_j = \bigwedge_{k \geq 1} \bigwedge_{n \geq j} P_n(k).$$

Therefore

$$I - Q_j = \bigvee_{k \geq 1} \bigvee_{n \geq j} (I - P_n(k)),$$

so using the normality of $\bar{f}$ and (Ref. 18;2.1.5) (which is perhaps *the* crucial step for which we seem to need $\bar{f}$ to be tracial),

$$\bar{f}(I - Q_j) \leq \sum_{k \geq 1} \sum_{n \geq j} \bar{f}(I - P_n(k))$$
$$\leq \sum_{k \geq 1} \sum_{n \geq j} 1/2^{k+n} = 1/2^j \to 0 \quad \text{as } j \to \infty$$

i.e., $\lim_{j\to\infty} \bar{f}(Q_j) = 1$. Since $Q_j \leq P_j(k)$ for all j and k, it is clear that

$$\lim_{N\to\infty} \left\| \left[\frac{1}{N} \sum_{n=0}^{N-1} (\alpha^n - T) a_k \right] Q_j \right\| = 0$$

for every k.

Now given any fixed a in A, j in $\mathbb{N}$ and $\epsilon > 0$, one can choose k such that $\|a - a_k\| < \epsilon/4$, and N_1 such that

$$\left\| \left[\frac{1}{N} \sum_{n=0}^{N-1} (\alpha^n - T) a_k \right] Q_j \right\| < \epsilon/2$$

for all $N > N_1$.

Therefore

$$\left\| \left[\frac{1}{N} \sum_{n=0}^{N-1} (\alpha^n - T) a \right] Q_j \right\|$$
$$\leq \left\| \frac{1}{N} \sum_{n=0}^{N-1} (\alpha^n - T)(a - a_k) \right\|$$
$$+ \left\| \left[\frac{1}{N} \sum_{n=0}^{N-1} (\alpha^n - T) a_k Q_j \right] \right\|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

The advantage of having $\{P_j | j \in \mathbb{N}\}$ independent of A' is to avail ourselves of the following ideas based on Segal's noncommutative integration theory.[19]

3. APPLICATION TO THE ERGODIC PROBLEM

Throughout this section we use the notation of Sec. 2 with the further assumption that $\bar{f}$ is tracial. We will also need the following notation from Refs. 7 and 11.

Definition. A sequence $\{P_j | j \in \mathbb{N}\}$ of projections P_j in M will be called an *exhaustion* if: (a) $P_{j+1} \supseteq P_j$ for all j, and (b) $\lim_{j\to\infty} \bar{f}(P_j) = 1$.

Definition: A subset S' of the state space S of A is said to contain *$\bar{f}$-almost every* state, or to be *of full $\bar{f}$-measure*, if there exists an exhaustion $\{Q_j | \in \mathbb{N}\}$ such that

$$S' \supseteq S(\{Q_j | j \in \mathbb{N}\})$$
$$\equiv \overline{\bigcup_{j \geq 1} \{f^r | f \in M_*, \operatorname{supp} f \subseteq Q_j\}},$$

where f^r is the restriction to A of f, and the closure is with respect to the w^*-topology of S. The complement of a set of full $\bar{f}$-measure is *of $\bar{f}$-measure zero*.

As demonstrated in Ref. 11, the collection $\mathcal{S}$ of all sets of full $\bar{f}$-measure is closed under countable intersection. Also, it is proven in Ref. 7 that if A (and therefore M) is Abelian and X is the set of pure states on A in the w^*-topology, so that $A \simeq C(X)$ and $\bar{f}$ is (integration with respect to) a regular Borel probability measure on X, then every set in $\mathcal{S}$ contains "$\bar{f}$-almost every" point of X in the usual measure theoretic sense. The main result of Ref. 11, which we need at this point, is the noncommutative generalization of the von Neumann–Maharam theorem, namely that any *-automorphism of M, for example α, is implemented by or induces a canonical point transformation on S defined $\bar{f}$-almost everywhere. (The point transformation is "essentially unique" in that any two such transformations would have to agree $\bar{f}$-almost everywhere.) Since the transformation is canonical we will use for it the intuitive notation $f \to \tilde{\alpha} f$.

The generalization of the above from a single *-automorphism to a group of *-automorphisms (in particular to the cyclic group $\{\alpha^n | n\in\mathbb{Z}\}$) is contained in Ref. 7. Thus given α on M, we have a canonical "orbit" $\{\tilde{\alpha}^n f | n\in\mathbb{Z}\}$ defined for $\bar{f}$-almost every f in S, defining a set $\tilde{S}\subseteq S$. Let $\tilde{\tilde{S}}\subseteq S$ be the set of full $\bar{f}$-measure defined by the exhaustion of Theorem II.

With the above notation we now prove

Theorem III: For $\bar{f}$-almost every f in S, the following limit exists:

$$w^* - \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \tilde{\alpha}^n f. \qquad (2)$$

Proof: Let $\{P_j | j\in\mathbb{N}\}$ be an exhaustion corresponding to $\tilde{S}\cap\tilde{\tilde{S}}$ (defined above) and let $f = w^* - \lim_\gamma f_\gamma$, where $f_\gamma \in M_*$ and $\text{supp} f_\gamma \subseteq P_j$, j fixed. For each a in A and integers N_1, N_2,

$$\left| \frac{1}{N_1} \sum_{n=0}^{N_1-1} \tilde{\alpha}^n f(a) - \frac{1}{N_2} \sum_{n=0}^{N_2-1} \tilde{\alpha}^n f(a) \right|$$

$$= \lim_\gamma \left| \frac{1}{N_1} \sum_{n=0}^{N_1-1} \tilde{\alpha}^n f_\gamma(a) - \frac{1}{N_2} \sum_{n=0}^{N_2-1} \tilde{\alpha}^n f_\gamma(a) \right|$$

$$= \lim_\gamma \left| \frac{1}{N_1} \sum_{n=0}^{N_1-1} [\tilde{\alpha}^n f_\gamma(a)] - f_\gamma(Ta) \right.$$

$$\left. - \frac{1}{N_2} \sum_{n=0}^{N_2-1} [\tilde{\alpha}^n f_\gamma(a) - f_\gamma(Ta)] \right|$$

$$= \lim_\gamma \left| f_\gamma \left[\frac{1}{N_1} \sum_{n+0}^{N_1-1} (\alpha^n a - Ta) \right.\right.$$

$$\left.\left. - \frac{1}{N_2} \sum_{n=0}^{N_2-1} (\alpha^n a - Ta) \right] \right|$$

$$\to 0 \quad \text{as } N_1, N_2 \to \infty.$$

The existence of the limit (2) is then evident from the completeness of S in the w^*-topology.

4. PHYSICAL ASPECTS

Lance's Theorem I is an ergodic theorem concerned with "time averages" of operators; Theorem III is an analogous result (but under the added assumption that $\bar{f}$ be tracial) for time averages of states. As was emphasized in the introduction, aside from their inherent interest there is an added significance for results of the latter form, determined by their widespread utility in physics. We need to comment further on this point.

Assume a C^*-dynamics on A, with a continuous or discrete time variable t. For simplicity further assume $\alpha^t = \gamma^t$ for all approriate t, where $\{\gamma^s | s\in\mathbb{R}\}$ is a strongly continuous one-parameter group of $*$-automorphisms of A, let $\bar{f}$ be a faithful state on A fixed by all α^t, and let π be the GNS representation of A associated with $\bar{f}$. If $\bar{f}$ is tracial (or KMS as defined below) then the α^t extend by continuity to $*$-automorphisms of the W^*-algebra $M = \pi(A)''$ and $\bar{f}$ extends by continuity to a faithful, normal, tracial (or KMS) state on M.

The "usefulness" of Theorem III then rests solely on the appropriateness of the assumption that $\bar{f}$ be tracial. Here however there is some difficulty. In any form of measure theory the concept of a set being "of $\bar{f}$-measure zero" is only useful to the extent that such sets are in practice negligable, which depends essentially on the particular $\bar{f}$. For quantum mechanical applications then, before Theorem III can be used effectively one must determine a physically relevant *tracial* $\bar{f}$, one for which sets "of $\bar{f}$-measure zero" would be convicingly small or negligable in relevant calculations. Such states must (and do) occur corresponding to the infinite temperature state, as can be seen by taking the limit $\beta\to 0$ in the canonical ensemble. Physically such a state can clearly be used to study qualitative features at high temperature. As this is the regime where classical and quantum mechanics coincide, our results describe, physically as well as mathematically, the overlap between classical and quantum behavior.

Finite temperature states, which are not tracial, therefore represent the next frontier, and it would be of great value if the condition that $\bar{f}$ be tracial could be dropped from Theorem III; but this also does not seem highly probable at present.

One intermediate problem however, which does not seem entirely hopeless and the solution of which would be of definite physical interest, would be to prove the results of this paper for $\bar{f}$ being a KMS state, where we define this latter notation as follows. For each a, b in A and $d > 0$, we define the functions

$$F_{ab}: s\in\mathbb{R} \to \bar{f}(b\gamma^s a)$$

$$G_{ab}: s\in\mathbb{R} \to \bar{f}([\gamma^s a]b)$$

$$e_d: s\in\mathbb{R} \to \exp(ds/2).$$

Then we say $\bar{f}$ is KMS if there exists $\beta > 0$ such that $\hat{F}_{ab} = e_{2\beta} \hat{G}_{ab}$, where $\hat{F}_{ab}$ and $\hat{F}_{ab}$ are the Fourier transforms of the respective functions considered as tempered distributions.[20–22] Noting that a trace state could be considered a KMS state corresponding to $\beta = 0$, we conclude with the question: Can Theorem II be extended by replacing the assumption that $\bar{f}$ be tracial with the assumption that $\bar{f}$ be KMS?

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