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Quaquaversal tilings and rotations

John H. Conway[1,*], **Charles Radin**[2,**]

[1] Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
[2] Department of Mathematics, University of Texas, Austin, TX 78712, USA

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Abstract. We construct a hierarchical tiling of 3 dimensional Euclidean space based on a triangular prism, using repeated rotations, about orthogonal axes, by angles $2\pi/m$ and $2\pi/n$. To analyze the structure of the tiling we are led to determine the group $G(m,n)$ generated by such a pair of rotations, for $m = n = 3$ and for $m = 3,\ n = 4$.

I. Introduction

This paper is concerned with certain patterns in 3 dimensional Euclidean space, and their symmetries. One step in analyzing these symmetries involves determining the group $G(m,n)$ generated by a pair of rotations about orthogonal axes, one by $2\pi/m$ and the other by $2\pi/n$, $2 \leq m \leq n$. Specifically, we determine presentations for the groups $G(3,3)$ and $G(3,4)$, which are perhaps the simplest cases aside from the finite groups, $G(4,4)$ and $G(2,n)$. (There were previous results for some cases [Swi] where $m = n = \infty$.)

The patterns we consider are tilings of space, by triangular prisms, which have a hierarchical structure. In our primary example there is only one type of tile, and it is shown to appear in the tilings in infinitely many orientations, in fact with orientations which are uniformly distributed in $SO(3)$. Furthermore, the number of such orientations which occur in a sphere of volume N in a tiling grows polynomially in N. As we shall see, only logarithmic growth is possible in analogous 2 dimensional tilings, such as the pinwheel [Ra1. Ra2], due to the commutativity of rotations in 2 dimensions.

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We also discuss two local properties of the tilings, concerned with the neighborhoods of tiles.

II. The quaquaversal tilings

Consider the triangular prism ("tile") made from a 1, $\sqrt{3}$, 2 right triangle, with depth 1. We want to define a "deflation" rule which decomposes this tile into eight congruent pieces ("small tiles"), each similar to the original. First we decompose the right triangle by joining the midpoint of the hypotenuse to the midpoints of the two legs and to the vertex of the right angle; see Fig. 1 for a labelling of the four subtriangles this produces. We now fatten this by depth 1/2 to make a collection C of four small tiles, labelled 1 to 4, each similar to the original tile. Next we modify this collection of four small tiles in two different ways. One modified collection, C_A, is obtained by rotating by $2\pi/4$ the small tiles labelled 2 and 3 about the axis joining the centres of their two square faces. (The square faces are perpendicular to those visible in Fig. 1.). Note that the modification does not change the set theoretic union of the collection of four (solid) small tiles; the collection C_A still has the shape of C, that of a triangular prism with a face consisting of a 1, $\sqrt{3}$, 2 right triangle and depth 1/2. The other modification, C_B, of C is obtained by rotating by $2\pi/3$ the small tiles 3 and 4, which together define an equilaterial triangular prism, about the axis joining the centers of the two triangular faces of the equilateral prism. Again, C_B has the same overall shape as C. Now we abut C_A and C_B along their triangular faces to yield a collection $\tilde{C}$ of eight small tiles, for which the set theoretic union is precisely the original tile. There is some minor ambiguity in the above description, which is clarified in Fig. 2, which shows two views of $\tilde{C}$; the four tiles labelled with "A" refer to C_A, the four labelled with "B" refer to C_B.

We interpret the above as follows: starting with a certain tile we end up with a decomposition of it into eight small tiles each similar to the original

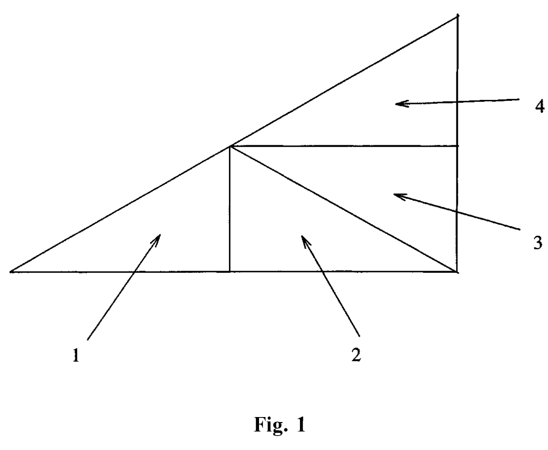

Fig. 1

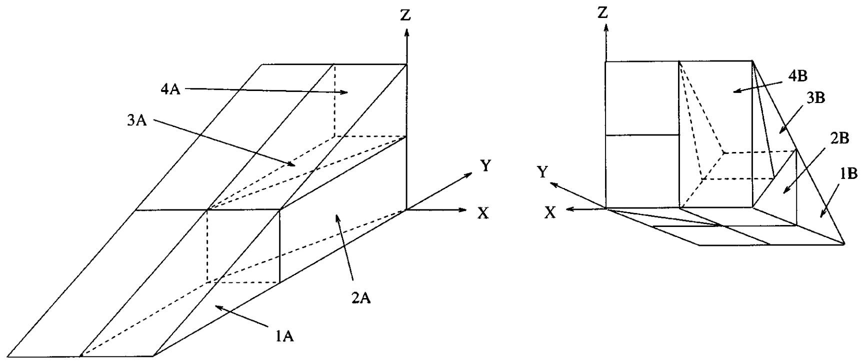

Fig. 2

but shrunk by a factor of 2. We call this a deflation of the tile. And we define deflation of any collection of tiles by deflating each separately. Now if we begin with a collection of n tiles, deflate them as above, and linearly expand about any point by a factor of 2, we end up with $8n$ tiles (of the original size). "Quaquaversal tilings" of space are produced by infinite repetition of this deflation-expansion process, as follows.

Start with a tile, do the deflation-expansion producing eight tiles, repeat the process on the set of eight tiles, producing 64 tiles, and again on these, producing 512 tiles. We will call this a "big" deflation-expansion, which, starting with one tile, produces a collection of 512 tiles. (As with the simpler deflation-expansion, it can, and will, be applied to a collection of tiles.) The collection $\hat{C}$ of 512 tiles has the overall shape of a tile, stretched by a factor of 2^3, and contains a tile T in its interior which has its faces parallel to the corresponding faces of $\hat{C}$. (We see the latter as follows. After the first deflation-expansion select the "$2B$" tile, as defined by Fig. 2, calling it T_1; after the next deflation-expansion, again choose, from the byproducts of T_1, the $2B$ tile, calling it T_2; in the last deflation-expansion, the $4A$ tile produced by T_2 is what we are calling T.) There is a unique fixed point $P \in T$ for the similarity which takes $\hat{C}$ to T. Therefore if we start with a single tile and perform the big deflation-expansion, expanding about the "fixed point" of the tile, the result is simply to add 511 tiles around the tile (which therefore can reasonably be called T), enlarging the volume of the region tiled *by a factor of* 2^9. Repeating this big deflation-expansion on the set of tiles produced from the previous action, always expanding about the fixed point of the original T, leads to (exponential) *extension* of the already constructed tiling of a region of space. A quaquaversal tiling of space is obtained in the limit of infinitely many repetitions. (Expanding about variable points can produce noncongruent tilings; this is not relevant in this paper, but for an analysis of this see [Ra2].)

Our next goal is to analyze the set of relative orientations of the tiles which appear in an expanding sphere of such a tiling.

Theorem 1. *The orientations of the tiles in a quaquaversal tiling are uniformly distributed in $SO(3)$.*

Proof. From the Peter-Weyl theorem, a sequence $\{g_n\}$ of elements of $SO(3)$ is uniformly distributed over $SO(3)$ if and only if $(1/N)\sum_{n=1}^{N} D_{ij}(g_n) \to 0$ for every matrix element D_{ij} of every continuous irreducible unitary representation D of $SO(3)$ other than the trivial one [K-N]. Using the methods of [Ra2] we can restrict attention to $N = 8^k$, corresponding to k iterates of deflation-expansion. In such a structure the orientations of the tile are the summands in the elements $[g'_1 + \ldots + g'_8]^k$ of the group ring of $SO(3)$, where $g'_1, \ldots, g'_8$ are the eight orientations of the small tiles with respect to that of their deflated parent. (Specifically, with respect to the x, y, z axes of Fig. 2 and using the notation e for the identity and R_x^θ for rotation by angle θ about the x axis, the eight orientations of tiles $1A, \ldots, 4A$ and $1B, \ldots, 4B$ are: $e, R_y^{2\pi/4}R_z^{2\pi/2}, R_y^{6\pi/4}$, e, and $e, R_z^{2\pi/2}, R_x^{2\pi/3}R_y^{2\pi/2}, R_x^{2\pi/3}$.)

Now

$$|([D(g'_1) + \cdots + D(g'_8)]^k)_{ij}| \le \|[D(g'_1) + \cdots + D(g'_8)]^k\| \tag{1}$$

So

$$([D(g'_1) + \cdots + D(g'_8)]^k)_{ij}/8^k \;\to\; 0 \tag{2}$$

as $k \to \infty$ if

$$\|[D(g'_1) + \cdots + D(g'_8)]^k\|/8^k = \|([D(g'_1) + \cdots + D(g'_8)]^k\|^{1/k}/8)^k \;\to\; 0 \tag{3}$$

But $\|[D(g'_1) + \cdots + D(g'_8)]^k\|^{1/k}$ has as its limit the spectral radius of $D(g'_1) + \cdots + D(g'_8)[\mathrm{R} - \mathrm{N}]$, so the limit in (2) can only be nonzero if that spectral radius in 8; it certainly cannot be larger than 8 since the norm cannot be larger than 8. We prove it is not 8 by contradiction as follows. Assuming the spectral radius is 8, there is a (unit length) eigenvector f of $D(g'_1) + \cdots + D(g'_8)$ with eigenvalue of absolute value 8, and since each $D(g'_j)f$ is of unit length, they must be the same vector for all $j = 1, \ldots, 8$, namely the same multiple of f. But then f defines a 1 dimensional space invariant under all the $D(g_j)$, and thus invariant for the representation of the group generated by the g_j. But since the representation is continuous, the space is also invariant for the representation of the closure of that group, which is all of $SO(3)$ (the closed subgroups of $SO(3)$ are known). But this would be a contradiction with the irreducibility of the representation D, unless it is the trivial representation. So the spectral radius of $D(g'_1) + \cdots + D(g'_8)$ cannot be 8, and the limit in (3) is 0, except for the trivial representation, proving that $\{g_n\}$ is uniformly distributed. □

Next we need to consider the algebraic aspects of the rotations in the deflation to analyze the rate of growth of the number of different orientations of the tile within an expanding sphere of a tiling.

III. Groups of rotations

Recalling the notation of the introduction, let $G(m, n)$ be the group generated by a pair of rotations about orthogonal axes, one by $2\pi/m$ and the other by $2\pi/n$, where $2 \le m \le n$.

Theorem 2. *$G(3, 4)$ is the free product $S_3 * (C_2)C_4$ of the symmetric group S_3 with the cyclic group C_4, amalgamated over C_2.*

Proof. Our proof represents rotations by conjugation of quaternions in the well-known way [Cur]. That is, with the quaternion notation:

$$q = q_1 + q_2 i + q_3 j + q_4 k = r[\cos(\phi) + \sin(\phi)v] \tag{4}$$

where q_m, r are real, v is a unit 3-vector, $0 \le \phi \le \pi$, $r \ge 0$ and the norm $N(q) = r^2 = {q_1}^2 + {q_2}^2 + {q_3}^2 + {q_4}^2$, we have the following. If $qq'q^{-1} \equiv \tilde{q} = \tilde{r}[\cos(\tilde{\phi}) + \sin(\tilde{\phi})\tilde{v}]$, then $\tilde{r} = r'$, $\tilde{\phi} = \phi'$ and the 3-vector $\tilde{v}$ is obtained from v' by rotation by 2ϕ about the axis containing v. As a map from quaternions of norm one to rotations this is a group homomorphism, with kernel $\{\pm 1\}$.

Consider the subring K of quaternions generated by i and $w = (-1 + j\sqrt{3})/2$. Note that $w^2 = w^* = (-1 - j\sqrt{3})/2$, and $w^3 = 1$. Also, i inverts $w : wi = iw^*$. If therefore we define $g = wi$ and $h = wg$ we will have $gi = hg = ih = -w$ and $ig = gh = hi = -w^*$, where we used $gg = hh = ii = -1$.

So the ring K is additively generated by $1, w, w^*$ together with g, h, i, where each of these triples adds to 0. Therefore as a subset of $\mathbf{R}^4$, K is the direct sum of two equilateral triangular lattices: integer multiples of $1, w$ and w^* in the plane P_1 spanned by 1 and j, and integer multiples of g, h and i in the plane P_2 spanned by i and k.

The furthest a point in the plane P_1 can be from a point of its lattice is when it is at the center of one of the equilateral triangles, when it is $1/\sqrt{3}$ from it. Similarly for P_2. So the furthest a point in $\mathbf{R}^4$ can be from the direct sum of the lattices is $\sqrt{2}/\sqrt{3}$, which happens to be less than 1.

So by a familiar argument K is a Euclidean domain, and thus a principal ideal domain. This implies a form of unique factorization, in the following sense.

Lemma. *Let X be an element of K that is not divisible by any rational integer greater than 1. If the norm $N(X)$ is the product $p_1 p_2 \ldots p_m$ of primes, then we can write $X = P_1 P_2 \ldots P_m$, where $N(P_1) = p_1$, $N(P_2) = p_2, \ldots, N(P_m) = p_m$.*

Given the order in which we take $p_1, p_2, \ldots, p_m$, *this expression is unique up to "unit passage" (that is, we may replace* P_j *and* P_{j+1} *by* $P_j u$ *and* $u^{-1}P_{j+1}$ *for any unit* u).

Sketch of a proof by induction on m. First assume the norm $N(X)$ of X is the prime p. Consider the right ideal generated by X and p. This must be principal, say (P), and P is unique up to multiplication by a unit since if $(P) = (P')$ then $P = P'q_1$ and $P' = Pq_2$. But then $P' = P'q_1q_2$ and cancellation justifies the claim. Then continue inductively. □

We return to the proof of Theorem 2. Now the rotations that define $G(3,4)$ are represented by the conjugations by w and $1+i$, and so any element of $G(3,4)$ is represented by the conjugation by some product of these, that product being unique up to a real factor. We can adjust the real factor so that we get an element of our ring that is not divisible by any rational integer greater than 1. From the geometrical form of K it has just three right ideals of norm 2, namely $(1+g), (1+h), (1+i)$, and we see that everything they (and the units) generate is uniquely of the form $(1+k_0)(1+k_1)\cdots(1+k_m)$ times a unit, where each of the k_j is one of g, h, i, and no two adjacent factors are equal.

Rewriting each of the k_j as $w^{-t}iw^t$ for some t, we get another normal form:

$$W_0(1+i)W_1(1+i)W_2(1+i)\cdots(1+i)W_k(\pm 1 \text{ or } \pm i) \tag{5}$$

which is (when we forget the sign) exactly the normal form for $S_3 * (C_2)C_4$. □

Corollary. $G(3,3)$ *is the free product* $C_3 * C_3$, *and is a normal subgroup of index* 4 *in* $G(3,4)$.

Proof. The result for $G(3,3)$ is obtained by specializing: the two generators are w and its conjugate by $1+i$, which we call W. Under conjugation by $1+i$, we see

$$w \to W \to w^* \to W^* \to w \tag{6}$$

So the group $\langle w, W\rangle$ is normal of index 4 in $\langle w, 1+i\rangle$, justifying the assertion that $G(3,3)$ is normal of index 4 in $G(3,4)$. □

Consider the orientations that appear in the structure of 8^k tiles produced by k iterations of the deflation-expansion process, as in the proof of Theorem 1. Using the above Corollary and just considering the 2^k tiles which at each stage of the hierarchy are within prisms of type $2A$ or $4B$, we see that the tiles must exhibit roughly 2^k different orientations, that is, the total number of orientations grows polynomially in the volume. This is in marked contrast to the situation for hierarchical tilings in the plane, such as the pinwheel, for which the number of different orientations in a structure produced by k iterations of deflation-expansion can be at most algebraic in k (because of the commutativity of the rotations), and therefore logarithmic as a function of the area.

IV. Two features of tile neighborhoods

This section contains an analysis of two local properties of the tilings. First we show that there is only a finite number of ways in which a tile is surrounded by abutting tiles in a quaquaversal tiling. This does not follow merely from the hierarchical nature of the tilings; we later give a different example to illustrate this.

We add some lines on the faces of the tile. Specifically: on the rectangular face of area 2 (henceforth called the "top face") draw a line between the midpoints of the edges of length 2; on the triangular faces draw lines from the midpoint of the hypotenuse to the midpoint of the leg of length $\sqrt{3}$ and to the vertex of the right angle; and on the rectangular face of area $\sqrt{3}$ (henceforth called the "bottom face") draw both diagonals and a line between the midpoints of the edges of length $\sqrt{3}$; and finally, draw lines along all edges of all faces except for the edges of length $\sqrt{3}$ of the triangular and bottom faces. Fig. 3 shows the added lines by heavy dashes.

We claim that in a quaquaversal tiling tiles abut in such a way that lines overlap lines wherever faces abut: that is, wherever faces intersect in positive area. (This of course implies our previous claim that tiles only abut in finitely many ways.) This is true by inspection for the eight abutting tiles in the deflation of a tile. One can then use induction if it remains true on further deflation, which is indeed the case as we see next from the assignment of lines.

Proposition 1. *In a quaquaversal tiling each triangular or bottom face of a tile lies in a plane which only consists of such faces, and the pattern of lines on that plane is an equilateral triangular lattice; each square or top face lies in a plane*

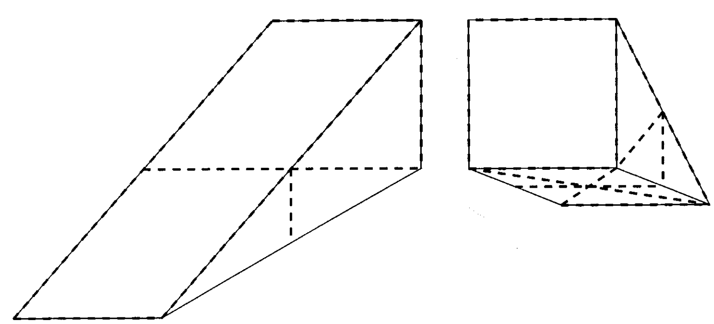

Fig. 3

which only consists of such faces, and the pattern of lines on that plane is a square lattice.

Sketch of a proof. The proof is by induction, as follows. One first sees by examination that in a single deflation top or square faces give rise to such faces of small tiles, and similarly for bottom and triangular faces. And then we see by induction that this remains true.

Next one notes that on deflation each square of lines on top and square faces is decomposed into four squares. As for the bottom and triangular faces, it is convenient to think of them as follows. The pattern of lines on a bottom face is the same as that of a pair of triangular faces making up the rectangle, and on deflation it produces the same pattern of lines as would the pair of triangular faces. And, creating an equilateral prism from a pair of tiles as above, the lines created by deflation of the two tiles just adds inscribed equilateral triangles within the four original equilateral triangles. □

This result can be strengthened by the addition of further lines; namely, on the triangular and bottom faces draw angle bisecters of the equilateral triangles. Now weaken the notion of "face" of a polyhedron to allow faces to simultaneously share an edge and be coplanar, and consider the lines on the prism to define 51 faces: 3 square and 48 triangular. With this convention we have the following corollary.

Corollary. *Using the faces defined by both sets of lines, a quaquaversal tiling is "full face to full face".*

We clarify the above result with an example of hierarchical tilings with different local properties. We begin with a triangular prism made from a $1, 2, \sqrt{5}$ right triangle, with depth 1. We assume it comes in two colors, red and black. We will give a deflation rule which decomposes each into ten small tiles, five red and five black. First decompose the right triangle in two ways as follows. Drop a perpendicular from the vertex V of the right angle to point P on the hypotenuse, from the midpoint M of the leg of length 2 to the point Q on the hypotenuse, and from M to the midpoint N of the line $[V, P]$. The "pinwheel" decomposition of the right triangle is obtained by

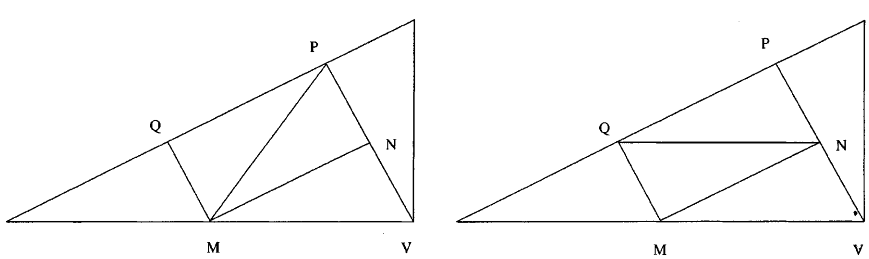

Fig. 4

adding the line $[M, P]$, while the "anti-pinwheel" decomposition is obtained by adding the line $[N, Q]$; see Fig. 4. By fattening each of these by depth 1 one makes two collections of five congruent prisms, all red from the pinwheel decomposition, all black from the anti-pinwheel decomposition. The deflation of the black tile is then made by abutting these two collections of prisms along triangular faces. This involves a choice. The deflation of the red tile is made similarly, but with the other choice. We then make "sandwich" tilings by repeated deflation-expansion, starting with some tile. We note that any such tiling will consist of parallel red and black layers. (By definition the colors appear in a Morse sequence [Que].)

Now it is known [Ra1] that in each red layer the tiles appear in infinitely many orientations. It is easy to see that in each black layer the tiles appear in only four orientations. Therefore wherever black and red layers meet the tiles must abut in infinitely many ways. This proves the following.

Proposition 2. *In a sandwich hierarchical tiling the tiles abut neighbors in infinitely many ways.*

We conclude this section by considering another local property of the quaquaversal tilings. First we note that, by construction, a quaquaversal tiling can be thought of as a tiling at infinitely many scales; that is, space can be seen as simultaneously tiled by structures which are the result of iterating the deflation-expansion process k times, for any fixed k. This could also be said to be true of the lattice tiling of space by unit cubes, which could be obtained using the deflation rule which decomposes a unit cube into eight congruent small cubes in the obvious way. An important difference between the cubic and quaquaversal tilings is that while a cubic tiling can be thought of as a tiling at infinitely many scales, these are nonunique, while this is not true for quaquaversal tilings. To see the nonuniqueness of the hierarchies of cubic tilings one simply notes that there are eight distinct ways to group together all the cubes of a tiling into appropriate sets of eight.

Proposition 3. *The hierarchical structure of quaquaversal tilings is unique.*

Sketch of a proof. This is easily seen by inspection, noting that a tile of type $3B$ differs in the way it abuts geometrically with its neighbors from each of the other seven tiles obtained in the deflation-expansion of a tile. □

V. Conclusion

Most work on hierarchical tilings has centered on planar models. The 3 dimensional quaquaversal tilings in this paper show that the algebraic aspects of $SO(3)$, very much richer than $SO(2)$, lead to new qualitative features such as the growth rate of orientations, treated in section II.

These growth rates have practical consequences. Hierarchical structures such as the Penrose [Gar], pinwheel and quaquaversal tilings exhibit "statistical rotational symmetries" [Ra3], whereby the *frequencies* of finite elements in the tilings (or more accurately, the frequency densities) are invariant under rotations without the tiling itself being invariant. (Theorem 1, together with methods from [Ra2], shows that quaquaversal tilings are statistically invariant under all rotations.) It has been suggested [Ra5] that such tilings could be useful in discrete models, or numerical solution of differential equations, since local rotational symmetry might by better preserved in such models than, say, with cubic grids. But for planar models logarithmic growth implies that this is impractical since too few orientations would appear in a pattern of reasonable size to implement the symmetry. The algebraic growth of our 3 dimensional quaquaversal tilings avoids this problem.

Hierarchical tilings originated thirty years ago in logic, to solve decidability problems of predicate calculus [Wan, G-S], and have since had significant interaction with research in condensed matter physics [S-O, Sen], discrete geometry [G-S] and ergodic theory [Ra4]. Most of this developed from examples of 2 dimensional tilings, such as the Penrose and pinwheel tilings. Through the rotation group, 3 dimensional tilings add a significant algebraic aspect to this wide ranging research, raising natural questions such as the structure of the groups $G(m,n)$.

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