The stationary problem has been studied by Calogero,[6] who gave the whole set of eigenvalues and the characterization of the eigenfunctions. Introducing the mean square radius r,

$$r^2 = \frac{1}{N} \sum_{i=2}^{N} \sum_{j=1}^{i-1} (x_i - x_j)^2, \quad (5.3)$$

we separate the problem into an angular part independent on $\omega(t)$ and a radial part of the form of Eq. (2.2), where the effective coupling constant, as in the previous case, depends on the eigenvalues of an operator acting only on the "angular" variables.

The problems (a) and (b) can then be solved by using the generating function method.

The raising operator method can be, in principle, applied to both problems, but the stationary problems are not yet solved by this method, owing to their intrinsic complications. The research for the complete algebra of the raising operators in case (b), in particular, for the three-body problem is in progress.

---

* Research partially supported by the Consiglio Nazionale delle Ricerche.

1 R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948); Phys. Rev. **84**, 108 (1951); J. Schwinger, *ibid.* **91**, 728 (1953); K. Husimi, Progr. Theoret. Phys. (Kyoto) **9**, 381 (1953); A. M. Dykhne, Zh. Eksp. Teor. Fiz. **38**, 570 (1960) [Sov. Phys. JETP **11**, 411 (1960)]; M. Kolsrud, Phys. Rev. **104**, 1186 (1956); R. M. Kulsrud, *ibid.* **106**, 205 (1957); F. Hetwek and A. Schluter, Z. Naturforschr. **12a**, 844 (1957); G. Backus, A. Lenard, and R. Zulsud, *ibid.* **15a**, 1007 (1960); S. Chandrasekhar, in *The Plasma in a Magnetic Field*, R. K. M. Landshoff, Ed. (Stanford U.P., Palo Alto, Calif., (1958), p. 3; M. Kolsrud, Kgl. Norske Videnskab. Selskabs, Forh., **31**, No. 5 (1958); A. Lenard, Ann. Phys. (N. Y.) **6**, 261 (1959); L. M. Garrido, Progr. Theoret. Phys. (Kyoto) **26**, 577 (1961); J. E. Littlewood, Ann. Phys. (N.Y.) **21**, 233 (1963); **29**, 13 (1964); H. R. Lewis, Phys. Rev. Letters **18**, 510 (1967); J. Math. Phys. **9**, 1976 (1968); H. R. Lewis and W. B. Reisenfeld, J. Math. Phys. **10**, 1458 (1969); B. Crosignani, P. Di Porto, and S. Solimeno, *ibid.* **10**, 1922 (1969); W. H. Louisell, A. Yariv, and E. Sigman, Phys. Rev. **124**, 1646 (1961); B. R. Mollow and R. J. Glauber *ibid.* **160**, 1076, 1097 (1967); B. R. Mollow, ibid. **162**, 1256 (1967); C. Marchioro, E. Presutti, and E. Scacciatelli, "Breaking of Coherence in a Monochromatic Photon Beam by an External Quadratic Time-Dependent Interaction" (unpublished).

2 V. S. Popov, A. M. Perelomov, Zh. Eksp. Theor. Fiz. **56**, 1375 (1969); **57**, 1684 (1969) [Sov. Phys. JETP **29**, 738 (1969); **30**, 911 (1970)].

3 E. Salusti and F. Zirilli, Nuovo Cimento Letters **4**, 999 (1970).

4 L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1958), Sec. 35.

5 F. Calogero, J. Math. Phys. **10**, 2191 (1969).

6 F. Calogero, J. Math. Phys. **12**, 419 (1971).

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 12, NUMBER 10 OCTOBER 1971

Relaxation of Local Thermal Deviations from Equilibrium

Gérard G. Emch* and Charles Radin

(Instituut voor Theoretische Fysica, Universiteit Nijmegen, The Netherlands)

(Received 31 March 1971)

We give precise conditions sufficient to guarantee that an infinite system will act as a thermal reservoir for any of its finite parts. In particular we show that these conditions are satisfied for the X-Y model. Further results on the ergodic behavior and general relaxation properties of the systems considered are also obtained directly from the C^*-algebraic methods used in the main body of the paper.

1. HEURISTIC STATEMENT OF THE PROBLEM

A physical system is ordinarily defined by the attribution of a Hamiltonian $H(\Omega)$ to every finite piece $\Sigma(\Omega)$ of it; here Ω denotes the spatial extension of $\Sigma(\Omega)$.

Given $\Sigma(\Omega)$, we cut out of it a piece $\Sigma(\Omega_0)$ and denote by $\Sigma(\Omega\backslash\Omega_0)$ the remaining part of $\Sigma(\Omega)$. One would presume, according to the phenomenological laws of thermodynamics, that if Ω is very large compared to Ω_0, then $\Sigma(\Omega\backslash\Omega_0)$ serves as a *thermal reservoir* for $\Sigma(\Omega_0)$. The traditional approach of statistical mechanics suggests that the mechanical description of this situation will be mathematically simpler (and thus more efficient) if the limit of large systems is taken. To this effect, we shall first suppose that $\Sigma(\Omega_0)$ is of *finite* extent and that $\Sigma(\Omega)$ is of *infinite* extent in all directions around Ω_0. We shall then discuss briefly the thermodynamical consequences of letting Ω_0 become very large.

To formulate more specifically the problem to which we want to address ourselves, we consider an initial situation where (i) the system $\Sigma(\Omega_0)$, cut off from the rest of $\Sigma(\Omega)$, is in the canonical equilibrium state ϕ_{Ω_0,β_0} corresponding to $H(\Omega_0)$ and the natural temperature β_0 and (ii) the system $\Sigma(\Omega)$ extends over the entire physical space and $\Sigma(\Omega\backslash\Omega_0)$, cut off from $\Sigma(\Omega_0)$, is in the state $\phi_{R,\beta}$ obtained as the thermodynamical limit (as Ω' becomes infinite) of the states $\phi_{\Omega'\backslash\Omega_0,\beta}$, which are defined as the canonical equilibrium states computed from $H(\Omega'\backslash\Omega_0)$ for the natural temperature $\beta \neq \beta_0$. One would then expect that this state of the composite system "relaxes" to the state $\phi_{S+R,\beta}$ obtained as the thermodynamical limit of the canonical equilibrium states $\phi_{\Omega',\beta}$ computed from $H(\Omega')$, these Hamiltonians taking into account the interactions between $\Sigma(\Omega_0)$ and $\Sigma(\Omega\backslash\Omega_0)$.

On a more modest level, one would at least expect that the time average (as well as the space average)

Downloaded 23 Mar 2004 to 146.6.139.170. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

of the initial state described above is equal to $\phi_{S+R,\beta}$.

If, furthermore, Ω_0 is taken to be large enough, one would expect that, *well inside* Ω_0, the system $\Sigma(\Omega_0)$ would, in the same sense as above, approach *its own* thermal equilibrium for the natural temperature β.

These expected properties are clearly of an ergodic character. Pathological cases where they would not be satisfied are likely to occur as well. The question to which we want to address ourselves is rather whether one can *prove* explicitly (i.e., without uncontrollable approximations or unwarranted statistical assumptions) that there are mechanical models that *do* exhibit these properties.

We shall present such a proof. Some related ergodic properties will be proven as well, along the principal line of argument. We shall then touch upon the question of an actual (i.e., pointwise in time) relaxation to equilibrium. The latter problem is that of ultimate physical interest. Some results are already known in this connection.[1,2] However, information on the latter type of problem is usually obtained through detailed computations which are restricted to the models analyzed (e.g., one makes use of the precise behavior of the excitation spectrum as the thermodynamical limit is taken). In contrast, the results presented in this paper are obtained by methods of a more general character, and as such, stand a better chance to extend further than the details specific to the models treated.

2. THE X-Y MODEL

We consider a one-dimensional lattice-spin system. To every site i in $\mathbb{Z}$ is attached a spin-$\frac{1}{2}$ particle σ_i, and hence a copy $\mathfrak{A}_i$ of the C^*-algebra of the 2×2 matrices. The observables attached to each finite region Ω in $\mathbb{Z}$ are therefore the Hermitian elements of the C^*-algebra $\mathfrak{A}_\Omega = \bigotimes_{i\in\Omega} \mathfrak{A}_i$, and the C^*-algebra of quasilocal observables on the infinite lattice is the C^*-inductive limit:

$$\mathfrak{A} = \overline{\bigcup_{\Omega\subset Z} \mathfrak{A}_\Omega}\,.$$

To every interval $[a,b] = \{i \in \mathbb{Z} \mid a \leq i \leq b\}$ of $\mathbb{Z}$ we attribute the Hamiltonian

$$H_{[a,b]} = -\sum_{a=1}^{b-1} (1+\zeta)\,\sigma_i^x \sigma_{i+1}^x + (1-\zeta)\,\sigma_i^y \sigma_{i+1}^y\,,$$

the time evolution $\alpha_{[a,b]}(t)$ defined by

$$\alpha_{[a,b]}(t)\,[A] = \exp(-\,iH_{[a,b]}t)A\,\exp(iH_{[a,b]}t) \quad \text{for all } A \in \mathfrak{A}_{[a,b]},$$

and the canonical equilibrium state $\phi_{[a,b]}$ defined by

$$\langle\phi_{[a,b]}\,;A\rangle = \mathrm{Tr}[\exp(-\,\beta H_{[a,b]})A]/\mathrm{Tr}[\exp(-\,\beta H_{[a,b]})] \quad \text{for all } A \in \mathfrak{A}_{[a,b]}.$$

From the fact that the interaction from which the Hamiltonian $H_{[a,b]}$ is built is of finite range, we know[3,4] that a time evolution and a Gibbs state can be naturally defined for the infinite system described by $\mathfrak{A}$. Specifically there exists a one-parameter, strongly continuous group of automorphisms $\alpha(t)$ of $\mathfrak{A}$ and a state ϕ on $\mathfrak{A}$ such that, for every finite Ω_0 in $\mathbb{Z}$ and $A \in \mathfrak{A}_{\Omega_0}$,

$$\lim_{a\to-\infty,\,b\to+\infty} \|\alpha(t)\,[A] - \alpha_{[a,b]}(t)\,[A]\| = 0,$$

$$\lim_{a\to-\infty,\,b\to+\infty} |\langle\phi;A\rangle - \langle\phi_{[a,b]}\,;A\rangle| = 0.$$

Furthermore, ϕ is uniformly clustering, i.e., for every $A \in \mathfrak{A}$ and $\epsilon > 0$ there exists a finite N such that

$$|\langle\phi;AB\rangle - \langle\phi;A\rangle\langle\phi;B\rangle| \leq \epsilon\,\|B\|$$

for all B in $\mathfrak{A}_{(\mathbb{Z}\setminus[-N,N])}$. Moreover, ϕ satisfies the KMS boundary condition with respect to $\alpha(t)$ (for definitions see Ref. 5). As a consequence[6] of the uniform clustering of ϕ, ϕ is[4] extremal with respect to the KMS condition; so ψ KMS and $\psi \leq \lambda\phi$ together imply $\psi = \phi$.

We now define the automorphism γ of $\mathfrak{A}$ by means of

$$\gamma[\sigma_i^z] = \sigma_i^z;\, \gamma[\sigma_i^x] = -\,\sigma_i^x;\, \gamma[\sigma_i^y] = -\,\sigma_i^y$$

for all i in $\mathbb{Z}$. Let us denote by $\mathfrak{A}_e$ the C^*-subalgebra of $\mathfrak{A}$ consisting of its "even" elements, i.e.,

$$\mathfrak{A}_e = \{A \in \mathfrak{A} \mid \gamma[A] = A\}.$$

Clearly $H_{[a,b]}$ belongs to $\mathfrak{A}_e$. As a consequence, γ commutes with each $\alpha_\Omega(t)$ and hence with $\{\alpha(t) \mid t \in \mathbb{R}\}$; also, ϕ is even (i.e., $\langle\phi;\gamma[A]\rangle = \langle\phi;A\rangle$ for all $A \in \mathfrak{A}$). Hence ϕ is determined by its restriction ϕ_e to $\mathfrak{A}_e$, and $\alpha(t)$ maps $\mathfrak{A}_e$ onto itself. Let $\alpha_e(t)$ denote the restriction of $\alpha(t)$ to $\mathfrak{A}_e$. From the corresponding properties of ϕ, one concludes immediately that ϕ_e is KMS with respect to $\alpha_e(t)$ and is uniformly clustering on $\mathfrak{A}_e$. One then checks that the arguments of Ref. 6, Properties 2.2 and 2.3, go through for $\mathfrak{A}_e$, and hence one concludes that ϕ_e is extremal KMS on $\mathfrak{A}_e$.

The arguments developed so far apply to any one-dimensional lattice-spin system with even, finite-range, lattice-invariant interaction. We now use a specific ergodic property of the X-Y model. As is well known, the Jordan–Wigner transformation[7] brings $H_{[a,b]}$ into a form which is quadratic in Fermi operators. It can be seen that in this form the interaction satisfies the assumptions of theorem II in Ref. 8, so that $\{\alpha_e(t) \mid t \in \mathbb{R}\}$ acts as a strongly asymptotically Abelian group of automorphisms of $\mathfrak{A}_e$, i.e.,

$$\lim_{|t|\to\infty} \|[A,\alpha_e(t)\,[B]]\| = 0 \quad \text{for all} \quad A,B \text{ in } \mathfrak{A}_e.$$

Therefore, if η is any invariant mean on $\mathbb{R}$, we have

$$\eta\langle\phi_e\,;C^*[A,\alpha_e(\cdot)[B]]C\rangle = 0$$

Downloaded 23 Mar 2004 to 146.6.139.170. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

for all A, B and C in $\mathfrak{A}_e$. Consequently,[9] the state ϕ_e on $\mathfrak{A}_e$ is not only extremal KMS, but also extremal time invariant. Hence $\psi_e \leq \lambda\phi_e$ and ψ_e time invariant together imply $\psi_e = \phi_e$. In particular, if ψ is even on $\mathfrak{A}$, time invariant, and satisfies $\psi \leq \lambda\phi$, we can conclude that $\psi = \phi$, a fact that we shall use repeatedly in the next section.

3. ERGODIC BEHAVIOR

Let $[c,d]$ be a finite interval in $\mathbb{Z}$. For each finite interval $[a,b]$ in $\mathbb{Z}$ such that $a < c-1$ and $b > d+1$, we define

$$H^c_{[a,b]} = H_{[a,c-1]} + H_{[c,d]} + H_{[d+1,b]} = H_{[a,b]} + V,$$

with V independent of a and b. We then define the state $\phi^c_{[a,b]}$ on $\mathfrak{A}_{[a,b]}$ by

$$\langle \phi^c_{[a,b]}; A\rangle = \mathrm{Tr}[\exp(-\beta H^c_{[a,b]})A]/\mathrm{Tr}(-\beta H^c_{[a,b]})$$

Using Araki's proofs,[4] it has been shown[10] that a state ϕ^c exists on $\mathfrak{A}$ such that

$$\lim_{a\to-\infty,\ b\to+\infty} |\langle \phi^c; A\rangle - \langle \phi^c_{[a,b]}; A\rangle| = 0$$

for all A in $\mathfrak{A}_{\Omega_0}$ and all finite Ω_0. Hence ϕ^c can be interpreted as the Gibbs state corresponding to the modified infinite chain obtained by cutting off the interaction between $c-1$ and c and between d and $d+1$. Moreover, there exists[10] a real constant λ such that $\phi^c \leq \lambda\phi$. Let us now write

$$\mathfrak{A}_S = \mathfrak{A}_{[c,d]}, \quad \mathfrak{A}_R = \mathfrak{A}_{\mathbb{Z}\setminus[c,d]}.$$

Since $\mathfrak{A}$ is isomorphic[11] to $\mathfrak{A}_S \otimes \mathfrak{A}_R$ and since ϕ^c_R(resp. ϕ^c_S), defined as the restriction of ϕ^c to $\mathfrak{A}_R$(resp. $\mathfrak{A}_S$), is the Gibbs state of the system $\Sigma(\mathbb{Z}\setminus[c,d])$(resp. $\Sigma([c,d])$) for the temperature β, we have $\phi^c = \phi^c_S \otimes \phi^c_R$. Since ϕ^c_S is faithful on $\mathfrak{A}_S$ (i.e., $\langle \phi^c_S; A^*A\rangle = 0$ implies $A = 0$) and since $\mathfrak{A}_S$ is finite dimensional, there exists a real constant λ_S depending only on ϕ^c_S such that $\psi_S \leq \lambda_S\phi^c_S$ for all states ψ_S on $\mathfrak{A}_S$. It then follows easily that

$$\psi \equiv \psi_S \otimes \phi^c_R \leq \lambda_S\phi^c \leq \lambda_S\lambda\phi$$

for all ψ_S on $\mathfrak{A}_S$. Since ϕ is invariant with respect to $\{\alpha(t)|t\in\mathbb{R}\}$, we have further that, for any invariant mean η on $\mathbb{R}$, the state $\eta\psi$ defined by

$$\langle \eta\psi; A\rangle = \eta\langle \psi; \alpha(\cdot)[A]\rangle$$

also satisfies $\eta\psi \leq \lambda_S\lambda\phi$. In particular, $(\eta\psi)_e \leq \lambda_S\lambda\phi_e$. Since ϕ_e is extremal time invariant, $(\eta\psi)_e = \phi_e$. Now suppose that ψ_S is even on $\mathfrak{A}_S$ and therefore that ψ is even on $\mathfrak{A}$. Since γ commutes with $\{\alpha(t)|t\in\mathbb{R}\}$, $\eta\psi$ is also even. We have thus proven that, for any even state ψ_S on $\mathfrak{A}_S$ and any invariant mean η on $\mathbb{R}$,

$$\eta(\psi_S \otimes \phi^c_R) = \phi$$

as states on the whole algebra $\mathfrak{A}$. Incidentally, $\langle \eta(\psi_S \otimes \phi^c_R); A\rangle$ can be computed to be

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \langle \psi_S \otimes \phi^c_R; \alpha(t)[A]\rangle dt.$$

We emphasize in particular that for any natural temperature β_0, the corresponding canonical equilibrium state ϕ^c_{S,β_0} on $\mathfrak{A}_S$ is even. Consequently, for any natural temperatures β_0 and β, the time average of the initial state $\phi^c_{\beta_0,\beta} \equiv \phi^c_{S,\beta_0} \otimes \phi^c_{R,\beta}$ on the cut system $S+R$ is equal to the canonical equilibrium state ϕ_β of the joint system for the temperature β, independently of β_0. (Since ϕ_β is[4] extremal lattice invariant, the same result using space averages is easy to prove.)

Hence the first ergodic property mentioned in Sec. 1 is given a precise meaning, and is proven to hold for the model considered: The time average (and space average) of $\phi^c_{\beta_0,\beta}$ coincides with ϕ_β for all finite temperatures β_0 and β.

It has been shown[10] that the Gibbs state ψ, corresponding to a situation where $H_{[a,b]}$ is modified by a certain type of perturbation, exists and satisfies $\psi \leq \lambda\phi$. This is in particular the case for the X-Y model and any local, fixed perturbation. If these perturbations are, moreover, even (such as

$$V = -\sum_{i\in[c,d]} B_i\sigma^z_i$$

or

$$\sum_{i\in[c,d]} [(1+\zeta_i)\sigma^x_i\sigma^x_{i+1} + (1-\zeta_i)\sigma^y_i\sigma^y_{i+1}]),$$

and switched off at $t=0$, then we notice that an argument analogous to that developed above shows that the time average over positive times (as well as the space average) of the corresponding Gibbs states ψ coincides with ϕ.

We now turn to the case where Ω_0 can be taken to be very large (but still finite). Let Ω_1 be any finite interval in $\mathbb{Z}$. We know that on $\mathfrak{A}$, and hence *a fortiori* on $\mathfrak{A}_{\Omega_1}$, ϕ_Ω converges to ϕ in the weak*-topology as Ω tends to infinity. Since Ω_1 is finite, $\mathfrak{A}^*_{\Omega_1}$ is a finite-dimensional linear space, and so its weak*- and norm-topologies coincide. Consequently, given $\epsilon > 0$ and Ω_1, there exists a finite $\Omega_0 \supseteq \Omega_1$ such that for all $\Omega \supseteq \Omega_0$

$$|\langle \eta\psi; A\rangle - \langle \phi_\Omega; A\rangle| \leq \epsilon\,\|A\|$$

for all A in $\mathfrak{A}_{\Omega_1}$ and all states ψ considered above. Hence we can conclude that well inside Ω_0 (namely in Ω_1) $\eta\psi$ is as close as one wants to the canonical equilibrium of $\Sigma(\Omega_0)$ for the temperature β.

4. RELAXATION PROPERTIES

To prove $\eta\psi = \phi$, we used a very much weakened form of the strong asymptotic Abelian character of the action of $\{\alpha_e(t)|t\in\mathbf{R}\}$ on $\mathfrak{A}_e$, namely

$$\eta\langle \phi_e; C^*[A, \alpha_e(\cdot)[B]]C\rangle = 0$$

for all A, B and C in $\mathfrak{A}_e$, and for the restriction

Downloaded 23 Mar 2004 to 146.6.139.170. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

ϕ_e of the Gibbs state ϕ to $\mathfrak{A}_e$. The strong asymptotic Abelian character of the evolution is evidently much more stringent a condition (and, as such, is less likely to hold in general). It nevertheless does hold for the X-Y model, so that we can considerably strengthen the results of the previous section. These further results will be obtained by means of a generalization of an argument published by Kastler,[12] which we now describe.

For any representation π of $\mathfrak{A}_e$ in some Hilbert space $\mathcal{H}$, we denote by $\mathfrak{Z}_\pi$ the von Neumann algebra $\pi(\mathfrak{A}_e)'' \cap \pi(\mathfrak{A}_e)'$. Its commutant $\mathfrak{Z}_\pi'$ is the von Neumann algebra generated by $\pi(\mathfrak{A}_e)''$ and $\pi(\mathfrak{A}_e)'$, and hence by $\pi(\mathfrak{A}_e)$ and $\pi(\mathfrak{A}_e)'$. Consequently, given any element B in $\mathfrak{Z}_\pi'$, any $\epsilon > 0$, and any Ψ_i $(i = 1, 2)$ in $\mathcal{H}$, there exists $B_0 = \sum_{i=1}^n \pi(A_k)B_k$, with A_k in $\mathfrak{A}_e$, B_k in $\pi(\mathfrak{A}_e)'$ and n finite, such that

$$\|(B - B_0)\Psi_i\| \leq \epsilon, \quad i = 1, 2.$$

Using now the strong asymptotic Abelian character of $\{\alpha_e(t) \mid t \in \mathbb{R}\}$ on $\mathfrak{A}_e$, we see that for each A in $\mathfrak{A}_e$ there exists a positive number T such that

$$\|[B_0, \pi(\alpha_e(t)[A])]\| \leq \epsilon$$

for all t with $|t| \geq T$. From these inequalities, we conclude that

$$|(\Psi_1, [B, \pi(\alpha_e(t)[A])]\Psi_2)| \leq \epsilon\ (2\|A\| + 1)\ \|\Psi_1\|\ \|\Psi_2\|,$$

which is to say that, for any B in $\mathfrak{Z}_\pi'$ and A in $\mathfrak{A}_e$, $[B, \pi(\alpha_e(t)[A])]$ tends to zero in the weak operator topology as $|t|$ tends to infinity. Let now π be primary and $\|\Psi_1\| = \|\Psi_2\| = 1$. We form the states ψ_i, $i = 1, 2$, defined on $\mathfrak{A}_e$ by

$$\langle \psi_i; A\rangle = (\Psi_i, \pi(A)\Psi_i)$$

and notice that there exists a unitary operator U in $\mathfrak{B}(\mathcal{H}) = \mathfrak{Z}_\pi'$ such that $\Psi_2 = U\Psi_1$ and hence

$$\langle \psi_1; \alpha_e(t)[A]\rangle - \langle \psi_2; \alpha_e(t)[A]\rangle$$
$$= (\Psi_2, [U, \pi(\alpha_e(t)[A])]\Psi_1).$$

We conclude from this that for any twovector states ψ_1 and ψ_2 on $\pi(\mathfrak{A}_e)$, which is assumed to be primary, we have

$$\lim_{|t| \to \infty} |\langle \psi_1; \alpha_e(t)[A]\rangle - \langle \psi_2; \alpha_e(t)[A]\rangle| = 0$$

for all A in $\mathfrak{A}_e$.

In particular, if ϕ_e is extremal KMS, if π is the representation of $\mathfrak{A}_e$ associated to ϕ_e (and is thus primary), and if $\psi_e \leq \lambda\phi_e$, then

$$\lim_{|t| \to \infty} \langle \psi_e; \alpha_e(t)[A]\rangle = \langle \phi_e; A\rangle$$

for all A in $\mathfrak{A}_e$.

Thus for the states ψ considered in the preceding section, which are even, we have not only $\eta\psi = \phi$, but actually

$$\lim_{|t| \to \infty} \langle \psi; \alpha(t)[A]\rangle = \langle \phi; A\rangle$$

for all A in $\mathfrak{A}$, i.e., these states actually relax to the canonical equilibrium state ϕ.

5. CONCLUSION

We have provided a positive answer to the question to which we addressed ourselves: Thermal baths can indeed work in the sense set forth in Sec. 1. We also mention that the methods used to prove this result apply to some other situations. Indeed, the essential ingredients are (a) the time-evolution acts in an asymptotically Abelian manner on the even subalgebra $\mathfrak{A}_e$ and (b) the initial state $\phi_{\beta_0,\beta}$ satisfies the following two properties: (i) $\phi_{\beta_0,\beta}$ is even, (ii) $\phi^c_{\beta_0,\beta} \leq c\phi_\beta$ for some positive number c. These conditions also hold[10] when (a′) a uniform magnetic field B in the z direction is added to the X-Y Hamiltonian and (b′) the initial state $\phi_\beta^{B'}$ is the canonical equilibrium state corresponding to β and $B'(i) = B(i)$ for all but a finite number of sites $i \in \mathbb{Z}$. Hence $\phi_\beta^{B'}(t)$ relaxes to the equilibrium state ϕ_B^B, in agreement with results obtained by Abraham *et al.*[1] and Tjon.[2] The exact solubility of the X-Y model also enabled these authors to analyze in detail the excitation spectrum as the thermodynamic limit is taken; they then use this analysis to compute the *rate* at which equilibrium is reached. The purpose of the present paper was, rather, to emphasize some immediate consequences of general ergodic properties, which we illustrate with the X-Y model.

ACKNOWLEDGMENTS

The authors would like to thank Dr. G. Gallavotti for discussions on his work[1] and for showing to us the preprint of Dr. H. Narnhofer.[8]

* On leave of absence from the Department of Physics and Astronomy of the University of Rochester, Rochester, New York.

1 D. B. Abraham, E. Baruch, G. Gallavotti, and A. Martin-Löf, Phys. Rev. Letters **25**, 1449 (1970).
2 J. A. Tjon, Phys. Rev. B **2**, 2411 (1970).
3 D. R. Robinson, Commun. Math. Phys. **7**, 337 (1968).
4 H. Araki, Commun. Math. Phys. **14**, 120 (1969).
5 R. Haag, N. M. Hugenholtz, and M. Winnink, Commun. Math. Phys. **5**, 215 (1967).
6 O. E. Lanford and D. Ruelle, Commun. Math. **13**, 194 (1969).
7 E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961).
8 H. Narnhofer, Preprint IHES, 1970 (unpublished).
9 H. Araki and H. Miyata, Publ. Res. Inst. Math. Sci. Kyoto Univ., Ser. A, 4, 373 (1968).
10 C. Radin, preprint, Nijmegen, 1971 (unpublished).
11 A. Guichardet, Ann. Sci. Ecole. Norm. Sup. **83**, 1 (1966).
12 D. Kastler, in *Cargèse Lectures in Theoretical Physics*, edited by F. Lurçat (Gordon and Breach, New York, 1967).

Downloaded 23 Mar 2004 to 146.6.139.170. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp