Statistical Symmetry

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1 Introduction

After a short discussion of traditional ideas about the symmetries of patterns in the plane and space, we introduce a slight extension of the traditional notions. This new "statistical symmetry" is being used in various ways, from modeling quasicrystals to constructing new forms of graph paper.

2 Periodic Tilings

It is easy to determine the symmetries of a regular n-gon, for instance a regular octagon as in Figure 1 Such a figure coincides with itself after a rotation about its center by the angle $2\pi/8$ (radians), or any integer multiple of that angle such as $4\pi/8$ or $6\pi/8$. The figure also has the symmetry of reflection about any of the four lines which bisect a pair of opposite sides, or the four which join opposite vertices. (The set of all symmetries of the octagon is called the dihedral group D_8.)

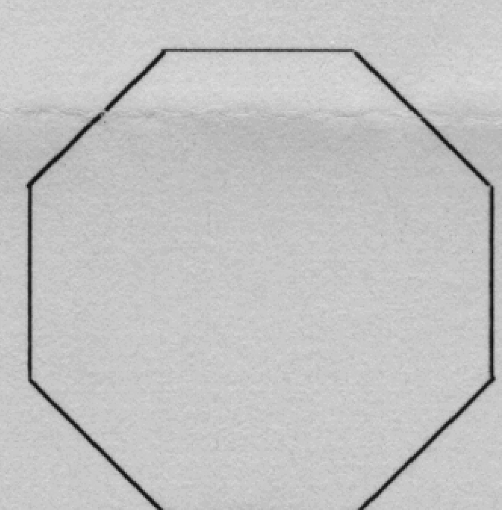

Figure 1: Octagon

More interesting are the symmetries of patterns which fill the whole plane, for example the "bathroom tiling" shown in part in Figure 2, made of regular hexagons. It is an interesting exercise to work out the symmetries of this pattern, which now includes some translations.

There has been a great deal of study of patterns in the plane such as that of Figure 2, that is patterns in which there is some "unit cell" such that the whole pattern is obtained by translating the cell by integer multiples of vectors in two different directions. (We will call such patterns "periodic". For the hexagonal tiling we can use a hexagon for the unit cell and can take as vectors those perpendicular to two opposite sides of one of the hexagons, and $\sqrt{3}$ times their length.) It has been shown that there are precisely 17 different sorts of periodic patterns in the plane; that is, there are precisely 17 different sets of symmetries that can occur for periodic patterns. These 17 sets are called "the wallpaper groups". (As a general reference see [1].)

3 Aperiodic Tilings

There is a simple method for producing periodic tilings, namely first make a cell and then "grow" a larger and larger pattern by repeatedly adding on certain translations of the cell. We now con-

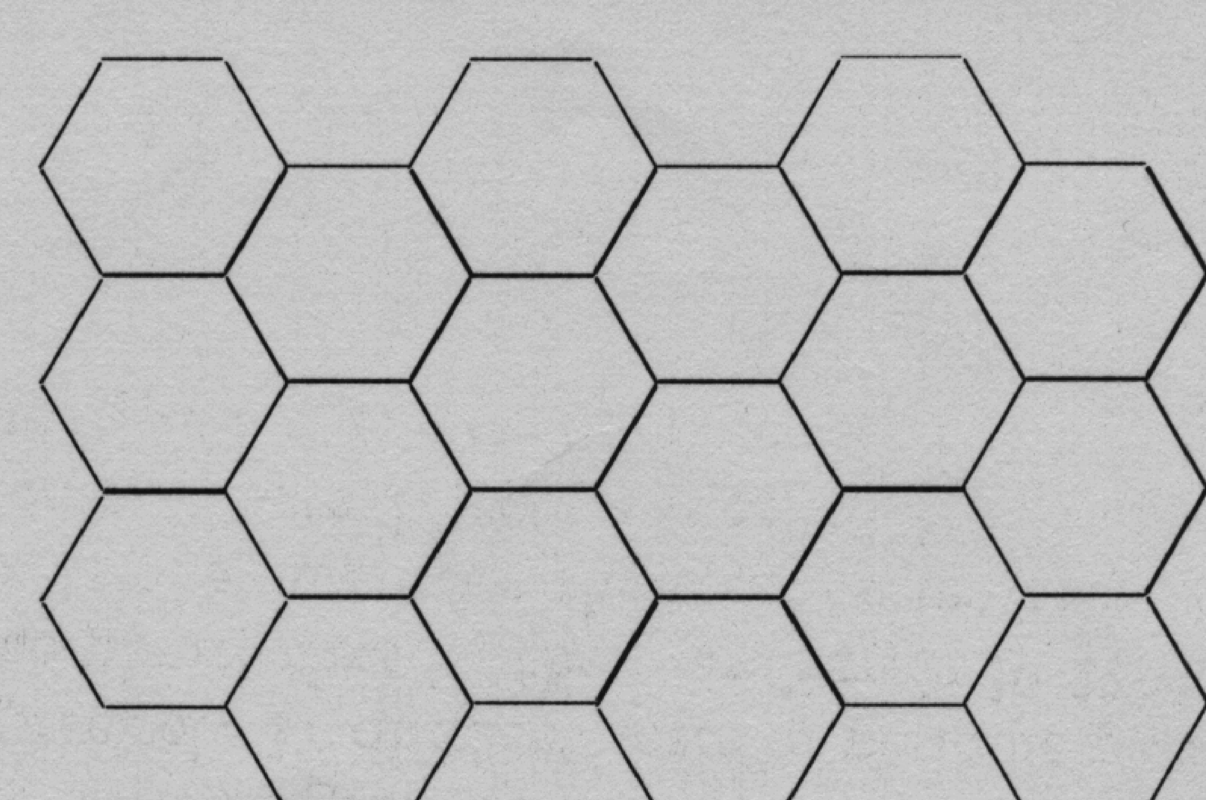

Figure 2: Tiling with Hexagons

*Research supported in part by NSF Grant No. DMS-9304269 and Texas ARP Grant 003658-113.

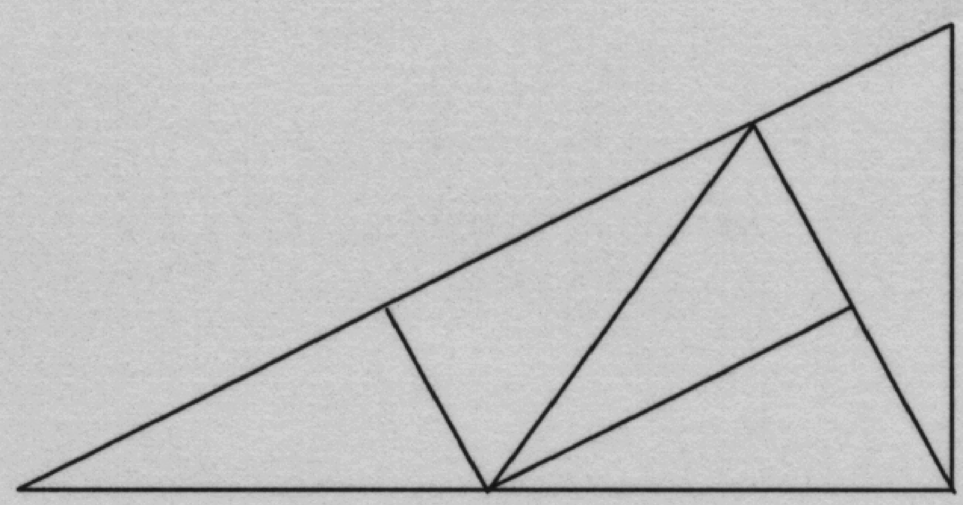

Figure 3: Hierarchy

sider patterns made by a different sort of algorithm, called "hierarchical" (or fractal). An interesting example is produced by starting with a right triangle with legs of length one and two (and hypotenuse $\sqrt{5}$). Divide it into five smaller triangles as in Figure 3, noting that the five smaller triangles are congruent to each other and similar to the original large triangle, and then expand the collection of triangles by stretching in all directions by a factor $\sqrt{5}$, so again the triangles each have legs of size one and two.

Repeat this process in each of the five triangles, obtaining now twenty-five triangles each congruent to the one we started with. If you do this process five times you get a pattern like Figure 4. If you keep repeating this algorithm forever you get a pattern filling the plane, of which a part appears in Figure 5. (If you look carefully at the twenty-five triangles after the second expansion, you will find one in the interior with edges parallel to the edges of the original triangle. Therefore if the expansions of our process are always taken about an appropriate point in this interior triangle, each two applications of the process amounts to simply adding more triangles around the original, and shows more explicitly in what way the process leads to a tiling of the plane.) Such a tiling of the plane is called a "pinwheel" [3].

There are better known tilings of the plane, created by Roger Penrose, made out of two shapes with a similar hierarchical growth algorithm; Figure 6 contains part of such a "Penrose tiling". For more on Penrose tilings, including the hierarchical algorithm, see [1].

These pinwheel and Penrose patterns and others like them, all of which we will call "hierarchical", have symmetry properties different from the periodic patterns considered above; this new kind of symmetry is the main focus of this article. To be specific, we claim that in some appropriate sense a Penrose tiling is symmetric under rotation by the angle $2\pi/10$ about any point, and a pinwheel tiling is symmetric under rotation by any angle about any

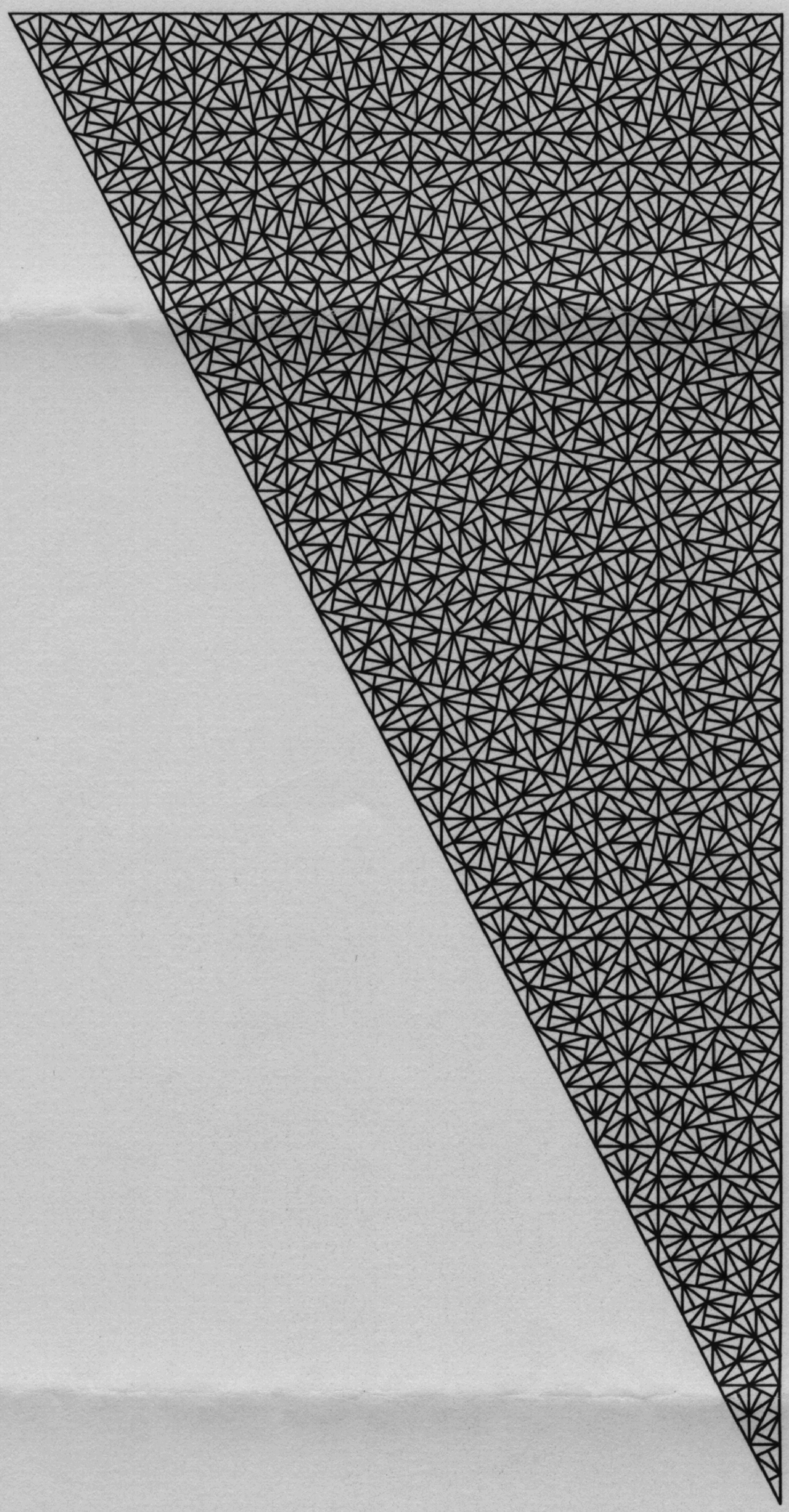

Figure 4: Result of 5 Subdivisions

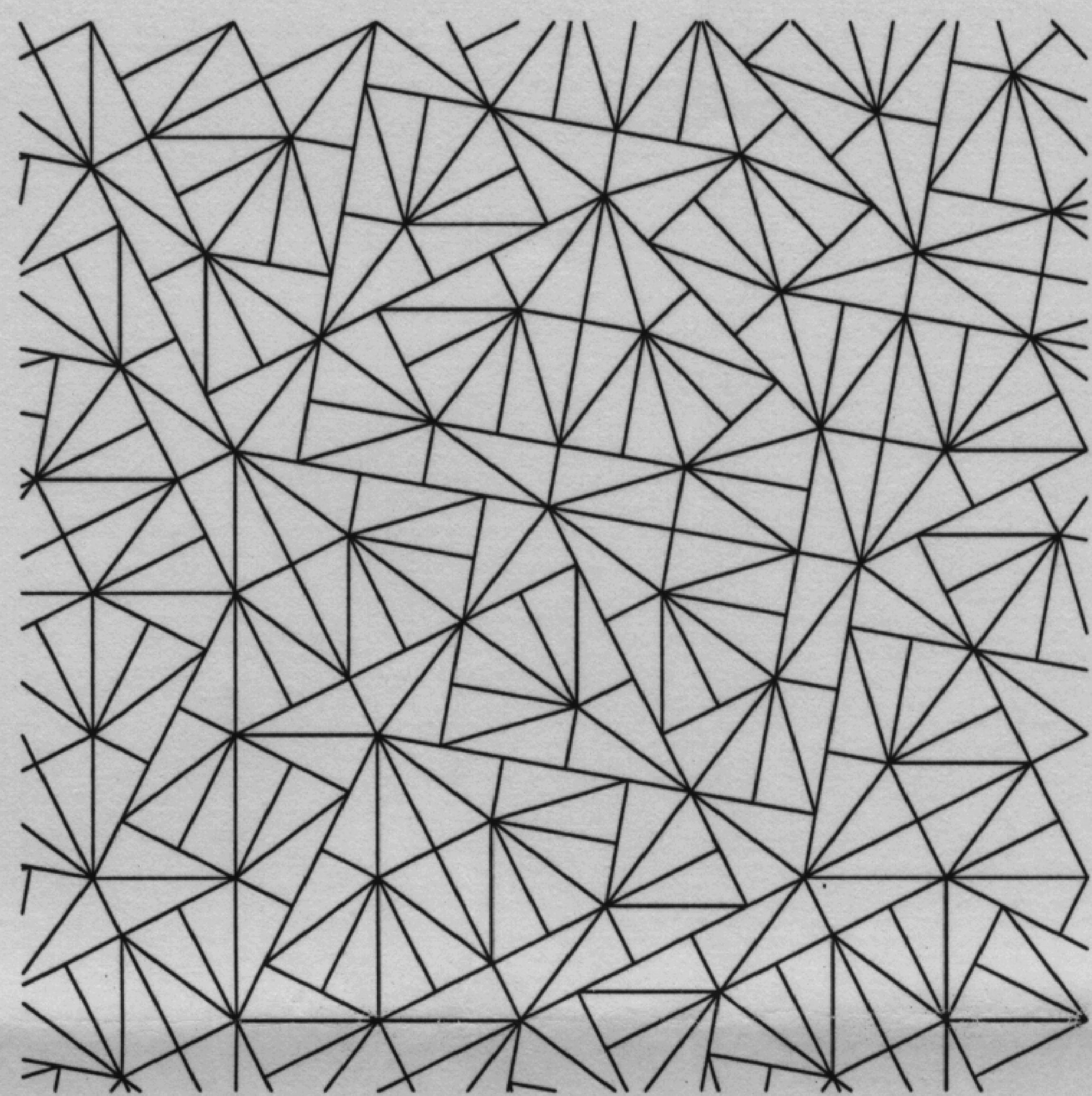

Figure 5: Pinwheel Tiling

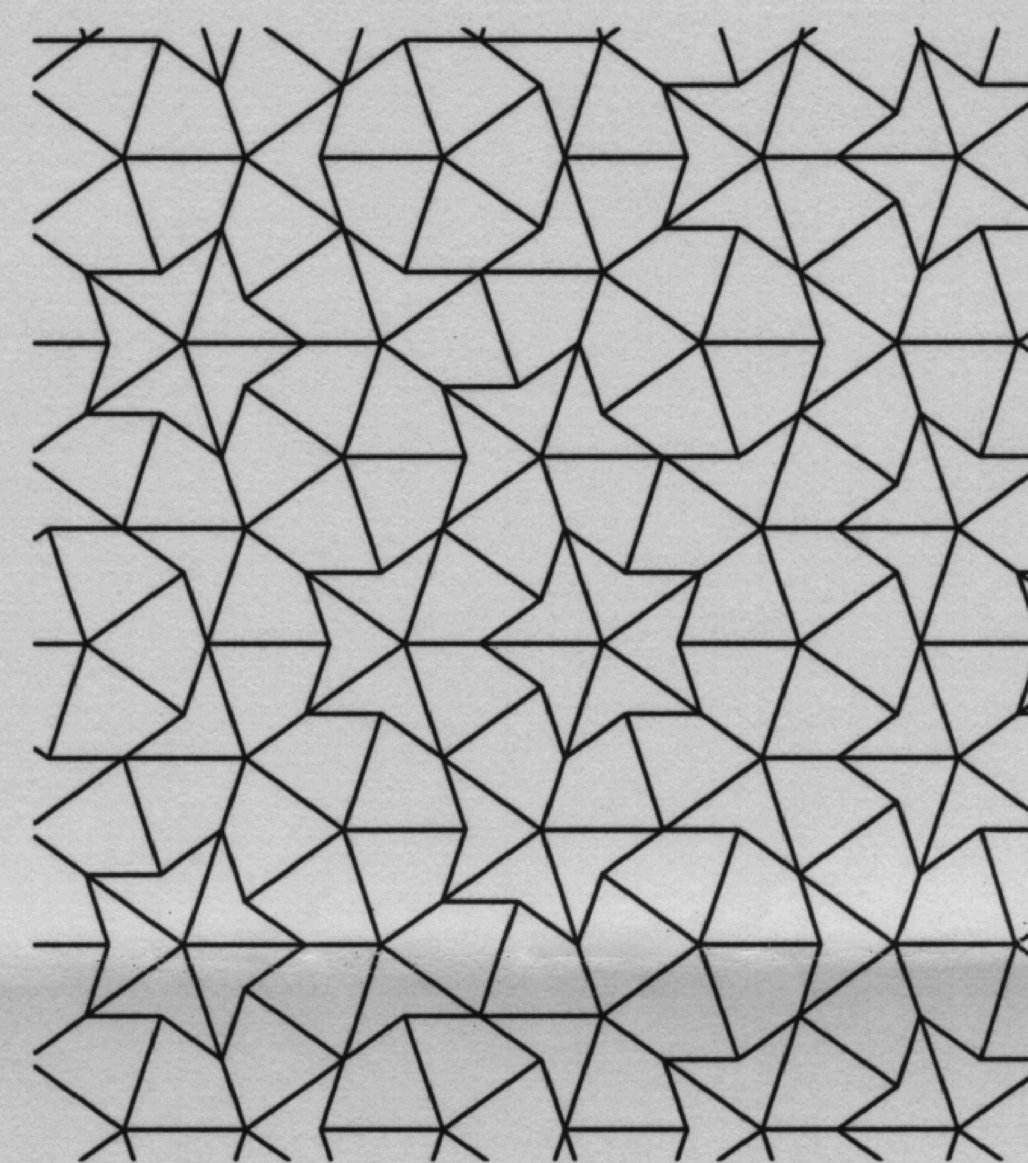

Figure 6: Penrose Tiling

point. Clearly we have some explaining to do!

We begin our explanation by introducing a notion of "frequencies" for patterns. Consider the Penrose tiling and fix some finite portion f of it, say the three tiles with heavy outline in Figure 7. Fix also some arc a of a unit circle, say the arc containing all angles from 0 to $\pi/20$ inclusive. Now take any large circle C in the tiling, of radius R and centered at a point P, and count the number of times the finite pattern f appears (completely) in the circle, where we only count occurrences of f for which the orientation is rotated from that of the original by some angle in a.

That means we count occurrences that have the original orientation as in Figure 7, or such an orientation rotated counterclockwise by any angle up to $\pi/20$. We claim that if one makes such a count, getting the number N, then one will find that $N/\pi R^2$ is approximately $\sqrt{5}-2$, and this approximation will be better and better the larger the size of the circle used. (We are assuming for definiteness that the area of the kite-like tile is 1.) We say that associated with the finite pattern f and arc a there is a "frequency" $F(f,a)$ which in this case is $\sqrt{5}-2$. Of course this is just one example, and we are really claiming that there are analogous frequencies for each finite pattern f and arc a.

4 Statistical Symmetry

We are now ready to clarify the new "statistical" notion of symmetry. We say the Penrose tilings have the statistical symmetry of rotation by $2\pi/10$ because if such a tiling is rotated about any point by $2\pi/10$ none of its frequencies change. Note that in a Penrose tiling all the edges of the tiles are parallel to one of five lines, and the tiles only appear in ten different orientations. The statistical symmetry we have claimed means that the tiles appear equally often in each of these orientations; more generally, not just individual kites and darts but any finite pattern in the tiling has this property [3].

A more surprising fact is that for a pinwheel tiling, roughly speaking each finite part f appears equally often in all directions! In particular picking some elementary triangle in a pinwheel tiling, about one eighth of any large region is filled with copies of that triangle with orientation within $\pm 2\pi/8$ of the original.

This notion of symmetry is weaker than the usual one, in that if a pattern is invariant in the ordinary sense under a rotation by angle α about some point (as is the hexagonal tiling under rotation by $2\pi/3$ about any vertex), then it is also statistically symmetric under rotation by α, while from the above examples the converse is not true in general. At this point we should give some indication why this notion of statistical symmetry is of interest.

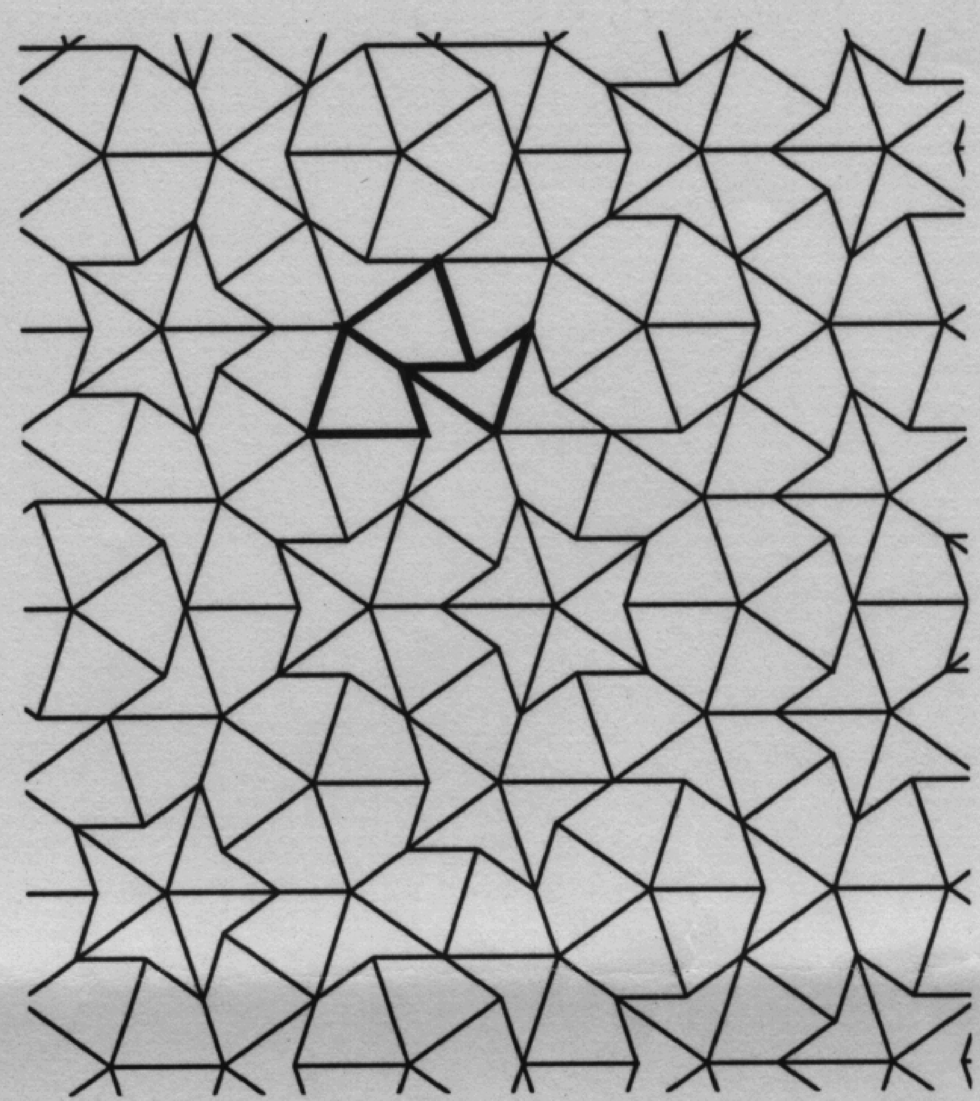

Figure 7: Tiling Frequencies

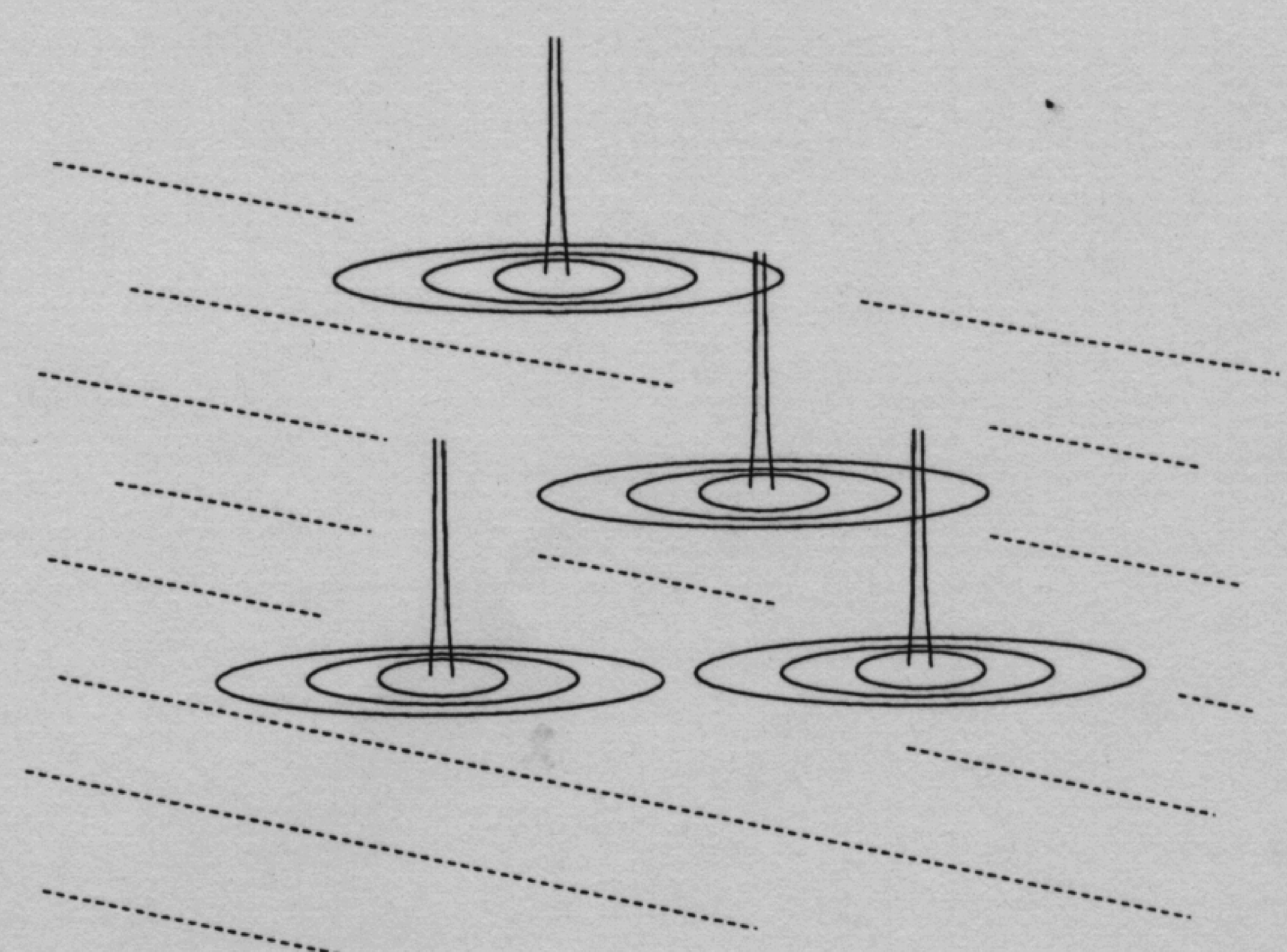

Figure 8: Expanding Circles off Trees

5 Diffraction Patterns

Imagine the surface of a large lake, out of which is growing a collection of isolated, thin trees. If a wind created a sequence of parallel waves on the lake surface, as these waves struck each tree circular rings of waves would appear to come out of the tree; see Figure 8.

The expanding rings from different trees eventually overlap and "interfere" with each other, producing complicated patterns of reinforcement, both positive (where the heights of two intersecting rings-waves are both high or both low) and negative (where one height is high and the other is low). This is similar to a mechanism which allows physicists to investigate the internal (atomic) structure

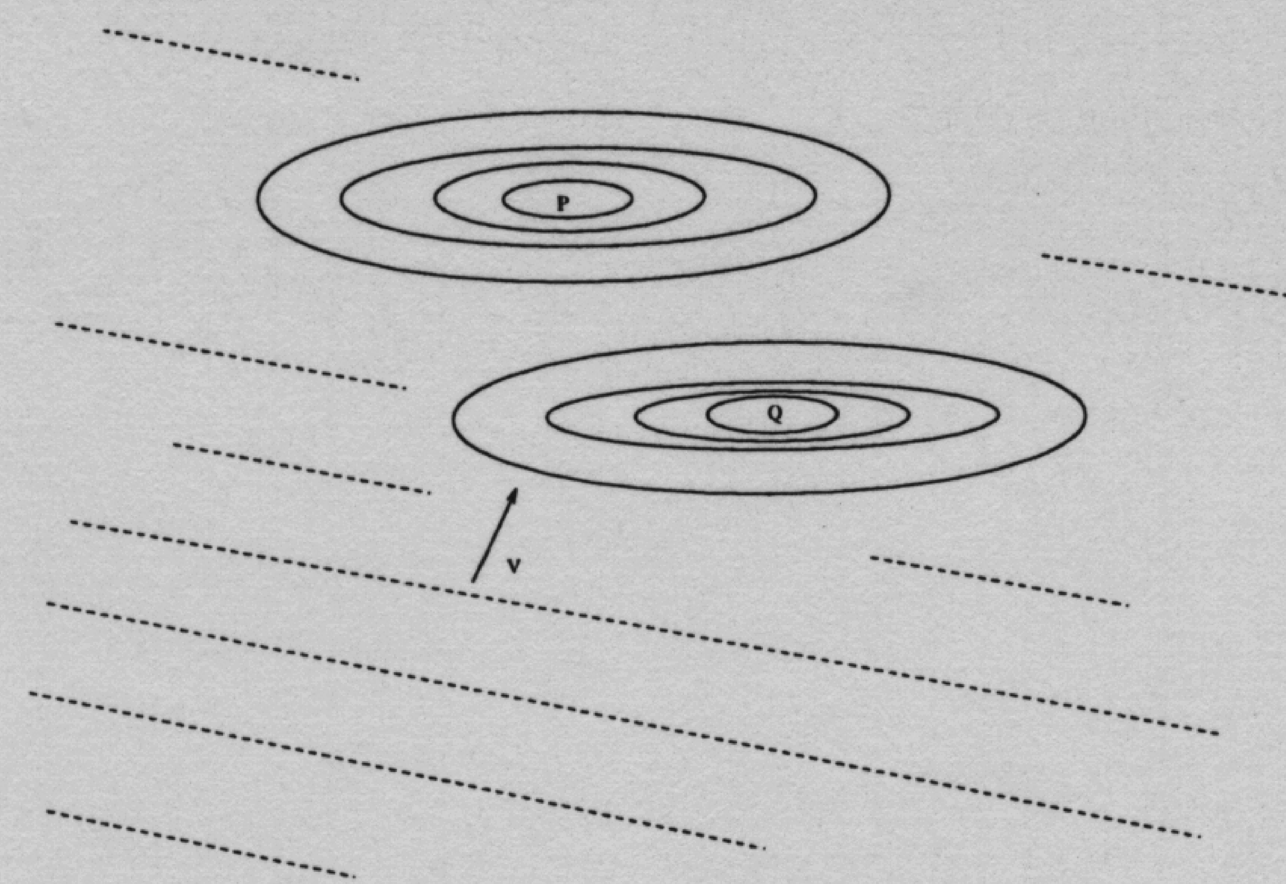

Figure 9: Expanding Spheres off Atoms

of solids. To "visualize" their internal structure, parallel plane waves of very short wavelength (such as X-rays) are sent into the material, creating expanding spheres of waves issuing from each atom; the expanding spheres from different atoms interfere, and the patterns of interference are recorded on appropriate film. (The result is called a "diffraction pattern" for the material.)

We analyze the situation as follows. Imagine we have two atoms in the target material, at positions P and Q, and there are parallel waves hitting these atoms, the waves moving in the direction of the unit vector V; see Figures 9 and 10. (We use two dimensional drawings for simplicity, with lines for the waves instead of planes, and circles instead of spheres.) The atoms emit expanding spheres of outgoing waves. At a point far away these spheres appear as plane waves, and we assume we will measure the size of the resulting waves at such a point in the direction of some unit vector W from the atoms. Comparing the distance traveled by the waves hitting the two atoms, we see there is an extra distance of size $V\cdot(P-Q)+(-W)\cdot(P-Q) = (V-W)\cdot(P-Q)$ for the wave going through P, as in Figure 10. Measuring distances by the natural scale of the wavelength of the waves, and recalling that waves reinforce optimally when corresponding maxima meet, we see that for maximum reinforcement this extra distance should be a multiple of a wavelength, which occurs only in special directions W.

So far we have considered how the spherical waves emanating from two atoms interfere. Now consider the total diffraction from the target, the sum of the contributions of all pairs of atoms. If the atoms in the target are arranged in a periodic array then there will be many copies of each vector of the sort $P-Q$ and one can then calculate precisely which directions W will show a large effect for given in-

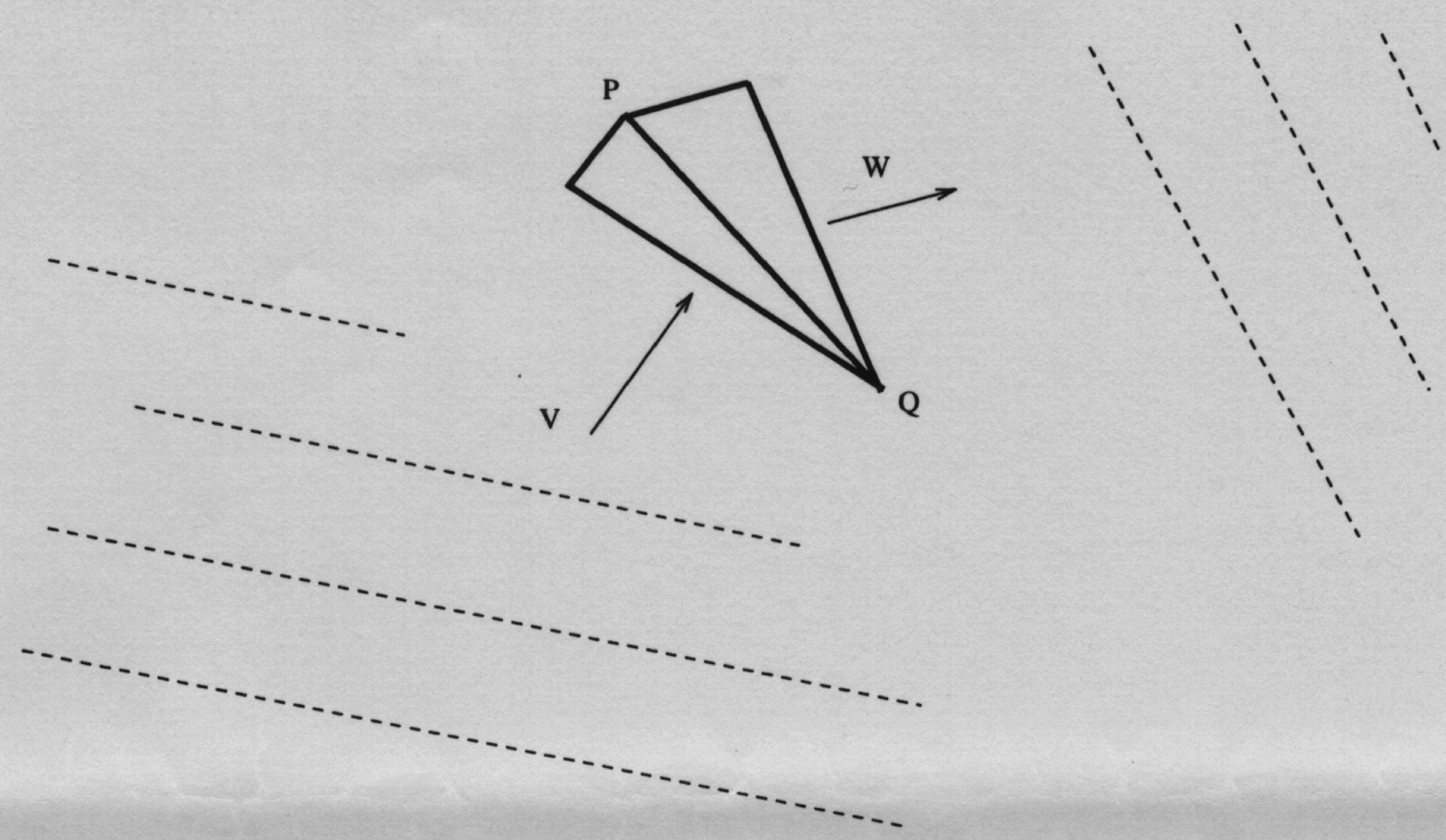

Figure 10: Diffraction

coming beam direction V. (Each individual X-ray picture samples a small range of directions W; the bright spots are due to positive reinforcement from many sources (atoms). For a discussion of diffraction see [2, 5].)

The relevance of all this to statistical symmetry is the following. To get bright spots in a diffraction pattern of a material does not require that the material be a periodic crystal, all one needs is a lot of vectors $P-Q$ creating the same contribution. That is, the effect is a consequence of the appropriate statistical nature of the locations of the atoms in the material, namely the frequencies of finite clusters of atoms. And in particular, a diffraction pattern will show an ordinary rotational symmetry if the frequencies of finite sets of atomic locations have the corresponding statistical symmetry. (As noted above this is automatic if the configuration of atoms itself has the rotational symmetry in the ordinary sense, but this is not necessary to achieve statistical symmetry.)

About ten years ago certain metallic alloys, called quasicrystals, were discovered which exhibit unusual diffraction patterns, unusual in that they have rotational symmetries never seen before and which in fact were known to be impossible for any ordinary solid. The location of the atoms in an ordinary solid is periodic in the sense discussed above, and it has been known for many years what sort of symmetries could be produced by diffraction of any such periodic pattern. In an attempt to understand what sort of atomic structure could be producing these unusual diffraction patterns, physicists used three dimensional versions of Penrose tilings, with what we would now call their statistical symmetry of rotation by $2\pi/10$, and showed that they could reproduce the unusual diffraction patterns with such models; see [5]. In this way the statistical symmetry of tilings can help us understand previously unknown atomic structures of solids. And furthermore, the pinwheel with its complete rotational symmetry suggests that there are even wilder structures possible for the atoms in solids.

6 Conclusion

Although we were led to the pinwheel to understand the structure of materials, now that we have it before us we find that it has other uses. For instance, imagine a piece of graph paper, made up of many little squares all lined up. Such a pattern can be used as a model of a sort of (planar) discrete world, where you can only travel along the lines. (This is sometimes called a "taxicab geometry", from analogy with the way a taxi travels along city streets.) There are all sorts of things one can imagine investigating about such a world. For a mathematician the isoperimetric problem comes to mind. That is, imagine we wanted to enclose a large region in such a world with the smallest possible amount of perimeter ("fencing material"), where the perimeter consists of lines in the pattern. What shape would be optimal for the region? The same problem in the ordinary plane, in which there are no constraining lines, goes back to the ancient Greeks and is called "Dido's problem", and of course has a circle for its solution. In taxicab geometry a little thought shows that for large regions the shape is not circular but square, oriented the same as the little squares. Of course what we are leading up to is: What is the optimal shape for "pinwheel geometry"? This is quite a bit harder, and has only recently been shown [4] to be—again a circle! (asymptotically, for large regions).

The pinwheel, with its statistical roundness, is a recent discovery and it must still be hiding lots of unknown but interesting properties. The above are just some of the first ones we have found. And then one can try to investigate statistical symmetry in three dimensions ...

References

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