Miles of Tiles*

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Introduction. This article corresponds closely to four lectures given in conjunction with the July 1994 workshop at Warwick; the informal format of those lectures is maintained.

Our main goal is to discuss the generalization of subshifts with $\mathbf{Z}^d$ actions, especially those of finite-type, to tiling systems with $\mathbf{R}^d$ actions. Our justification for this generalization is the natural action of the Euclidean group, in particular rotations, on tiling systems. The new "statistical" form of rotational symmetries which appear are a significant addition to the mathematics of subshifts, and of decided value in applications, which we discuss.

There are four sections to this article. We begin slowly, warming up within the familiar context of subshifts. The two themes in section I) are first the contrast between subshifts of finite-type and substitution subshifts, and second the entropy of uniquely ergodic subshifts of finite-type. Sections II) and III) generalize the subject in two ways. In II) we make the leap from subshifts to tiling systems, and discuss further the connection between finite-type and substitution systems. The main examples used for illustration are the pinwheel and Penrose tilings of the plane; see Figures 1 and 2. Section III) is devoted to the use of intuition from physics (statistical mechanics) to finite-type systems, and in particular leads into the subject of rotational symmetries in such systems. All the previous discussion becomes focused in section IV) with a discussion of statistical symmetries and their use in ergodic theory and in the analysis of patterns in space. Again the main illustrations are the pinwheel and Penrose tilings of the plane.

I. The Wang/Berger phenomenon

First some notation. Throughout this section $\mathcal{A}$ will denote a fixed, finite alphabet of cardinality at least 2, T^t will denote translation by $t \in \mathbf{Z}^d$ on $\mathcal{A}^{\mathbf{Z}^d}$ (compact in the product topology), and X will be a subshift: that is, a closed, translation invariant (nonempty) subset of $\mathcal{A}^{\mathbf{Z}^d}$. An element of $\mathcal{A}$ will be called a letter, and an element of X will be called a configuration.

We will be concerned with two types of subshifts, substitution and finite-type, and we consider them first in the classical regime, $d = 1$.

a) $d = 1$.

We begin with substitution subshifts. We define $\mathcal{B}$, the set of "words", as $\bigcup_{K\ge 1} \mathcal{A}^K$. To define a substitution subshift one must have a substitution function, $F : \mathcal{A} \longrightarrow \mathcal{B}$. One then extends F to be a map from $\mathcal{B}$ to itself. (If $b = a_1 a_2 \cdots a_K,\ F(b) \equiv F(a_1)F(a_2)\cdots F(a_K)$.) $\mathcal{B}_F$ is then defined by

$$\mathcal{B}_F \equiv \{b \in \mathcal{B} \,:\, b = F^k(a),\ \text{for some } a \in \mathcal{A}\} \qquad 1)$$

If $b = F^k(a)$ we call b a letter of "level k". Finally, we define X_F, the substitution subshift associated with F, by

$$X_F \equiv \{x \in \mathcal{A}^{\mathbf{Z}} : \text{ for each } t, j \in \mathbf{Z},\ (x_{t+1}\ x_{t+2}\ \cdots x_{t+j}) \subset b,\ \text{ for some } b \in \mathcal{B}_F\} \quad 2)$$

We now record some simple facts about substitution subshifts.

Lemma 1. Each $x \in X_F$ "decomposes" into adjacent letters of level 1, and therefore also into adjacent letters of level k, for any fixed k.
(The proof is a simple exercise using compactness.)

Lemma 2. Assume there exists p such that for all $a \in \mathcal{A}$, $F^p(a)$ contains all $a' \in \mathcal{A}$. Then X_F is uniquely ergodic – that is, there is one and only one translation invariant Borel probability measure on X_F.
(We will sketch the proof in the last section.)

Lemma 3. Assume the decomposition of Lemma 1 is unique for each $x \in X_F$. Then for all $x \in X_F$, $T^t x = x$ implies $t = 0$.
(We will give the proof in the last section.)

To clarify the uniqueness assumption of Lemma 3 we leave it as an exercise to see that there is not uniqueness for the substitution:

$$\mathcal{A} = \{0, 1\},\ F(0) = 101,\ F(1) = 010 \quad 3)$$

We now consider finite-type systems. For fixed positive integer K and nonempty subset $\phi \subset \mathcal{A}^K$ we define the finite-type subshift X_ϕ associated with ϕ by

$$X_\phi \equiv \{x \in \mathcal{A}^{\mathbf{Z}} :\ \text{ for all } t \in \mathbf{Z},\ (x_{t+1}, x_{t+2}, \cdots x_{t+K}) \in \phi\} \quad 4)$$

(Note the difference from substitution systems for which restrictions are made for subwords of all sizes.)

Lemma 4. If X_ϕ is nonempty there exists $\tilde{x} \in X_\phi$ and $\tilde{t} \neq 0$ such that $T^{\tilde{t}}\tilde{x} = \tilde{x}$.

Note the difference between Lemmas 3 and 4. Since the assumptions in these lemmas are very weak, we make the intuitive conclusion that these two classes of subshifts, the substitutions and the finite-type subshifts, are essentially disjoint.

b) $d \geq 2$.

We now define $\mathcal{B}$, the set of "words", as $\bigcup_{K \subset \mathbf{Z}^d} \mathcal{A}^K$. And again, to define a substitution subshift one must have a substitution function, $F : \mathcal{A} \longrightarrow \mathcal{B}$. But now there can be a problem extending F to a function from $\mathcal{B}$ to itself. This problem will naturally disappear when we extend our format to tilings. (We will see then that basically substitutions are representations of similarity transformations of Euclidean space, and the discreteness of $\mathbf{Z}^d$ creates artificial difficulties such as this extension problem.) As an example consider the following.

Let $d = 2$, $\mathcal{A} = \{a, b, c, d\}$, and define the substitution function F by

$$F(a) = \begin{matrix} a & c \\ b & d \end{matrix}; \quad F(b) = \begin{matrix} b & d \\ a & c \end{matrix}; \quad F(c) = \begin{matrix} c & a \\ d & b \end{matrix}; \quad F(d) = \begin{matrix} d & b \\ c & a \end{matrix} \qquad 5)$$

This F has a natural extension from $\mathcal{B}$ to itself. And then we can define the subshift associated with F as for $d = 1$. It is easy to check that Lemmas 1, 2 and 3 are still true in this more general setting.

For subshifts of finite-type there is less of a problem; we just fix some bounded $K \subset \mathbf{Z}^d$ and nonempty $\phi \subset \mathcal{A}^K$, and define the subshift X_ϕ associated with ϕ as before. The big difference is that **Lemma 4 is not true for $d \geq 2$.** Specifically, in 1966 Robert Berger [Ber] answered a question of his thesis advisor, the philosopher Hao Wang, by giving an example with $d = 2$ and an alphabet $\mathcal{A}$ with over 20,000 letters, of a subshift X_ϕ of finite-type such that for all $x \in X_\phi$, $T^t x = x$ implies $t = 0$. After much intermediate research, for example by Raphael Robinson [Rob], this situation was clarified by Shahar Mozes in 1989 [Moz].

Theorem 1. Given a substitution subshift $X_F \subset \mathcal{A}^{\mathbf{Z}^2}$ which is uniquely ergodic with unique measure μ_F, (and satisfies some weak conditions we do not list), there exists an alphabet $\tilde{\mathcal{A}}$ and some $\phi \subset \tilde{\mathcal{A}}^K$ such that

i) X_ϕ is uniquely ergodic, with unique measure μ_ϕ;

ii) $(X_\phi, \mathbf{Z}^2, \mu_\phi)$ and $(X_F, \mathbf{Z}^2, \mu_F)$ are metrically conjugate.

Considering the correspondence of this result with Lemmas 3 and 4 we conclude that although for $d = 1$ the class of substitution subshifts and the class of subshifts of finite-type are basically disjoint, for $d \geq 2$ the former is basically a subset of the latter! We call this the Wang/Berger phenomenon, the starting point for research in what is now called aperiodic tilings, or forced tilings. We will follow up this path in later sections, but first we want to add one fundamental fact about subshifts of finite-type, a result by Jacek Miękisz and the author [Rad1].

Theorem 2. If the subshift of finite-type $(X_\phi, \mathbf{Z}^d)$ is uniquely ergodic then it has zero entropy.

sketch of the proof. The case of $d = 1$ is immediate from Lemma 4. Assume for simplicity of notation that $d = 2$ and K is a 2×2 subset of $\mathbf{Z}^2$. Let S_N be the square set of N lattice points of $\mathbf{Z}^2$ centered at the origin. Consider the cylinder sets based on S_N, that is, sets of configurations with fixed values in S_N. Think of the letters of the alphabet as colors, and the fixed values of such configurations as (colored) "pictures". We will assume the entropy is $\alpha > 0$, and obtain a contradiction. From Birkhoff's theorem, as a function of N there are roughly $e^{\alpha N}$ pictures on S_N which appear with positive frequency in all configurations. Consider a picture frame of unit thickness around S_N; there are at most $e^{\beta\sqrt{N}}$ different ways to color in the frame. So for large N there is some colored frame D "compatible with" more than N^2 different pictures on S_N, that is, such that each of those pictures and the colored frame D together appear with positive frequency in all configurations. Fix some configuration x in the support of the measure and large N with some picture C on S_N inside the colored frame D. Consider S_{9N}. It "contains" at most roughly N

different pictures of shape $\sqrt{N} \times \sqrt{N}$, so there is some picture C', compatible with D, which doesn't appear in S_{9N} in x. The picture P on S_N appears with positive frequency in all configurations, even ignoring overlapping copies of it. Change the configuration x to a new configuration x' by replacing each of these copies of a C inside a D, in the center of these $P's$, by a C'. On the other hand it is also clear that the frequency of C' in x' is higher than it is in x, and this is incompatible with the assumption that X_ϕ is uniquely ergodic. This contradiction proves that the entropy is zero. ■

We conclude this section with some comments on the assumption of unique ergodicity. This assumption is unusual when considering subshifts of finite-type, because in the classical regime of $d = 1$ it follows from Lemma 4 that only trivial subshifts would have the property. Now the classical regime of $d = 1$ corresponds to, or originated from, considerations of time evolution, wherein configurations can correspond to very different conditions, and be very different one from another. We assert that the regime $d \geq 2$ corresponds to a different paradigm, namely space rather than time translation. And there the assumption of unique ergodicity is much more natural, corresponding as it does to "essentially" only one global pattern – that is, locally unique if not globally. So the study of subshifts of finite-type with $\mathbf{Z}^d$ action is the study of spatial patterns, and the assumption of unique ergodicity is a natural one of nondegeneracy [Rad3].

II. The pinwheel and Penrose tilings

One of the key ideas in the last section was that interesting subshifts of finite-type could be constructed which are metrically conjugate to given substitution subshifts. And in fact Mozes has given a rather general prescription for this. Before we discuss this further, it is appropriate to make the transition to tilings.

First some notation. We will consider (finite) alphabets $\mathcal{A}$ whose letters are polyhedra in $\mathbf{R}^d$. A "tiling (from $\mathcal{A}$)" will then be a countable collection of "tiles" (sets which are congruent to letters of $\mathcal{A}$), which is simultaneously a covering and a packing of $\mathbf{R}^d$.

Now although in the last section I presented the work of Berger *et al* as results about subshifts, in fact this early work was formulated in terms of tilings of the plane. The letters in $\mathcal{A}$ are there all basically unit squares, but each has a different pattern of bumps and dents on the edges so that, like jigsaw puzzle pieces, they only fit together in certain ways. This is easily seen to determine a set $\phi \subset \mathcal{A}^K$, where K is a 2×2 subset of $\mathbf{Z}^2$, and thus a subshift X_ϕ of finite-type. We will refer to such tilings as "square-ish".

In order to study symmetry properties it is necessary to generalize the above mathematics to include polyhedra which are not basically squares or cubes. We will introduce the ideas through two examples. In both cases there is a substitution system and an associated finite-type system. As with subshifts, a substitution tiling system will be a special set of tilings associated with a substitution function, and a finite-type tiling system will consist of **all** tilings by the letters in the given alphabet, the finite-type condition consisting of the geometric restriction that the tiles fit together and cover space. We will begin by describing the two substitution systems.

For the Penrose tilings the alphabet $\mathcal{A}$ for the substitution system consists of two letters, called the "kite" and the "dart"; see Figure 3. The family $\mathcal{B}$ of words is the set of all finite collections of tiles (that is, sets congruent to letters of $\mathcal{A}$), whose union is connected and simply connected, and in which tiles have pairwise disjoint interiors. The substitution function $F : \mathcal{A} \longrightarrow \mathcal{B}$ should be thought of as obtained by first associating with each letter (shown by dotted lines in Figure 4) a collection of "shrunken tiles" (shown by solid lines in Figure 4), small by a factor $(1+\sqrt{5})/2$, and then expanding the collection about the origin by $(1+\sqrt{5})/2$. With this convention we define the family $\mathcal{B}_F$ of words of the form $F^k(a)$, $a \in \mathcal{A}$, $k \geq 1$. Finally we define the substitution system X_F associated with F to be the set of all tilings x of the plane such that every subword of x is congruent to a subword of some word in $\mathcal{B}_F$.

For the pinwheel tilings the alphabet for the substitution system consists of a $1, 2, \sqrt{5}$ right triangle and its reflection, as in Figure 5. The substitution function, indicated in Figure 6, should again be considered a composite of two maps. The substitution system X_F is defined as above.

We have thus specified two substitution systems. The main question in this section is how to associate finite-type systems with such substitution systems, in analogy with the special case of square-ish tilings (*i.e. subshifts*) discussed in the last section. The vague idea is that one wants a new alphabet similar to the given one, but altered so that the tiles can *only* fit together in the special way the original letters do in the given substitution tilings.

For the Penrose system the answer is easy. By examining small words in any tiling one sees that the kite and dart can be prevented from abutting incorrectly (for example from forming a rhombus, copies of which could then tile periodically) by adding bumps and dents to them as in Figure 7. And with these letters one can only tile the plane as in the substitution.

So for the Penrose substitution system it is easy, by examining small words in the tilings, to create a new alphabet which defines a finite-type system conjugate to the original. It is then natural to ask whether a similar analysis can be made for the pinwheel tilings. The answer is no, and we will present a simplified version of an argument proving this, an argument due to Ludwig Danzer. Before we do this however, we consider the following generalization of the above analysis.

Given a tiling x let W be the set of all words in x with upper bound D on the diameter. We use W to create the dynamical system X_W, defined to contain all tilings y by the letters in x such that all words of diameter at most D in y are congruent to words in W. It is then not hard to construct a new alphabet of polyhedra such that the dynamical system consisting of all tilings by these letters is conjugate to X_W. One aspect of this generalized method should be emphasized.

For the Penrose system we examined words of two letters in the substitution system, and ended up with an alphabet for the finite-type system of the same cardinality as the original, just distortions of the original letters. But in this generalized method, where we examine larger words in the substitution system, we may end up with a much larger alphabet than for the substitution system.

We now return to the pinwheel substitution system. Consider the periodic tiling P of which the unit cell is either of the two words called B and C in Figure 8.

$F^5(a)$ is shown in Figure 9. In this word one sees several copies of B and C. Now given words of diameter D in a pinwheel tiling, apply the substitution at least $2D\sqrt{5}$ times to P, creating a new periodic tiling P', with B and C leading to two unit cells of P' which we call B' and C'. It is easy to see that all words of diameter at most D in P' are interior to one of these two unit cells, and *so must appear in the pinwheel.* This proves the above assertion since any finite-type rule, based on a set of words of a certain size appearing in the pinwheel, will be compatible with the periodic tiling P' – that is, P' would belong to the space of tilings, in contradiction to Lemma 3.

Now as we noted there is a finite-type pinwheel system conjugate to the pinwheel substitution, although it cannot be obtained directly from words of fixed size by the above method. Furthermore, the technique used by Mozes for square-ish tiles has also proved to be too simple. Next we will sketch a method which works for the pinwheel [Rad4].

First an overview. The letters in the new alphabet $\mathcal{A}'$ come in two families, one for each of the two triangles in the original $\mathcal{A}$. The letters in each family can be thought of as perturbed versions of the original triangle, that is, the original triangle with some pattern of bumps and dents on each edge. But this is not the best way to think of them; it is preferable to view each letter as a triangle with certain information associated with each vertex. (It is proven in [Rad4] that the information can transformed to bumps and dents.) The finite-type system is then defined by requiring that the triangles may abut if and only if the information associated with vertices of abutting triangles is "consistent" in a certain precise sense. Roughly speaking, the information can be thought of as a picture of what "should be" the environment of that vertex, referring implicitely to any triangles abutting the given one at that vertex. (By "consistent" we mean that their pictures agree.) We write "should be" because the information can only be thought of as a wish list, and it is necessary to prove that in any tiling with such marked triangles all the wish lists are automatically satisfied. Next we want to indicate what information is included at vertices.

Before this can be done we will pick out one tiling from the substitution system and define a certain hierarchical structure in it. The substitution function (Figure 6) associates five triangles with each letter of $\mathcal{A}$. We give them labels, A through E, as in Figure 10. Next we construct a certain tiling of the plane, called "the tesselation", as follows. Start with a letter sitting in the plane and use the substitution. Reorient the word obtained so the "C" triangle is sitting over the original letter. Apply the substitution to this word, reorienting again so the "C" associated with the first "C" again lies over the original letter. Repeating this process indefinitely produces the tesselation.

We introduce the following notation with reference to the above process. We say the original letter is a letter of level 0. The first application of the substitution produces a word of five letters, a letter of level 1, containing the five letters of level 0. The five letters of level 0 are given labels A, B, C, D or E. When the substitution is applied again, the letter of level 1 is (by choice) seen to be a letter with label C inside a letter of level 2, *etc.* So the tesselation can be understood to be a tiling of the plane by letters of all levels, each labelled by A through E. Finally we introduce the notion of "complete edges" for these letters. Referring to the three edges of the letters as "small", "medium" and "large" (or S, M and L respectively) in the obvious way, we define the complete edges to be the small edges of letters (of any level) with label A, B, C and D, the medium edges of letters with label C, D and E, and the large edges of letters with label B and C. These happen to be the "internal" edges in the letters of level n as these sit in a letter of level $n+1$; see Figure 10.

It is easy to see from the construction that in the tesselation each edge of each letter of level 0 is geometrically part of precisely one complete edge. Next we show how this fact is used in the information associated with the letters of the alphabet $\mathcal{A}'$ of the finite-type system.

The information associated with each edge of the letters of $\mathcal{A}'$ is incorporated in a tree-like structure. The highest level in the tree is the variable μ. With each of the three vertices of the letter, labelled S, M and L referring to the size of the angle of the vertex, there is the variable μ_S or μ_M or μ_L. These variables each contain other variables, some of which contain other variables *etc.* A complete list of the variables involved is:

$$\mu = (\mu_S, \mu_M, \mu_L)$$

$$\mu_j = (A_1, E_1^1, E_1^2; A_2, E_2^1, E_2^2; \cdots, A_e, E_e^1, E_e^2)$$

$$3 \le e \le 8$$

$$A_j \in (S, M, L, \pi)$$

$$E_k^j = (J, N, P, F, G, H)$$

$$J \in (S, M, L)$$

$$N \in (A, B, C, D, E)$$

$$P \in (A, B, C, D, E)$$

$$F \in (S, M, L, Z, R)$$

$$G = (G_A^{\alpha 1M}, G_E^{\alpha 1S}, G_B^{\alpha 1S}, G_D^{\alpha 1S}, G_D^{\alpha 1M}, G_A^{\alpha 2M}, G_E^{\alpha 2S}, G_B^{\alpha 2S}, G_D^{\alpha 2S}, G_D^{\alpha 2M}, G_A^{\epsilon 1M}, G_E^{\epsilon 1S}, G_B^{\epsilon 1S}, G_D^{\epsilon 1S}, G_D^{\epsilon 1M}, G_A^{\epsilon 2M}, G_E^{\epsilon 2S}, G_B^{\epsilon 2S}, G_D^{\epsilon 2S}, G_D^{\epsilon 2M})$$

$$G_m^{jkn} \in \{+1, -1\}$$

$$H = (H_A^{1M}, H_E^{1S}, H_B^{1S}, H_D^{1S}, H_D^{1M}, H_A^{2M}, H_E^{2S}, H_B^{2S}, H_D^{2S}, H_D^{2M})$$

$$H_m^{kn} \in \{+1, -1\}$$

There do not exist letters with all possible values of these variables; there are restrictions, forty two of them, which spell out explicitly which combinations of values may occur. It would be hard to keep track of the variables if they did not have simple geometric meaning, so we next indicate this interpretation of the variables.

The basic structure of a variable such as μ_S is a sequence of triples: an A followed by two E's. This refers to the letters of $\mathcal{A}'$ which "should" abut the given letter, and gives their geometric relationships. The A's refer to the sizes of the angles of each of the abutting letters at the vertex in question. Each edge which "should" emanate from the vertex is split into two to enable reference to each of the two letters sharing that edge, and this is what the pair of E's refer to in each triple.

Now each E_k^j is a complicated quantity, but essentially all the information in it relates to certain complete edges. For example the variable J gives the size (small, medium or large) of the complete edge that the letter edge is part of; N gives the label of the letter, (A through E), of whatever level, that that complete edge is part of; and P gives the label of the letter (A through E) that that letter, of whatever level, is part of. The variable F describes how the complete edge which contains the given level 0 letter behaves as it meets the vertex in question; it might end at an angle of size S, M or L, or it might not end in which case F has the value z, or it might not end and not even have a vertex there (value R.) This takes care of all the variables except for G and H. The "gun" variables G "shoot" information out of vertices. The information is oscillatory – that is, the value of $+1$ or -1 does not carry intrinsic meaning, but is like a phase, and what is significant is if two variables meet at a vertex in phase or out of phase. The variables travel along channels prepared, along certain complete edges, by the H variables. The gun variables are needed in the induction proof which shows that in any tiling with the letters from $\mathcal{A}'$ each of these letters is contained in a unique letter of level k for every $k \geq 1$. This means that letters of level 0 must group together uniquely in groups of 5, then letters of level 1 must also, *etc.* The gun variables are needed to force this grouping. Roughly, the label which a given letter of level k is supposed to have is kept in a certain geometric place, and that information has to travel around in a reliable way to ensure that the neighboring letters of that level are correct.

So much for our sketch of the proof that the pinwheel is finite-type; for the full story one must read [Rad4]. We end this section with the hope that just as the simpler technique of Mozes works quite generally for square-ish substitutions, some version of the pinwheel technique will work for a large class of substitutions of general shapes, thereby giving an understanding for the way hierarchical ("fractal") structures could originate from local, finite-type, rules.

III. Statistical mechanics and tilings

This section is concerned with certain ideas from statistical physics which give an unconventional and useful viewpoint to the general area of subshifts of finite-type with $\mathbf{Z}^d$ action, or finite-type tiling systems with $\mathbf{R}^d$ action. We begin with a short, elementary introduction to (classical) statistical mechanics.

Heuristically, assume we wish to analyze a piece of matter composed of many atoms (for simplicity we assume they are all of the same element), at various temperatures $\tau > 0$ and chemical potentials c. Assume there are N of these atoms constrained in a cube L of volume $|L|$, which means their positions constitute N variable points in L, or a point in L^N.

We somehow assign a "(potential) energy" $E^L(x;c)$ to all possible configurations x of L^N. (This will be discussed further below. We are ignoring kinetic energy and quantum effects as their contributions to the following are relatively minor and easy to insert if desired.) Then for each $\tau > 0$ and c we consider

$$f_L(x) \equiv \frac{\exp[-E^L(x;c)/\tau]}{\int_{L^N} \exp[-E^L(x;c)/\tau]\, d^N x} \qquad 6)$$

as a probability density for x. We take the (weak-$*$) limit of this probability measure as $L \to \mathbf{R}^d$, getting a Gibbs measure $\mu_{\tau,c}$ on configurations in $\mathbf{R}^d$. We are mainly interested in the weak-$*$ limit $\tau \downarrow 0$, giving $\mu_{0,c}$.

Let X be the collection of all countable subsets of $\mathbf{R}^d$ such that no two points are less than unit distance apart. Let H be the subcollection of finite subsets in X which contain the origin.

Define an "interaction" as a function $e : H \longrightarrow \mathbf{R}$ such that $e(h) = 0$ if $|q| \geq r$ for some $q \in h$. (Here r is some fixed positive number, the "range of the many-body interaction".) For each cube L define E^L on X:

$$E^L(x;c) \equiv \sum_{q \in x \cap L} \{e[T^{-q}x \cap D_{r,0}] - c\} \qquad 7)$$

where $D_{r,0}$ is the open ball centered at the origin 0 of radius r, and as usual T denotes translation. Finally, define $X_{GS} \subset X$ by: $x \in X_{GS}$ if for every L, and every $x' \in X$ such that $x \cap (\mathbf{R}^d \backslash L) = x' \cap (\mathbf{R}^d \backslash L)$, it follows that $E^L(x;c) \leq E^L(x';c)$. GS stands for "ground state", and the notation ignores the dependence on c. Note: it can be proven [Rad2] under rather general conditions on E^L that $\mu_{0,c}(X_{GS}) = 1$.

It is not hard to check that X_{GS} is always nonempty, and closed under translation. We will postpone until the next section the definition of the natural topology for spaces like X_{GS}; suffice it to say that two configurations are "close" if they differ in some large neighborhood of the origin by a small rigid motion. We note that X_{GS} is compact and metrizable in that topology [R-W].

Consider the following example. Let $d = 2$, $c = 0$ and

$$e(h) \equiv \begin{cases} -25 + 24|h_1 - h_2|, & \text{if } h = \{h_1,\ h_2\} \text{ and } 1 \le |h_1 - h_2| \le 25/24; \\ 0, & \text{otherwise} \end{cases} \qquad 8)$$

Then if x is the triangular lattice as in Figure 11 a), $x \in X_{GS}$. Also, if x' is the configuration in Figure 11 b) then still $x' \in X_{GS}$. Notice the horizontal "fault line" in x'. The ergodic measures on X_{GS} each have as support the translations of x rotated by some fixed angle.

We now discuss one way in which configurations enter into physical theory – how they respond to beams of waves directed at them. We will assume the points $\{t_j\}$ in a configuration $x \in X_{GS}$ represent the centers of atoms, all with the same electron density. Let the electron density at $t \in \mathbf{R}^3$ of an atom centered at t_j be

$$\rho_j(t) = k(t - t_j) = \begin{cases} 1/10 - |t - t_j|, & |t - t_j| \le 1/10 \\ 0, & |t - t_j| > 1/10 \end{cases} \qquad 9)$$

Define $\psi : X_{GS} \longrightarrow \mathbf{R}$ by:

$$\psi(x) = \sum_{t_j \in x} k(-2t_j) \qquad 10)$$

Assume X_{GS} is uniquely ergodic, with measure μ_{GS}; this is a nondegeneracy assumption, essentially equivalent to assuming that the value of c is not that of a phase transition [Rad2]. Fix $x = \{t_j\} \in X_{GS}$ and an increasing sequence of cubes L_i. Steven Dworkin has shown [Dwo1] that if a plane wave of wavelength s in the direction W_0 is scattered off the atoms in L_i the intensity I_i scattered in the direction W satisfies:

$$\frac{I_i}{|L_i|} \xrightarrow[|L_i| \to \infty]{} d\langle E_\lambda \psi,\, \psi \rangle \qquad 11)$$

where $\lambda = (W - W_0)/s$ and $\{E_\lambda : \lambda \in \mathbf{R}^3\}$ is the spectral family associated with translations on X_{GS}. (That is, $\{E_\lambda\}$ is the spectral family, guaranteed by Stone's theorem [R-N], of the unitary representation of the translations on the complex Hilbert space of complex valued functions on X_{GS} square integrable with respect to the unique invariant measure on X_{GS}.)

This is intimately connected with the history of research on tilings as follows. Assume we have some finite-type tiling system X_F such as the Penrose or pinwheel. With each of the letters in the alphabet we associate one or more "atoms", enough to avoid accidental symmetries (in a sense to be clarified below). Then for each tiling of the tiling system we can associate a configuration of points in $\mathbf{R}^d$, the centers of the atoms associated with each tile. If the atoms are placed generically in the letters, the system of configurations $\tilde{X}_F$ thus produced is naturally isomorphic to the tiling system X_F, including of course the action $\mathbf{R}^d$ on it. Now $\tilde{X}_F$ is intuitively like a ground state system X_{GS} in that configurations belong to $\tilde{X}_F$ if and only if they satisfy the local rules of the tiles, which is very much like the local rules of a ground state system. This idea occurred to physicists and is the basis of their interest in Penrose tilings to model quasicrystals. They use 3 dimensional versions of the Penrose system to which they associate "atoms" as above, creating a space $\tilde{X}_F$ of configurations of "atoms". And they do this precisely because the dynamical spectrum of the 2 or 3 dimensional Penrose systems has unusual rotational symmetries, namely symmetry under rotations by $2\pi/10$ about certain axes. Such symmetries are impossible for the scattering intensities of ordinary crystals, but are in fact the hallmark of certain real materials called quasicrystals. To summarize, tilings such as those of Penrose are of interest in the modeling of quasicrystals because of the symmetries of their dynamical spectra. (There is also interest in the *smoothness* of their spectra; see [Rue], [B-R] and [Rad2]. And another related direction suggested by physics is that of "deceptions" of tilings; see [D-S] and [Dwo2].)

Before ending this section we make one more observation. Consider the dynamical system of configurations under space translations together with a Gibbs measure $\mu_{\tau,c}$. We have been concerned with the case $\tau = 0$ and have been led to interesting uniquely ergodic finite-type systems, with zero entropy. (The zero entropy result of the first section has been generalized to tiling systems by Jiunn-I Shieh [Shi].) It can be proven that entropy increases with τ, and that at high temperature or entropy the positions become statistically independent (the particles become "free"). The high entropy regime is the one best understood in ergodic theory and physics, with results on Bernoulli systems *etc.* But the low or zero entropy regime seems to contain a great deal of new and exciting possibilities, such as these tiling systems; just as the physics is more interesting in that regime, so too seems to be the mathematics.

IV. A new form of symmetry

We will now draw upon elements from the last two sections to elucidate a new form of geometrical symmetry and its relevance for ergodic theory [Rad6]. We will illustrate the idea by tilings of d dimensional Euclidean space by polyhedral tiles.

One usually says that a tiling (or other pattern) in space has a certain symmetry if it is invariant under the corresponding rigid motion of the space. The new form of symmetry we will describe is different, and is of a statistical nature. We will use the Penrose and pinwheel tilings as motivation, but first we need some notation.

Let X be some tiling dynamical system made from some finite alphabet $\mathcal{A}$ of polyhedra in $\mathbf{R}^d$. (We do not require at this stage that X be either a finite-type or substitution system, but only require that it be locally finite in the following sense. Given any "radius" $r > 0$, look at all the clusters of tiles that are a distance less than r from the center of mass of one of the tiles in any tiling in X. We require that there be only finitely many possible such clusters up to congruence. This is automatic for substitution systems.) Since we distinguish tilings which are rigid motions of one another there is a natural representation on X of G, the connected part of the Euclidean group, which we denote $T^g : X \longrightarrow X$ for $g \in G$. We put a topology on X using a neighborhood basis with three parameters: $x \in X$, a neighborhood Θ of the identity of G, and a neighborhood R of the origin of $\mathbf{R}^d$. The corresponding open set of tilings is:

$$C_{x,\Theta,R} \equiv \{x' \in X \,:\, T^g x'|_R = x|_R, \text{ for some } g \in \Theta\} \tag{12)}$$

where "$|_R$" refers to restriction to R. It is not hard to show that X is compact and metrizable in this topology [R-W].

We noted that the Euclidean group is naturally represented on X. Now we assume another feature of the Penrose and pinwheel tilings, the representation of a similarity. That is, we assume that for each $x \in X$ there is a *unique* $y(x) \in X$ associated, made from tiles which are larger by some factor $\gamma > 1$. Let σ be the similarity on $\mathbf{R}^d$ which takes t to t/γ, and let $T^\sigma : X \longrightarrow X$ be defined by $T^\sigma x = (1/\gamma) y(x)$. This homeomorphism is thought of as a representation of the similarity σ since it has the correct relations with the representation of G, for example:

$$(T^\sigma)T^t(T^\sigma)^{-1} = T^{\sigma(t)} = T^{t/\gamma} \tag{13)}$$

for all translations t. One use of this is the following. Assume for some tiling x_0 that $T^{t_0}x_0 = x_0$. Then for all n, $T^{\sigma^n(t_0)}[T^{\sigma^n}x_0] = T^{\sigma^n}x_0$. But $T^{\sigma^n(t_0)} = T^{t_0/\gamma^n}$. And this implies that $t_0 = 0$ since otherwise $T^{\sigma^n}x_0$ would be invariant under a translation by an amount smaller than the size of any tile.

We need one more set of ideas before we can consider the new symmetry. We must examine the content of Birkhoff's theorem in this tiling context. Assume μ is an ergodic Borel probability measure on X. Then for each basic open set $C = C_{x,\Theta,R}$ we have:

$$\mu(C) = \int_X \chi_C(x')\, d\mu = \lim_{L\to\mathbf{R}^d} \frac{1}{|L|}\int_L \chi_C(T^t x')\, dt \tag{14)}$$

where χ_C is the characteristic function for C. Consider the fraction

$$\frac{1}{|L|}\int_L \chi_C(T^t x')\, dt \tag{15)}$$

It is roughly

$$\frac{\epsilon^d \times (\text{the number of times the "picture of } C \text{ appears" in } x')}{|L|} \tag{16)}$$

where the "picture of C" refers to the part of the tiling in the definition of C, and the ϵ^d refers to the amount of translation and rotation the picture is allowed to move, in the definition of C.

In summary, this fraction is, up to a factor, just the *frequency* with which the picture of C appears in x'. Now although in general Birkhoff's theorem only holds for μ almost all x', if X is uniquely ergodic under translations then the convergence is not only valid for all x' but is in fact uniform in x' [Fur]. This is relevant because of the following, in which X_{pin} refers to the pinwheel system and X_{Pen} refers to the Penrose system.

Theorem 3. X_{pin} is uniquely ergodic. $X_{Pen} = \bigcup_{\alpha\in[0,2\pi/10)} X_\alpha$, where the X_α are uniquely ergodic and rotations of one another.

We will give a general technique for proving such results but first we want to discuss its use. Note that any rotation on X can be lifted to a rotation on the set of translation invariant probability measures on X. So if there is only one such measure, as the theorem claims is the case for the pinwheel, then that measure must be invariant under all rotations of X. A similar argument shows for the Penrose system that all translation invariant measures are invariant under rotations by multiples of $2\pi/10$. And from the above analysis of the geometric meaning of such measures we obtain the following statistical interpretation of these symmetries. For the Penrose tilings, any finite cluster of tiles appears in each tiling with the *same frequency* in each of 10 equally separated orientations. For the pinwheel, given any two equal-size intervals I_1 and I_2 of orientations, and any finite cluster of tiles in a tiling, the cluster will appear in that (or any other) tiling having an orientation in I_1 with the *same frequency* as it does with an orientation in I_2. In other words, the rotational symmetries are not leaving invariant tilings, but translation invariant measures on a space of tilings; and the invariance of all such measures is equivalent to the invariance of the above "frequencies" of finite portions of the tilings. Notice that this is a fundamentally new meaning of symmetry for a tiling or pattern, justified at least by its application in the structural study of materials [Rad6].

We now sketch the proof of a general argument [Rad5] for proving unique ergodicity of 2 dimensional tilings, which implies Theorem 3. We assume a general tiling system X as above, and will make a few assumptions as needed.

To prove unique ergodicity of X, given 15), we need to prove that for each finite cluster of tiles and interval of orientation of that cluster (with respect to some arbitrary standard), there is an associated frequency which is independent of the tiling and independent of rotation of the interval. We begin by ignoring the orientations, or what is the same thing, only considering the case where the interval of orientations is the full circle. One key idea is a refinement of the requirement concerning the similarity σ. We further require that this representation of σ be effected by a substitution in the usual way. Namely, the "small size" tiling x is obtained from the "large size" tiling $y(x)$ by simultaneously replacing each of the tiles in $y(x)$ by an appropriate word of small size letters. When such a word is expanded about the origin by the factor γ one gets a tile or letter "of level 1", *etc.* (We note that in simple examples like the pinwheel the letters of all levels are geometrically similar; this is not the case for the Penrose tilings, and is not necessary for the method.) If the finite alphabet of X is $\mathcal{A} = \{a_1, a_2, \ldots, a_K\}$ we introduce the notation that the letters of all levels, obtained by starting with a_k at level 0, are tiles or letters of "type k". The above hierarchical assumption on X is useful in that all we need prove now is that each given finite cluster of tiles has the same frequency of occurance *within each tile of level* $N >> 1$, and the frequency is independent of the type of the level N tile. (The reasoning is that we are just neglecting the occurances of the cluster on the edges between the tiles of level N, and for large N this is negligible.)

We now introduce the $K \times K$ matrix M for which M_{jk} is the number of type j tiles (level 0) in a type k tile of level 1. It follows that M^N_{jk} is the number of type j tiles in a type k tile of level N. For the pinwheel,

$$M = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \qquad 17)$$

Next we assume there is some p such that $M^p_{jk} > 0$ for all j, k. It then follows from the Perron-Frobenius theorem that there is some $\delta > 0$, and functions g, h, such that for all j, k:

$$\frac{M^N_{jk}}{\delta^N} \xrightarrow[N\to\infty]{} g(j)h(k) > 0 \qquad 18)$$

Therefore

$$\frac{M^N_{jk}}{M^N_{j'k}} \xrightarrow[N\to\infty]{} \frac{g(j)}{g(j')} > 0 \qquad 19)$$

The important points are that the limits in 19) exist, and that they are independent of k, which are precisely what was needed to prove the result for clusters consisting of single tiles. (The generalization to larger clusters is routine and we refer to [Rad5] for the details.) This is a well-known argument for substitution subshifts [Que], and suffices to prove the result on Penrose systems in Theorem 3; one just enlarges the Penrose alphabet to include the kite and dart rotated by multiples of $2\pi/10$ and only considers tiles which are *translates* of letters, not *congruent* to letters. But to deal with the pinwheel, or other tilings in which rotations play an essential role, we must now deal with the orientations. To do this we generalize the matrix M above to a family of matrices parametrized by $m \in \mathbf{Z}$:

$$M(m)_{jk} \equiv \sum_\ell \exp(im\alpha_{\ell k}) \qquad 20)$$

where $\alpha_{\ell k}$ is the angle of the ℓ^{th} copy of the tile of type j in a tile of type k of level 1. Note that $M(0)$ is just the previous matrix M and that $M(1)$ keeps track of the angles of the constituents of the level 1 tiles. We now make the further assumption that there is some tile of level q which contains two tiles of level 0 with irrational relative orientation. (For the pinwheel $q = 2$ suffices.) With this assumption we can use the Weyl criterion on equidistribution of points on a circle, and an old matrix result of Wielandt, to prove that the orientations of tiles (and then finite clusters of tiles) are equally distributed in every tiling in X; see [Rad5] for details. ■

At this point we would like to mention three open problems that are suggested by the above. First, although the method of this section makes sense in any dimension $d \geq 2$, we do not have a tool as effective as Weyl's criterion in higher dimensions. And yet tiling in higher dimensions could well bring new phenomena since their rotation groups are more interesting. This suggests the study of some 3 dimensional version of the pinwheel, in which the added richness of rotations in 3 dimensions comes into play. And finally there is the obvious need to generalize to a wide class of substitutions the method used to solve the pinwheel.

Summary. The main theme of this article has been the generalization of subshifts of finite-type (with $\mathbf{Z}^d$ actions) to tiling systems (with $\mathbf{R}^d$ actions), and in particular the new role played by rotations. This new marriage of groups of rotations with ergodic theory should prove beneficial in many ways. It clearly enriches ergodic theory, with at least new "symmetry" invariants, but also provides new lines of research. And in the other direction ergodic theory seems to provide a new way to analyze the geometric symmetries of patterns in space, through "statistical" symmetries, in which the invariance of a pattern is replaced by the invariance of the frequencies of all its finite subpatterns [Rad6].

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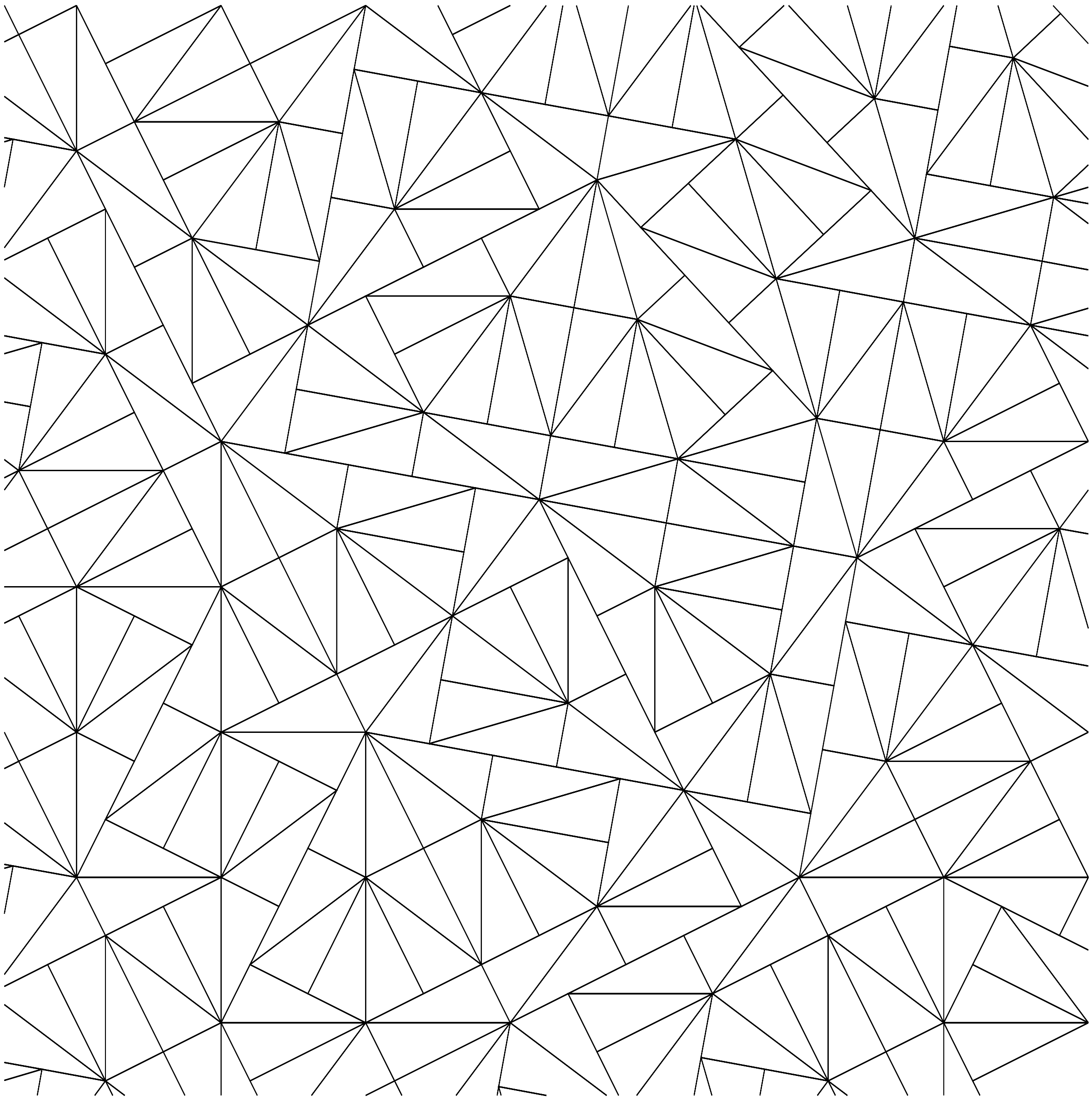


Figure 1. A pinwheel tiling

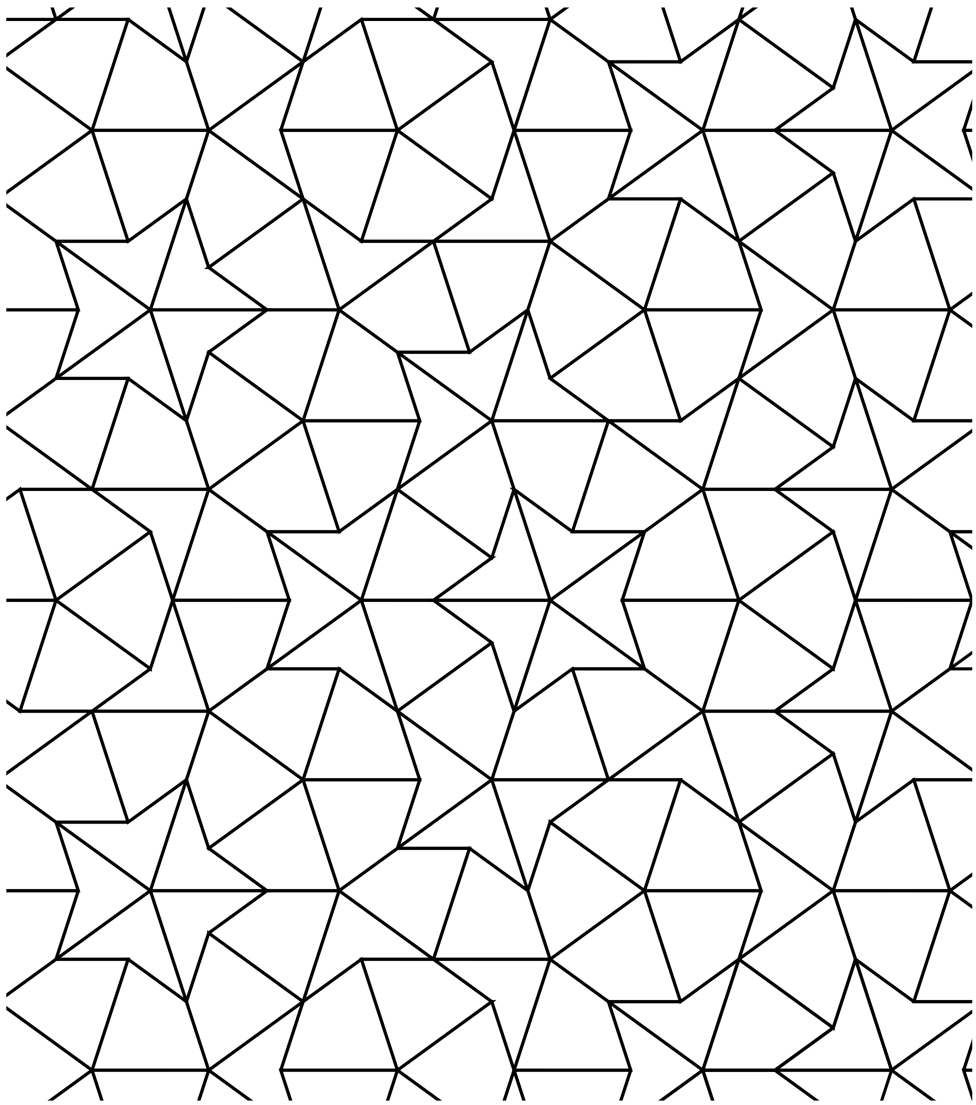
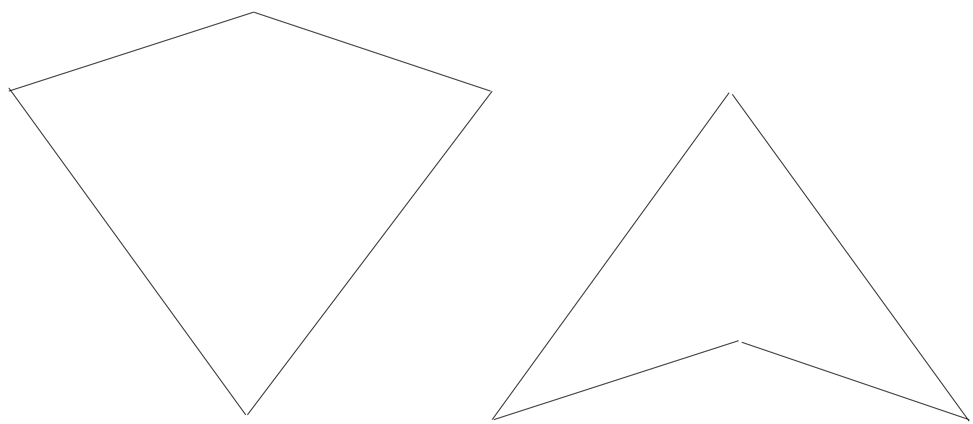


Figure 2. A Penrose tiling



kite dart

Figure 3. Penrose letters for substitution

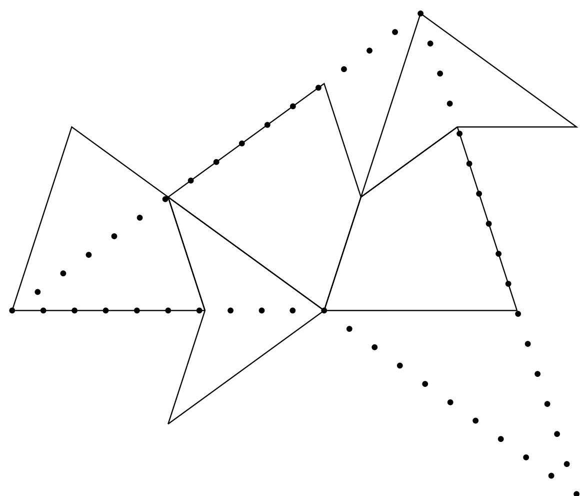


Figure 4a. Penrose substitution for dart

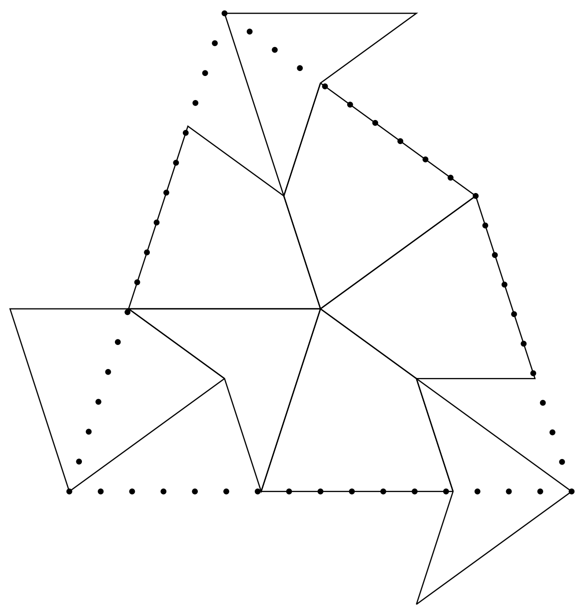


Figure 4b. Penrose substitution for kite

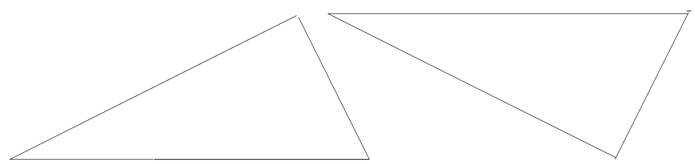


Figure 5. Pinwheel letters for substitution

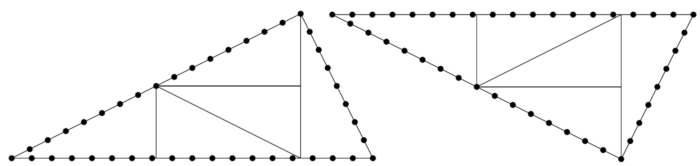
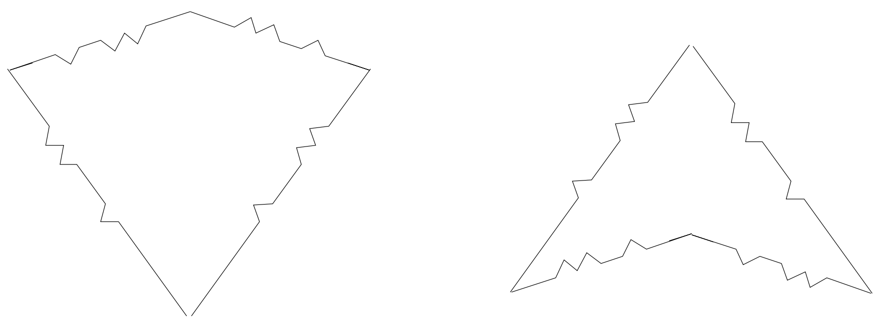
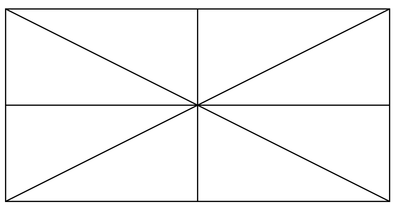


Figure 6. Pinwheel letters of level one for substitution

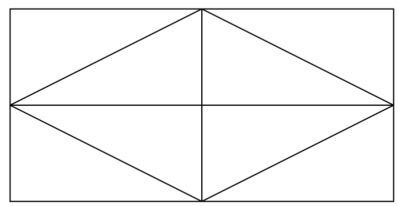


kite dart

Figure 7. Penrose letters for finite-type



unit cell B



unit cell C

Figure 8.

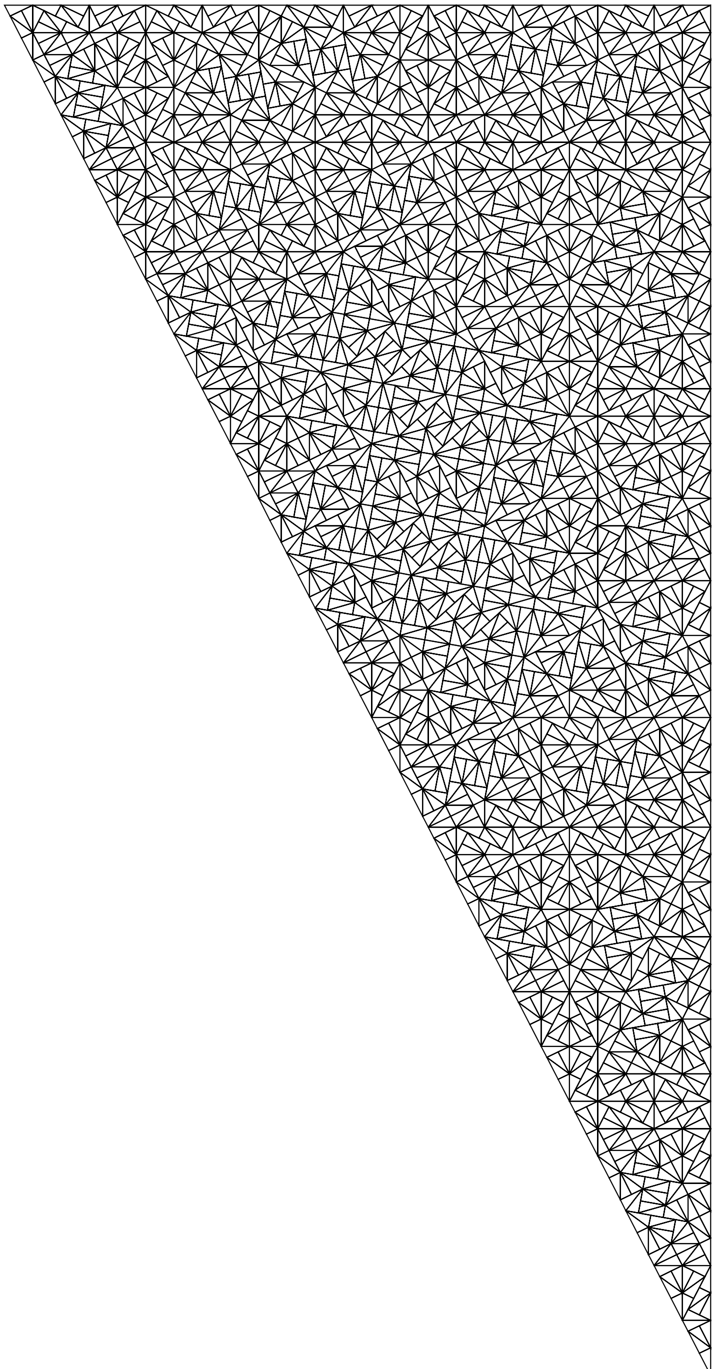


Figure 9. Pinwheel letter of level 5 for substitution

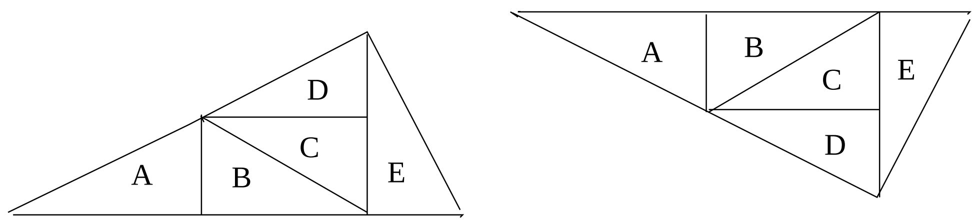
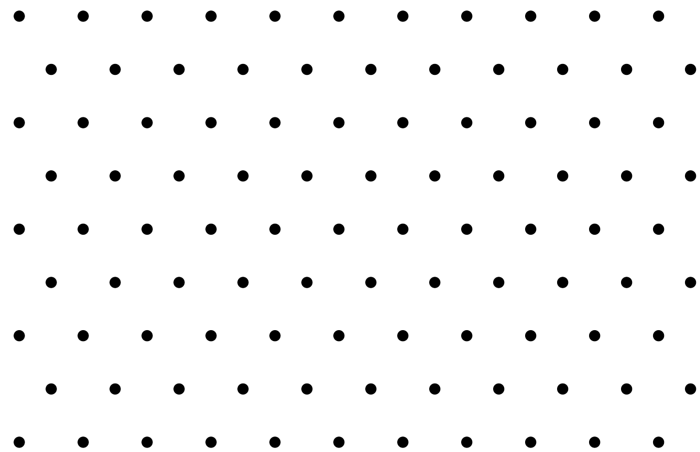
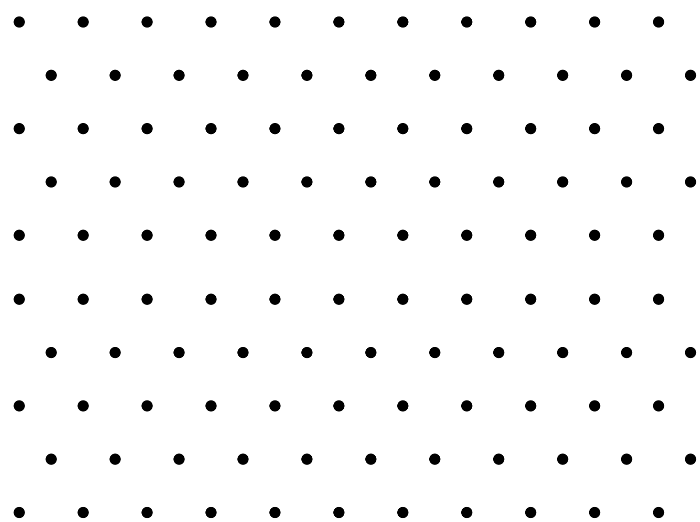


Figure 10. Tiles with labels A through E in tesselation



a) no fault



b) fault

Figure 11. Two ground state configurations