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# Asymptotic Expansions of Integrals and the Method of Steepest Descent

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**Abstract**

This paper gives an introduction to some of the most well-known methods used for finding the asymptotic expansion of integrals. We start by defining asymptotic sequences and asymptotic expansion. The classical result Watson's lemma is discussed and a proof of Laplace's method is presented. The theory of the method of steepest descent, one of the most widely used techniques in asymptotic analysis is studied. This method is then applied to calculate the asymptotics of the Airy function and the linearized KdV equation.

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# 1 Introduction

Many problems in mathematics and physics have solution formulas that can be represented as integrals depending on a parameter, and one is often interested in the behavior of the solution as the parameter $\lambda$ goes to infinity. The most frequently used way of evaluating how these integral representations behave is to find an asymptotic expansion in the parameter $\lambda$. In many problems these solutions can be written as an integral of the general form

$$F(\lambda) = \int_C e^{\lambda R(z)} g(z) dz \tag{1.1}$$

where $C$ is a contour in the complex plane. Moreover, many integral transforms such as the Laplace transform and the Fourier transform have this form. This paper will give a brief introduction to some of the most common methods for finding the asymptotic expansion of such integral representations, with focus on the method of steepest descent and some applications of it.
For more detailed information about this topic the text [8] by Peter Miller is especially recommended. See also the following well-known texts: [6] by Carl M. Bender and Steven A. Orszag, [5] by Norman Bleistein and Richard A. Handelsman, [1] by Mark J. Ablowitz and A. S. Fokas.

In Section 2 the definition of asymptotic expansion and some notation will be brought up before going into the part of the methods where the classical result known as Watson's lemma will be presented. In Section 3 we explain Laplace's method, a technique for finding the (dominant) contribution of the asymptotics of a real valued integral coming from a point or points on the interval where $R(t)$ attains its maximum.
In Section 4 the method of steepest descent is discussed, which can be considered a generalization of Laplace's method used for complex integrals. In Section 5 two applications are presented, the asymptotics of the Airy function and the asymptotics of the linearized KdV equation, both computed using the method of steepest descent.

# 2 Elementary asymptotics and Watson's lemma

## 2.1 Asymptotic expansion

To begin with there are a couple of definitions and expressions that needs to be brought up to clarify the material in upcoming sections. The asymptotic behavior of a function is expressed as an asymptotic expansion given a sequence of functions, so to be able to form an asymptotic expansion one needs asymptotic sequences.

For the following definitions let $D$ be a subset of the complex plane.

**Definition 2.1.** *(Asymptotic sequence.) A sequence of functions* $\{\phi_n(z)\}_{n=0}^{\infty}$ *is called an asymptotic sequence as* $z \longrightarrow z_0$ *from* $D$ *if whenever* $n > m$*, we have* $\phi_n(z) = o(\phi_m(z))$ *as* $z \longrightarrow z_0$ *from* $D$*.*

**Example 2.2.** The sequence of functions $\{z^n\}_{n=0}^{\infty}$ is an asymptotic sequence as $z \longrightarrow 0$, and $\{z^{-n}\}_{n=0}^{\infty}$ is an asymptotic sequence as $z \longrightarrow \infty$. □

**Definition 2.3.** *(Asymptotic expansion.) Let* $\{\phi_n(z)\}_{n=0}^{\infty}$ *be an asymptotic sequence as* $z \longrightarrow z_0$ *from* $D$*. Then the sum*

$$\sum_{n=0}^{N} a_n \phi_n(z)$$

*is said to be an asymptotic approximation as* $z \longrightarrow z_0$ *from* $D$ *of a function* $f(z)$ *if*

$$f(z) - \sum_{n=0}^{N} a_n \phi_n(z) = o(\phi_N(z))$$

*as* $z \longrightarrow z_0$ *from* $D$*. If* $\{a_n\}_{n=0}^{\infty}$ *is a sequence of complex constants such that the above is true for each* $N$*, then the formal infinite series*

$$\sum_{n=0}^{\infty} a_n \phi_n(z)$$

*is called an asymptotic series and is said to be an asymptotic expansion of* $f(z)$ *as* $z \longrightarrow z_0$ *from* $D$*.*

Normally the asymptotic expansion of $f(z)$ is written in the following way:

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$$

as $z \longrightarrow z_0$.
Note that the symbol $\sim$ is used here. That is because the series does not necessarily need to converge and the use of an equality sign would therefore be incorrect. Even if the the series is divergent it will still say a lot about the behavior of the function $f(z)$.

**Example 2.4.** Consider a function $f$ being $C^\infty$ in a neighborhood of the origin. Then from Taylor's theorem we have the asymptotic expansion

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

as $x \longrightarrow 0$. Indeed,

$$f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n = E_n(x),$$

where for $x \in [-R, R]$ the remainder term is

$$E_n(x) = \frac{f^{(n+1)}(s)}{(n+1)!} x^{n+1}$$

for some $s$ between 0 and $x$. This implies that $E_n(x) = O(x^{n+1})$, which is stronger than $o(x^n)$.

□

Note that a given function may have several asymptotic expansions, and that an asymptotic series does not need to represent a specific function. However, the coefficients $a_n$ in an asymptotic expansion of a function with respect to a given asymptotic sequence are unique.

**Example 2.5.** Suppose that

$$F(\lambda) \sim \sum_{n=0}^{\infty} \frac{a_n}{\lambda^n}$$

as $\lambda \longrightarrow \infty$ and $\operatorname{Re}\lambda > 0$.
If the term $e^{-\lambda}$ is added to $F(\lambda)$ the same asymptotic expansion is still valid, i.e.

$$e^{-\lambda} + F(\lambda) \sim \sum_{n=0}^{\infty} \frac{a_n}{\lambda^n}$$

as $\lambda \longrightarrow \infty$, with $\operatorname{Re}\lambda > 0$.
We say that $e^{-\lambda}$ is "beyond all orders" with respect to the asymptotic sequence $\{\lambda^{-n}\}_{n=0}^{\infty}$.

□

## 2.2 Watson's lemma

There are many different techniques to obtain asymptotic expansions of integrals. One elementary method is integration by parts. In this section a particular case of the integral (1.1) will be considered, namely

$$F(\lambda) = \int_0^T e^{-\lambda t}\phi(t)dt. \tag{2.1}$$

This corresponds to $R(t) = -t$, so $R(t)$ has a maximum at the left endpoint of the interval $[0, T]$.

**Example 2.6.** *(Integration by parts.)* Suppose

$$F(\lambda) = \int_0^T e^{-\lambda t} g(t)dt,$$

where $g \in C^\infty$ and $T \in \mathbb{R}_+$. By using integration by parts it follows that the asymptotic expansion of this integral is

$$F(\lambda) \sim \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{\lambda^{n+1}} \tag{2.2}$$

as $\lambda \longrightarrow \infty$. Indeed, making the first step with this well-known method one obtains the following:

$$F(\lambda) = -\frac{g(t)e^{-\lambda t}}{\lambda}\Big|_0^T + \int_0^T \frac{g'(t)e^{-\lambda t}}{\lambda}dt.$$

Continuing repeating this gives

$$F(\lambda) = \sum_{n=0}^{N} \left( \frac{g^{(n)}(0)}{\lambda^{n+1}} - \frac{g^{(n)}(T)e^{-\lambda T}}{\lambda^{n+1}} \right) + \int_0^T \frac{g^{(n+1)}(t)e^{-\lambda t}}{\lambda^{n+1}}dt, \tag{2.3}$$

where the integral in (2.3) is $O(\lambda^{-(N+2)})$ as $\lambda \longrightarrow \infty$. This implies (2.2)

□

Clearly this method is not suited for all types of integrals of the form (1.1). Another way of finding the asymptotic expansion for integrals is the following classical result known as Watson's lemma, that is a generalization of the example above.

**Proposition 2.7.** *(Watson's lemma.) Suppose that $T \in \mathbb{R}_+$ and that $g(t)$ is a complex-valued, absolutely integrable function on $[0, T]$:*

$$\int_0^T |g(t)|dt < \infty.$$

*Suppose further that $g(t)$ is of the form $g(t) = t^\sigma \phi(t)$ where $\sigma > -1$ and $\phi(t)$ has an infinite number of derivatives in some neighborhood of $t = 0$. Then the exponential integral*

$$F(\lambda) = \int_0^T e^{-\lambda t} g(t)dt$$

*is finite for all* $\lambda > 0$ *and it has the asymptotic expansion*

$$F(\lambda) \sim \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)\Gamma(\sigma+n+1)}{n!\lambda^{(\sigma+n+1)}} \tag{2.4}$$

*as* $\lambda \longrightarrow +\infty$.

For the proof of this proposition, see [8].

**Remark 2.8.** In many situations $T = +\infty$. In this case the conditions are unnecessarily restrictive. If $T = +\infty$ it suffices to assume that $\phi(t) = O(e^{at})$ as $t \to \infty$ for some $a \in \mathbb{R}$, and the proposition still holds true.

□

**Remark 2.9.** From the definition of the gamma function

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt, \quad \operatorname{Re} z > 0, \tag{2.5}$$

it follows that

$$\Gamma(z+1) = z\Gamma(z), \tag{2.6}$$

and in particular $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. So in case $\sigma = 0$ the asymptotic expansion expression (2.4) is reduced to

$$F(\lambda) \sim \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{\lambda^{n+1}},$$

which is the same result as (2.2). □

Note that the integral expression (2.1) and the Laplace transform of $f(t)$ defined by

$$\mathcal{L}(f)(s) = \int_0^{\infty} e^{-st}f(t)dt,$$

are of the same form, with $T = \infty$. So in this case one can get an asymptotic expansion of a Laplace transform for large $s$ immediately. For many exponential integrals Watson's lemma can be used by making adjustments of the variables so that the required structure of the lemma is met, and then apply it to obtain a full asymptotic expansion.

**Example 2.10.** Consider an integral of the form

$$F(\lambda) = \int_{-\alpha}^{\beta} e^{-\lambda t^2}\phi(t)dt, \tag{2.7}$$

where as in Proposition 2.7 the function $\phi$ is assumed to be infinitely differentiable in a neighborhood of 0. Let $\gamma = \min(\alpha, \beta)$, then

$$F(\lambda) - \int_{-\gamma}^{\gamma} e^{-\lambda t^2}\phi(t)dt = o(\lambda^{-n}),$$

for all $n \in \mathbb{N}$, as $\lambda \longrightarrow \infty$.
Because removing the part of the interval that not is around the point $t = 0$ will give an error beyond all orders with respect to $\left\{\lambda^{-n}\right\}_{n=0}^{\infty}$. Now, using symmetry

$$\int_{-\gamma}^{\gamma} e^{-\lambda t^2}\phi(t)dt = \int_{0}^{\gamma} e^{-\lambda t^2}\phi(t)dt + \int_{-\gamma}^{0} e^{-\lambda t^2}\phi(t)dt$$

$$= \int_{0}^{\gamma} e^{-\lambda t^2}(\phi(t) + \phi(-t))dt = 2\int_{0}^{\gamma} e^{-\lambda t^2}\phi_{\text{even}}(t)dt, \tag{2.8}$$

where

$$\phi_{\text{even}}(t) = \frac{(\phi(t) + \phi(-t))}{2}.$$

Making the substitution $t = \sqrt{s}$, the integral (2.8) becomes

$$\int_{0}^{\gamma^2} e^{-\lambda s}\frac{\phi_{\text{even}}(\sqrt{s})}{\sqrt{s}}ds. \tag{2.9}$$

By writing $\phi_{\text{even}}(t)$ as its Taylor series we obtain

$$\phi_{\text{even}}(t) \sim \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{\phi^{(n)}(0)}{n!}t^n + \frac{\phi^{(n)}(0)}{n!}(-t)^n\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{\phi^{(n)}(0)}{n!}(1 + (-1)^n)t^2 = \sum_{n=0}^{\infty}\frac{\phi^{(2n)}(0)}{(2n)!}t^{2n},$$

which gives us

$$\phi_{\text{even}}(\sqrt{s}) \sim \sum_{n=0}^{\infty}\frac{\phi^{(2n)}(0)}{(2n)!}s^n.$$

The integral (2.9) is now suitable for using Watson's lemma and we observe that $\sigma = -1/2$ and $T = \gamma^2$ . Thus we obtain the asymptotic expansion

$$F(\lambda) \sim \sum_{n=0}^{\infty}\frac{\phi^{(2n)}(0)\Gamma(n + 1/2)}{(2n)!\lambda^{n+1/2}} \tag{2.10}$$

as $\lambda \longrightarrow \infty$.
This asymptotic expansion can be written in another form. Using the relation (2.6) for the Gamma function in (2.10), we get

$$\Gamma(n + 1/2) = \frac{(2n)!}{2^{2n}n!}\Gamma(1/2).$$

From the definition of the Gamma function in (2.5), $\Gamma(1/2)$ is written as a Gaussian integral

$$\Gamma(1/2) = \int_{0}^{\infty} t^{-1/2}e^{-t}dt = \int_{-\infty}^{\infty} e^{-s^2}ds,$$

with $t = s^2$. Integrating using polar coordinates, it follows that

$$\Gamma(1/2)^2 = \left(\int_{-\infty}^{\infty} e^{-s^2}ds\right)^2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy = \int_{-\pi}^{\pi} \int_{0}^{\infty} re^{-r^2} drd\theta$$

$$= 2\pi \int_{0}^{\infty} re^{-r^2} dr = \pi \int_{0}^{\infty} e^{-\omega} d\omega = \pi.$$

So $\Gamma(1/2) = \sqrt{\pi}$. Thus the asymptotic expansion formula becomes

$$F(\lambda) \sim \sqrt{\frac{\pi}{\lambda}} \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(0)}{2^{2n} n!} \lambda^{-n} \tag{2.11}$$

as $\lambda \longrightarrow \infty$. □

This asymptotic expansion (2.11) illustrates the case when the exponent $R(t)$ attains a maximum at an interior point of the interval of integration and as we will see this result have resemblance to the leading order behavior obtained by Laplace's method. This topic will be discussed in the next section.

# 3 Laplace's Method

Consider an integral of the form

$$F(\lambda) = \int_a^b e^{\lambda R(t)} g(t) dt, \tag{3.1}$$

where $(a, b)$ is finite or infinite and real, as $\lambda \longrightarrow +\infty$ . Laplace's method is a way of finding approximations of integrals of this form. The idea is that the main contributions to $F(\lambda)$ as $\lambda \longrightarrow \infty$ will come from a neighborhood of the point or points where $R(t)$ has its maximum value on the interval $(a, b)$, and the rest of the contributions of the integral that are not close to the maximum will be exponentially small in comparison.

Suppose first that $R(t)$ has a maximum attained at a single interior point on the interval $(a, b)$, called $t_{\max}$. Assume furthermore that $R(t)$ and $g(t)$ are infinitely differentiable functions in a neighborhood of $t = t_{\max}$ and $R''(t_{\max}) < 0$.

Since we know that the main contribution of the integral (3.1) comes from the neighborhood of $t_{\max}$, we can remove the rest of the interval because it's contribution is beyond all orders with respect to $\left\{\lambda^{-n}\right\}_{n=0}^{\infty}$. We write $\tilde{R}(t) = R(t) - R(t_{\max})$ and choose $\delta$ as small as we please. Hence, the integral that contributes to the asymptotic expansion is

$$e^{\lambda R(t_{\max})} \int_{t_{\max}-\delta}^{t_{\max}+\delta} e^{-\lambda \tilde{R}(t)} g(t) dt$$

$$= e^{\lambda R(t_{\max})} \int_{-\delta}^{\delta} e^{-\lambda \tilde{R}(t_{\max}+\tau)} g(t_{\max} + \tau) d\tau. \tag{3.2}$$

For this integral we wish to make the change of variables,

$$\tilde{R}(t_{\max} + \tau) = -s^2, \tag{3.3}$$

in order to have (3.2) in the same form as the integral in Example 2.10 in the previous section. First we need to solve (3.3) for $\tau$ as a smooth function of $s$ near the point $s = 0$. Let

$$\psi(s, \tau) = \tilde{R}(t_{\max} + \tau) + s^2.$$

The implicit function theorem in this case fails because $\frac{\partial \psi}{\partial \tau}(0, 0) = 0$. We go around this problem by using a new variable $v$, defined by $\tau = sv$. We then have

$$\frac{\tilde{R}(t_{\max} + sv)}{s^2} = -1. \tag{3.4}$$

By expanding $\tilde{R}(t_{\max} + \tau)$ in a neighborhood of $\tau = 0$, we get

$$\tilde{R}(t_{\max} + sv) = \frac{1}{2} R''(t_{\max})(sv)^2 + O((sv)^3).$$

Using the expression above in the formula (3.4), one obtains

$$\frac{1}{2} R''(t_{\max}) v^2 + O(sv^3) = -1.$$

Observe that this equation is satisfied by the point $(0, v_0)$, where

$$v_0 = \sqrt{\frac{-2}{R''(t_{\max})}}. \tag{3.5}$$

Note that we here choose $v_0$ positive. Now, let

$$f(s,v) = \begin{cases} \dfrac{\tilde{R}(t_{\max} + sv)}{s^2} + 1, & \text{if } s \neq 0, \\ \dfrac{1}{2} R''(t_{\max}) v^2 + 1, & \text{if } s = 0. \end{cases} \tag{3.6}$$

Then $f \in C^\infty$. Since $\frac{\partial f}{\partial v}(0, v_0) = R''(t_{\max}) v_0 \neq 0$, by the implicit function theorem the relation $f(s,v) = 0$ locally near $(0, v_0)$ defines $v$ as a function of s. Now introduce the change of variables $\tau(s) = sv(s)$. Then the integral in (3.2) becomes

$$e^{\lambda R(t_{\max})} \int_{-\alpha}^{\beta} e^{-\lambda s^2} \phi(s) ds,$$

where $\alpha = \sqrt{-\tilde{R}(t_{\max} - \delta)}$ and $\beta = \sqrt{-\tilde{R}(t_{\max} + \delta)}$. We also have $\phi(s) = g(t_{\max} + sv(s))(sv'(s) + v(s))$. This is now an integral of the same form as (2.7) in the last section. By applying (2.11), we find that the asymptotic expansion is

$$F(\lambda) \sim e^{\lambda R(t_{\max})} \sqrt{\frac{\pi}{\lambda}} \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(0)}{2^{2n} n! \lambda^n} \tag{3.7}$$

as $\lambda \longrightarrow \infty$.
Note that it is possible to calculate as many terms in the asymptotic expansion (3.7) as one likes, but the higher order derivatives of $\phi$ will quickly lead to long calculations. For the first term of (3.7), we calculate $\phi(0)$:

$$\phi(0) = g(t_{\max}) v(0) = \sqrt{\frac{-2}{R''(t_{\max})}} g(t_{\max}).$$

It follows that the leading order behavior of $F(\lambda)$ is

$$F(\lambda) = \int_a^b e^{-\lambda R(t)} g(t) dt = e^{\lambda R(t_{\max})} \left( \sqrt{\frac{-2\pi}{\lambda R''(t_{\max})}} g(t_{\max}) + O(\lambda^{-3/2}) \right) \tag{3.8}$$

as $\lambda \longrightarrow \infty$. See [8] for more terms of the expansion.

For an interval containing many interior points $(t_1, ..., t_N)$ at which the maximum is attained, a similar result holds; one only has to sum the contributions from each maximum point. Thus the leading order term in this case will be given by

$$F(\lambda) = e^{\lambda R(t_{\max})} \left( \sum_{j=1}^{N} \sqrt{\frac{-2\pi}{\lambda R''(t_j)}} g(t_j) + O(\lambda^{-3/2}) \right)$$

as $\lambda \longrightarrow \infty$.

In case the maximum is obtained at one of the endpoints of the interval, the evaluation of the leading order behavior is even simpler. By replacing $R(t)$ with the first terms of its series expansion near the endpoint $R(t_{\max}) + R'(t_{\max})(t - t_{\max})$ will give the following results:

If the unique maximum is at the left boundary $t = a$ of the interval $[a, b]$ and $R'(a) < 0$, then

$$F(\lambda) = -e^{\lambda R(t_{\max})} \left( \frac{g(a)}{\lambda R'(a)} + O(\lambda^{-2}) \right)$$

as $\lambda \longrightarrow \infty$.
If the unique maximum is at the right boundary point $t = b$ and $R'(b) > 0$, then

$$F(\lambda) = e^{\lambda R(b)} \left( \frac{g(b)}{\lambda R'(b)} + O(\lambda^{-2}) \right)$$

as $\lambda \longrightarrow \infty$.

In the case when there are contributions coming from both interior and boundary points the dominant contribution will come from the interior points since the contribution from the boundary points is much smaller in comparison. With Laplace's method there is no limit in how many terms of the asymptotic expansion one can obtain, but the larger the number of terms there is the more effort it takes to calculate those terms. So it is in theory possible to obtain the full asymptotic expansion of an integral using this method.

Throughout the rest of this paper only the leading order term of the asymptotic expansion of integrals like (1.1) will be calculated since it gives an approximation that for most applications is sufficient.

**Example 3.1.** Consider

$$n! = \Gamma(n+1) = \int_0^\infty e^{-t} t^n dt.$$

By making the variable substitution $t = sn$ and rewriting $s^n$ as a exponential function $e^{n(\log(s))}$, the integral takes the form

$$\Gamma(n+1) = n^{n+1} \int_0^\infty e^{n(\log(s) - s)} ds.$$

The integral is now of the right form needed to be able to apply Laplace's method. Set $R(s) = \log(s) - s$. Since $R'(s) = \frac{1}{s} - 1$ the maximum is at a point $s_{\max} = 1$ and since $R''(s_{\max}) = -1$ the formula (3.8) can be used and thus the leading order behavior is

$$n! \sim n^{n+1} e^{-n} \left( \sqrt{\frac{2\pi}{n}} + O(n^{-3/2}) \right)$$

as $n \longrightarrow \infty$.
This result is the first term in Stirling's formula that is used for approximating factorials.

□

# 4 Method of Steepest Descent

## 4.1 The idea

So far Laplace type integrals of real-valued functions have been studied, but for many problems and applications the integral is in the complex plane. Consider an integral of the form

$$F(\lambda) = \int_C e^{\lambda h(z)} g(z) dz = \int_a^b e^{\lambda h(z(t))} g(z(t)) z'(t) dt \tag{4.1}$$

as $\lambda \longrightarrow +\infty$.
Here $C$ is a smooth curve in the complex plane and $z(t) = x(t) + iy(t)$, $t \in [a, b]$, a parametrization of the contour. We moreover assume that $h(z)$ and $g(z)$ are complex-valued entire functions of $z$.

Write $h(z) = R(x, y) + iI(x, y)$ for $z = x + iy$. Suppose that $I(x, y)$ is constant along $C$. Then the function $F(\lambda)$ can be rewritten in the form

$$F(\lambda) = e^{i\lambda I} \int_a^b e^{\lambda R(x(t), y(t))} u(t) dt + ie^{i\lambda I} \int_a^b e^{\lambda R(x(t), y(t))} v(t) dt, \tag{4.2}$$

where $u(t) = \operatorname{Re} g(z(t))z'(t)$ and $v(t) = \operatorname{Im} g(z(t))z'(t)$. The asymptotic expansion of each of the integrals can here be evaluated using Laplace's method since they are real. Note that it is highly unusual for an integral to have the property that $I(x, y)$ is constant from the beginning, so in most cases it is not possible to use Laplace's method directly.

There is however a way of finding the asymptotic expansion of complex-valued integrals when $I(x, y)$ not is constant along $C$. For a general contour $C$, the strategy is to deform $C$ into another new contour $C'$ along which the imaginary part $I(x, y)$ of $h(z)$ is constant.
By Cauchy's theorem it is possible to make contour deformations without affecting the value of the integral. When a suitable deformed contour $C'$ has been obtained an asymptotic expansion of the integral can be computed using Laplace's method. This method is called the method of steepest descent, or the saddle point method for reasons to be explained.

**Remark 4.1.** Deformation of the contour is still feasible even if $g(z)$ is merely meromorphic, or even has branch cuts, by applying the residue theory. □

## 4.2 Steepest descent paths and saddle points

Let $z_0 = x_0 + iy_0$ be a point in the complex plane and suppose that $h(z) = R(x, y) + iI(x, y)$ is analytic at $z_0$. When the value of $R(x, y)$ is declining from its value in the point $z_0$ in a direction emanating from $z_0$, that direction is a descent direction.
A curve emanating from $z_0$ with the tangents being descent directions for every

point on the curve is a descent path. The descent paths with the most rapid descent are called steepest descent paths. We desire to find these paths because it turns out that the curves where $I(x, y)$ is constant at the same time are steepest descent paths.

The gradient of a function indicates the direction in which the function increases most rapidly. As $I(x, y)$ is constant on $C'$, then the gradient of $I(x, y)$:

$$\nabla I(x, y) = \frac{\partial I}{\partial x}(x, y)\mathbf{i} + \frac{\partial I}{\partial y}(x, y)\mathbf{j}$$

is normal to $C'$. Since $h(z)$ is analytic the Cauchy-Riemann equations are satisfied

$$\frac{\partial R}{\partial x}(x, y) = \frac{\partial I}{\partial y}(x, y),$$
$$\frac{\partial R}{\partial y}(x, y) = -\frac{\partial I}{\partial x}(x, y).$$

This implies that

$$\frac{\partial R}{\partial x}(x, y)\frac{\partial I}{\partial x}(x, y) + \frac{\partial R}{\partial y}(x, y)\frac{\partial I}{\partial y}(x, y) = 0, \tag{4.3}$$

or, in other words, $\nabla R(x, y) \cdot \nabla I(x, y) = 0$.

It follows that $\nabla R(x, y)$ is perpendicular to $\nabla I(x, y)$. Consequently $\nabla R(x, y)$ will be tangent to $C'$. Thus, a properly oriented curve $I(x, y) = I(x_0, y_0)$ represents a curve of steepest descent of $R(x, y)$. This explains the name "Method of steepest descent".

We now explain why this method is also called the saddle point method. Suppose $t_0$ is a critical point of $R(x(t), y(t))$, that is

$$\frac{d}{dt}R(x(t), y(t))\bigg|_{t=t_0} = 0,$$

or equivalently

$$\nabla R(x(t), y(t)) \cdot (x'(t), y'(t))|_{t=t_0} = 0.$$

Since $(x'(t_0), y'(t_0)) \neq \bar{0}$, this implies that $\nabla R(x_0, y_0) = \bar{0}$ where $(x_0, y_0) = (x(t_0), y(t_0))$. Using the Cauchy-Riemann equations we find that

$$\frac{\partial R}{\partial x}(x_0, y_0) = 0, \quad \frac{\partial I}{\partial x}(x_0, y_0) = 0$$
$$\frac{\partial R}{\partial y}(x_0, y_0) = 0, \quad \frac{\partial I}{\partial y}(x_0, y_0) = 0.$$

Hence both vectors $\nabla R(x, y)$ and $\nabla I(x, y)$ will vanish at this point;

$$\nabla R(x_0, y_0) = \nabla I(x_0, y_0) = \bar{0}. \tag{4.4}$$

So $h'(z_0) = 0$. From the result (4.4) it follows that the main contribution comes from critical points of $h(z)$. In particular, if $R(x, y)$ has a maximum on $C'$ at

$z_0$, then $z_0$ is a critical point of h.

Note that $R(x, y)$ and $I(x, y)$ are solutions to Laplace equation and therefore are harmonic functions (which can be proved by Cauchy-Riemann's equations). For harmonic functions the maximum principle states the following:

**Theorem 4.2.** *(Weak Maximum Principle.) Let $D$ be a bounded domain , and let $u(x, y) \in C^2(D) \cap C(D)$ be a harmonic function in $D$. Then the maximum (minimum) of $u$ in $D$ is achieved on the boundary $\partial D$.*

**Theorem 4.3.** *(Strong Maximum Principle.) Let $u$ be a harmonic function in the domain $D$. If $u$ attains its maximum (minimum) at an interior point of $D$, then $u$ is constant.*

So $R(x, y)$ cannot have a maximum at $(x_0, y_0)$, i.e. the critical point $(x_0, y_0)$ of $R(x, y)$ is a saddle point. Because of the importance of the saddle points in the evaluation of integrals with contours without endpoints, the method of steepest descent is also called the saddle point method.

Near the saddle point $z = z_0$ the Taylor expansion of $h(z)$ has the form

$$h(z) = h(z_0) + \frac{h^{(p)}(z_0)}{p!}(z - z_0)^p + O((z - z_0)^{p+1}). \tag{4.5}$$

By setting $z = z_0 + re^{i\theta}$ and $h^{(p)}(z_0) = ae^{i\alpha}$, the leading terms of the Taylor expansion of $h(z)$ in (4.5) may be written in the form

$$h(z) - h(z_0) = \frac{ae^{i\alpha}}{p!} r^p e^{ip\theta} + O(r^{p+1}) =$$

$$= \frac{r^p a}{p!}(\cos(\alpha + p\theta) + i \sin(\alpha + p\theta)) + O(r^{p+1}).$$

This gives the following when dividing it into real and imaginary parts

$$R(x, y) = \frac{r^p a}{p!} \cos(\alpha + p\theta) + O(r^{p+1})$$

and

$$I(x, y) = \frac{r^p a}{p!} \sin(\alpha + p\theta) + O(r^{p+1}).$$

As discussed before it is known that the steepest descent curves that go through the saddle point have the property that $\operatorname{Im}(h(z) - h(z_0)) = 0$, therefore

$$\sin(\alpha + p\theta) = 0.$$

This implies that the direction of the steepest descent curves comes from the following angles:

$$\theta = -\frac{\alpha}{p} + \frac{n\pi}{p}, \tag{4.6}$$

where $n$ is any odd number $n = 1, 3, 5, ..., 2p-1$. The steepest ascent directions are obtained from the even numbers $n = 0, 2, ..., 2p-2$. We will only be interested in the case of simple saddle points, i.e. $p = 2$.

**Example 4.4.** Let $h(z) = i\sin(z)$. Then since $h'(z) = i\cos(z)$ and the saddle points are

$$z_m = \frac{\pi}{2} + m\pi, \quad m \in \mathbb{Z}.$$

Consider $z_0 = \frac{\pi}{2}$. Then $h''(z_0) = -i$ and thus we obtain $\alpha = -\frac{\pi}{2}$. So by (4.6) the steepest descent directions are

$$\theta = -\frac{\pi}{4} + \frac{n\pi}{2}, \quad n = 1, 3.$$

□

## 4.3 Leading order behavior of $F(\lambda)$

Recall the expression (4.2) where the Laplace's method could be used because the imaginary part of $h(z)$ was set to be constant on $C$. Now suppose that $C$ is deformed to go where the imaginary part is constant along a steepest descent path and through a saddle point as described earlier. Because the imaginary part is constant, $h(z) - h(z_0)$ is real on this deformed contour $C'$.

Let the maximum value of the real-part of $h(z)$ appear in a point denoted $t_0$ on the interval $[a, b]$ and that the saddle point $z_0 = z(t_0)$ is simple (that is $h'(z_0) = 0$ and $h''(z_0) \neq 0$). Furthermore let $C'$ have the parametrization $z(t)$, making it resemble the interval $[a, b]$. The integral (4.1) thus becomes

$$F(\lambda) = e^{\lambda h(z_0)} \int_a^b e^{\lambda(h(z(t))-(h(z_0)))} g(z(t))z'(t)dt.$$

From the leading order behavior expression (3.8) when the maximum value comes from an interior point, we have that

$$\int_a^b e^{\lambda(h(z)-(h(z_0)))} g(z(t))z'(t)dt =$$

$$= \sqrt{\frac{-2\pi}{\lambda(z'(t_0))^2 h''(z_0)}} g(z_0)z'(t_0) + O(\lambda^{-3/2}). \tag{4.7}$$

Because

$$\frac{d^2}{dt^2}(h(z) - h(z_0)) = (z'(t))^2 h''(z(t)) + h'(z(t))z''(t),$$

it follows that

$$\frac{d^2}{dt^2}(h(z) - h(z_0)) = (z'(t))^2 h''(z(t)), \tag{4.8}$$

since the second term is zero because $h'(z_0) = 0$. Note that (4.8) is a negative real number. We rewrite

$$\sqrt{\frac{-2\pi}{(z'(t_0))^2 h''(z_0)}} z'(t_0) = \sqrt{\frac{2\pi}{|h''(z_0)|}}$$

and with $z'(t_0) = |z'(t_0)|e^{i\theta(z_0)}$ we thus obtain the following formula for the asymptotic leading order behavior of $F(\lambda)$:

$$F(\lambda) = \lambda^{-1/2} e^{\lambda h(z_0)} \left( e^{i\theta(z_0)} \sqrt{\frac{2\pi}{|h''(z_0)|}} g(z_0) + O(\lambda^{-1}) \right) \tag{4.9}$$

as $\lambda \longrightarrow \infty$.

# 5 Applications

## 5.1 Airy functions

These are special functions named after George Biddell Airy that are solutions to the differential equation

$$y''(x) - xy(x) = 0, \tag{5.1}$$

called the Airy equation. The Airy functions appear in many situations in both mathematics and physics[1]. For example, in quantum mechanics, they describe the behavior of the wave function near the classical turning points. To later be able to examine the asymptotic behavior of the Airy function, suppose that the solutions to the differential equation (5.1) can be written as contour integrals of the form

$$y(x) = \frac{1}{2\pi i}\int_C q(z)e^{xz}dz. \tag{5.2}$$

Substituting the integral (5.2) into Airy's equation (5.1) we obtain

$$\int_C z^2 q(z)e^{xz}dz - \int_C xq(z)e^{xz}dz = 0. \tag{5.3}$$

Then, by using integration by parts, we get

$$\int_C z^2 q(z)e^{xz}dz - q(z)e^{xz}\Big|_C + \int_C q'(z)e^{xz} = 0.$$

This is done because we don't want the parameter $x$ as a factor in the second integral of (5.3). Let's choose $q(z)$ so that it solves the differential equation

$$q'(z) + z^2 q(z) = 0,$$

i.e. take $q(z) = e^{-z^3/3}$. We also need to choose the contour $C$ in such a way that the integrand disappears at the end-points of $C$. Therefore $C$ is set to go to infinity in a region where $\text{Re}\,(z^3) = r^3\cos(3\theta) > 0$ (with $z = re^{i\theta}$). This gives three sectors defined by

$$\frac{-5\pi}{6} < \theta < \frac{-\pi}{2}, \qquad \frac{-\pi}{6} < \theta < \frac{\pi}{6}, \qquad \frac{\pi}{2} < \theta < \frac{5\pi}{6}.$$

In each of these sectors the integrand goes to zero as $z$ goes to infinity. The integrand of (5.2) thus is $e^{xz-z^3/3}$ and we have a solution to the differential equation (5.1) of the form

$$y(x) = \frac{1}{2\pi i}\int_C e^{xz-\frac{z^3}{3}}dz. \tag{5.4}$$

[1] Airy functions and applications to physics, Olivier Valle'e Manuel Soares.

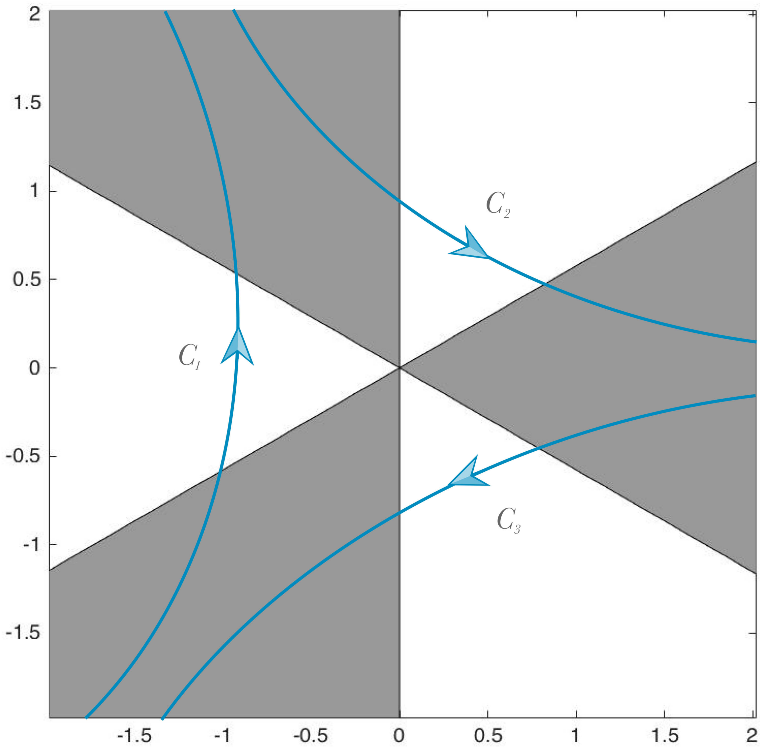


Figure 1: The sectors $-\frac{5\pi}{6} < \theta < -\frac{\pi}{2}, -\frac{\pi}{6} < \theta < \frac{\pi}{6}, \frac{\pi}{2} < \theta < \frac{5\pi}{6}$ are shown in grey and the contours $C_1, C_2$, and $C_3$ in blue color.

To get non-trivial solutions, we let $C$ go from infinity in one sector and end at infinity in another. Because there are three sectors we have essentially three different choices of contours, $C_1, C_2$, and $C_3$. Each choice gives a solution $y_i$. Since (5.1) only has two linearly independent solutions these are of course linearly dependent; clearly $y_1 + y_2 + y_3 = 0$. See Figure 1.

The solution $y_1$ is called the Airy function of the first kind and is denoted by

$$\mathrm{Ai}\,(x) = \frac{1}{2\pi i}\int_{C_1} e^{xz - z^3/3} dz. \tag{5.5}$$

The Airy function of the second kind, also known as the Bairy function, is defined by

$$\mathrm{Bi}\,(x) = iy_2 - iy_3,$$

or in other words is the integral (5.4) over $C_2 - C_3$. These functions are linearly independent solutions to the differential equation (5.1).

### 5.1.1 Asymptotics as $x \longrightarrow +\infty$

Let us write the parameter $x$ in polar form as $x = re^{ik}$. We here consider the asymptotic behavior of $\mathrm{Ai}\,(re^{ik})$ as $r \longrightarrow \infty$ for a fixed value of $k \in (-\pi, \pi)$. This includes the case $x \longrightarrow +\infty$. Since the integral (5.5) is not of the form (4.1) we first introduce the change of variables $z = \sqrt{r}u$, which gives the integrand the appropriate form:

$$\mathrm{Ai}\,(re^{ik}) = \frac{\sqrt{r}}{2\pi i}\int_{C_1} e^{r^{3/2}(e^{ik}u - u^3/3)}du \tag{5.6}$$

where $r \longrightarrow \infty$.
To simplify the calculations later we set $\lambda = r^{3/2}$ and consider the integral

$$I(\lambda) = \int_{C_1} e^{\lambda(e^{ik}u - \frac{u^3}{3})}du.$$

Observe that this is precisely of the form (4.1). We here have that

$$h(u) = e^{ik}u - \frac{u^3}{3}. \tag{5.7}$$

We know that saddle points are obtained from $h'(u) = e^{ik} - u^2 = 0$, which gives two saddle points at $u = \pm e^{ik/2}$. We will be using the notation

$$u_R = e^{ik/2}, \qquad u_L = -e^{ik/2}$$

for the two saddle points since the former is located in the right halfplane and the latter in the left. Now, writing $h(u_R) = \frac{2}{3}e^{3ik/2}$ in terms of cosine and sine we get

$$\mathrm{Re}\,(h(u_R)) = \frac{2}{3}\cos(\frac{3k}{2}), \qquad \mathrm{Im}\,(h(u_R)) = \frac{2}{3}\sin(\frac{3k}{2}). \tag{5.8}$$

In the same way we get

$$\mathrm{Re}\,(h(u_L)) = -\frac{2}{3}\cos(\frac{3k}{2}), \qquad \mathrm{Im}\,(h(u_L)) = -\frac{2}{3}\sin(\frac{3k}{2}). \tag{5.9}$$

The formulas (5.8) and (5.9) will be helpful in determining from where the dominant contribution to the asymptotic expansion will come. We observe that,

$$\mathrm{Re}\,(h(u_L)) < \mathrm{Re}\,(h(u_R)) \quad \text{for} \quad -\frac{\pi}{3} < k < \frac{\pi}{3},$$

$$\mathrm{Re}\,(h(u_R)) < \mathrm{Re}\,(h(u_L)) \quad \text{for} \quad -\pi < k < -\frac{\pi}{3}, \quad \frac{\pi}{3} < k < \pi,$$

and

$$\mathrm{Re}\,(h(u_R)) = \mathrm{Re}\,(h(u_L)) \quad \text{for} \quad k = \pi.$$

For the imaginary part we have

$$\mathrm{Im}\,(h(u_R)) = \mathrm{Im}\,(h(u_L)) \quad \text{for} \quad k = 0, \quad k = -\frac{2\pi}{3}, \quad k = \frac{2\pi}{3}.$$

To get a better view of how the steepest descent paths and the level curves of $R(x, y)$ depend on the parameter $k$, we start by considering some particular values of $k$:

Consider first $k = 0$. For this value of $k$ we deform $C_1$ to the steepest descent path that goes through $u_L$, which is part of the contour $I(x, y) = \mathrm{Im}\,(h(u_L)) = 0$. Note that it is not possible to deform the contour $C_1$ to the steepest descent path through the saddle point $u_R$. This value of $k$ is interesting because it is one of the few cases where the steepest descent curves can be drawed by hand. Examining $h(u)$ in (5.7) with $k = 0$, we find that

$$\mathrm{Im}\,(h(u)) = b\left(\frac{b^2}{3} - a^2 + 1\right), \quad u = a + ib.$$

Clearly, the zero set of this expression is the union of the hyperbola and line in Figure 2.

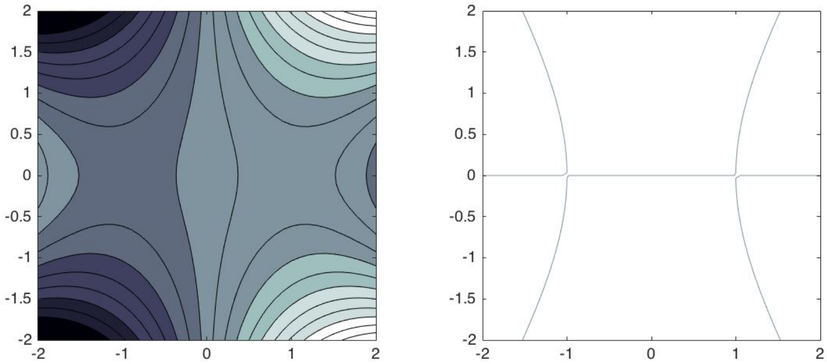


Figure 2: For $k = 0$. Left: Contour plot for $\mathrm{Re}\,(h(u))$, darker color indicate lower level curves. Right: The steepest descent paths.

For $k = \frac{\pi}{2}$ the contour $C_1$ is still suitable for deformation and the dominant contribution is coming from $u_L$, see Figure 3.

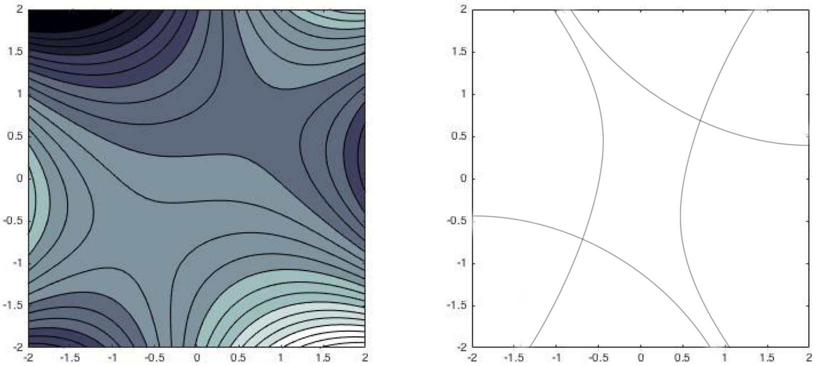


Figure 3: For $k = \frac{\pi}{2}$. Left: Contour plot for $\mathrm{Re}\,(h(u))$, darker color indicate lower level curves. Right: The steepest descent paths.

When $k = \frac{2\pi}{3}$ the steepest descent curve look as in Figure 4. In this case the steepest descent contour will go through $u_L$ to $u_R$ and then turn left to go along the descent path down from this saddle point. The real part of $u_L$ is greater than the real part of $u_R$, so the dominant contribution comes from $u_L$.

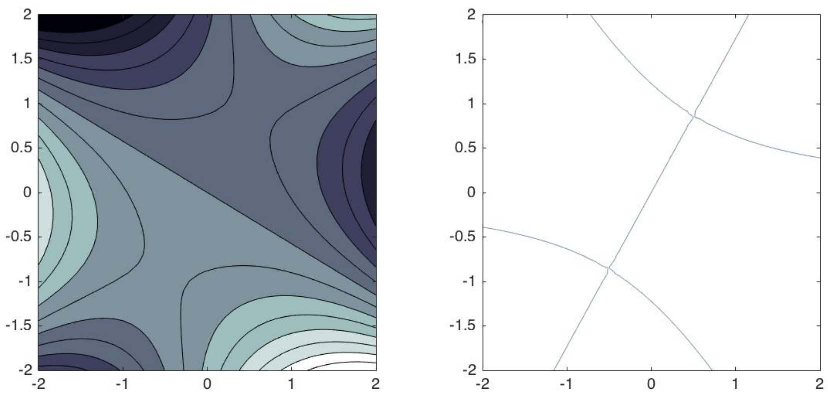


Figure 4: For $k = \frac{2\pi}{3}$. Left: Contour plot for $\mathrm{Re}\,(h(u))$, darker color indicate lower level curves. Right: The steepest descent paths.

Back to finding the leading order formula. We have

$$h''(u) = -2u \quad \text{so} \quad h''(u_L) = 2e^{ik/2},$$

which gives $\alpha = \frac{k}{2}$. By the formula (4.6) the steepest descent directions are

$$\theta = -\frac{k}{4} + \frac{n\pi}{2},$$

for odd $n$. Formula (4.9) gives that the leading order behavior of $I(\lambda)$ is

$$I(\lambda) = \lambda^{-1/2} e^{\lambda h(u_L)} \left( e^{i\theta(u_L)} \sqrt{\frac{2\pi}{|h''(u_L)|}} g(u_L) + O(\lambda^{-1}) \right)$$

$$= \sqrt{\pi} \lambda^{-1/2} e^{-\frac{2\lambda e^{3ik/2}}{3}} e^{i(-\frac{k}{4}+\frac{\pi}{2})} (1 + O(\lambda^{-1})) \tag{5.10}$$

as $\lambda \longrightarrow \infty$.

Hence by setting $\lambda = r^{3/2}$ and using the leading order expression of $I(\lambda)$ from above in the integral (5.6), the Airy function has the asymptotics

$$\mathrm{Ai}\,(x) = \frac{1}{2x^{1/4}\sqrt{\pi}} e^{-\frac{2x^{3/2}}{3}} (1 + O(|x|^{-\frac{3}{2}})) \tag{5.11}$$

as $x \longrightarrow \infty$.

Note that $x \in \mathbb{C}$ and that both $x^{1/4}$ and $x^{3/2}$ are the principal branches in (5.11). In a similar way the leading order expression for the Bairy function is obtained to be

$$\mathrm{Bi}\,(x) = \frac{1}{x^{1/4}\sqrt{\pi}} e^{\frac{2x^{3/2}}{3}} (1 + O(|x|^{-\frac{3}{2}})) \tag{5.12}$$

as $x \longrightarrow \infty$.

### 5.1.2 Asymptotics as $x \longrightarrow -\infty$

When $k = \pi$ we have steepest descent paths with the appearance as in Figure 5. We here have two contours to consider. The first is going through $u_L$ ending in the right sector and the second is beginning in the right sector going through $u_R$. The contributions from both of these saddle points must be taken into account when evaluating the asymptotics. This is because the contributions from both saddle points are of the same order. Note that the leading order asymptotics is oscillatory, since $h(u_L)$ and $h(u_R)$ are both purely imaginary.

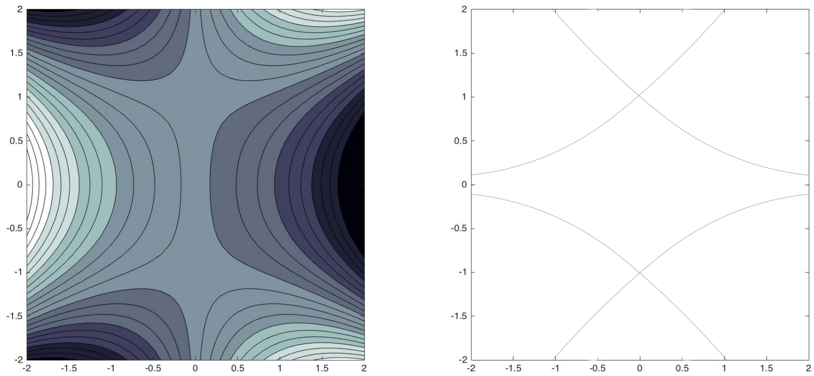


Figure 5: For $k = \pi$. Left: Contour plot for $\mathrm{Re}\,(h(u))$, darker color indicate lower level curves. Right: The steepest descent paths.

For this value of $k$ we choose a contour $C_1'$ containing both the steepest descent curve through $u_L$ and the steepest descent curve through $u_R$. We write

$$C_1' = \gamma_L + \gamma_R.$$

By (5.6) we have

$$\mathrm{Ai}\,(re^{i\pi}) = \frac{\sqrt{r}}{2\pi i}\int_{C_1'} e^{r^{3/2}(e^{i\pi}u - u^3/3)}du$$

$$= \frac{\sqrt{r}}{2\pi i}\left(\int_{\gamma_L} e^{r^{3/2}(e^{i\pi}u - u^3/3)}du + \int_{\gamma_R} e^{r^{3/2}(e^{i\pi}u - u^3/3)}du\right)$$

$$= \mathrm{Ai}_{u_L}(re^{i\pi}) + \mathrm{Ai}_{u_R}(re^{i\pi}). \tag{5.13}$$

From the formula (5.10) with $k = \pi$ the leading order behavior of $I_{u_L}(\lambda)$ is

$$I_{u_L}(\lambda) = \sqrt{\pi}\lambda^{-1/2}e^{-\frac{2\lambda e^{3i\pi/2}}{3}}e^{i(-\frac{\pi}{4}+\frac{\pi}{2})}(1 + O(\lambda^{-1})) \tag{5.14}$$

as $\lambda \longrightarrow \infty$.
Since $\lambda = r^{3/2}$, the first term in (5.13) has the leading order asymptotics

$$\mathrm{Ai}_{u_L}(x) = \frac{1}{2i|x|^{1/4}\sqrt{\pi}}e^{i\left(\frac{2|x|^{3/2}}{3}+\frac{\pi}{4}\right)}(1 + O(|x|^{-\frac{3}{2}}) \tag{5.15}$$

as $x \longrightarrow -\infty$.

Because
$$h''(u) = -2u \quad \text{and} \quad h''(u_R) = -2e^{ik/2},$$
we obtain $\alpha = -\frac{k}{2}$ for this saddle point. By the formula (4.6) the steepest descent directions are
$$\theta = \frac{\pi}{4} + \frac{n\pi}{2}.$$
By the formula (5.10) the leading order behavior of $I_{u_R}(\lambda)$ becomes

$$I_{u_R}(\lambda) = \sqrt{\pi}\lambda^{-1/2} e^{\frac{2\lambda e^{3i\pi/2}}{3}} e^{i(-\frac{\pi}{4}+\pi)}(1 + O(\lambda^{-1})) \tag{5.16}$$

as $\lambda \longrightarrow \infty$.
Again using that $\lambda = r^{3/2}$, we find that the second term in (5.13) has the leading order asymptotics

$$\mathrm{Ai}_{u_R}(x) = -\frac{1}{2i|x|^{1/4}\sqrt{\pi}} e^{-i\left(\frac{2|x|^{3/2}}{3}+\frac{\pi}{4}\right)}(1 + O(|x|^{-\frac{3}{2}})) \tag{5.17}$$

as $x \longrightarrow -\infty$.

From (5.15) and (5.17) we obtain the asymptotic expansion of $\mathrm{Ai}\,(x)$:
$$\mathrm{Ai}\,(re^{i\pi}) = \mathrm{Ai}_{u_L}(re^{i\pi}) + \mathrm{Ai}_{u_R}(re^{i\pi})$$
$$= \frac{1}{2i|x|^{1/4}\sqrt{\pi}} \left( e^{i\left(\frac{2|x|^{3/2}}{3}+\frac{\pi}{4}\right)} - e^{-i\left(\frac{2|x|^{3/2}}{3}+\frac{\pi}{4}\right)} \right)(1 + O(|x|^{-\frac{3}{2}})).$$
Thus we finally get

$$\mathrm{Ai}\,(x) = \frac{1}{|x|^{1/4}\sqrt{\pi}} \sin\left(\frac{2|x|^{3/2}}{3} + \frac{\pi}{4}\right)(1 + O(|x|^{-\frac{3}{2}})) \tag{5.18}$$

as $x \longrightarrow -\infty$.
Similarly for the Bairy function, we obtain

$$\mathrm{Bi}\,(x) = \frac{1}{|x|^{1/4}\sqrt{\pi}} \cos\left(\frac{2|x|^{3/2}}{3} + \frac{\pi}{4}\right)(1 + O(|x|^{-\frac{3}{2}})) \tag{5.19}$$

as $x \longrightarrow -\infty$.

Note the drastic difference in the leading order behavior of the Airy and Bairy functions when $k = \pi$ compared to the previous case. The asymptotic behavior of a function in the complex plane may differ between particular sectors even though the function considered is entire. This is known as the Stokes phenomenon. In our case this happens precisely along the negative real line. For more detail about the Stokes phenomenon, see [6].

Figure 6 below shows a plot of the Airy and Bairy functions.

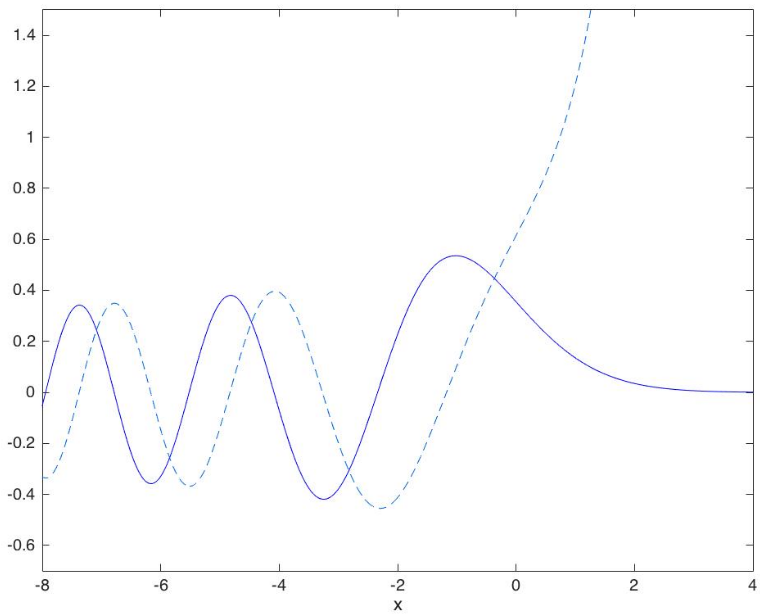


Figure 6: The Airy function (solid line) and the Bairy function (dashed line).

## 5.2 The KdV equation

The Korteweg–de Vries equation,

$$u_t + u_{xxx} + 6uu_x = 0, \tag{5.20}$$

named after Diederik Korteweg and Gustav de Vries is describing the behavior of waves in shallow water. The behavior of solutions of the KdV equation is still an area of active research. We will here examine the Cauchy problem for the linearized KdV equation,

$$u_t + u_{xxx} = 0, \tag{5.21}$$

with the initial data $u(x,0) = g(x)$. We assume that $g$ belongs to $C_0^\infty(\mathbb{R})$, that is the class of infinitely differentiable functions with compact support, which implies that $\hat{g}$ can be extended to an entire function. One can solve this equation using the Fourier transform. Indeed, taking the Fourier transform with respect to $x$ we obtain

$$\hat{u}_t + (i\omega)^3\hat{u} = 0 \quad \text{and} \quad \hat{u}(\omega,0) = \hat{g}(\omega).$$

This implies that

$$\hat{u}(\omega,t) = \hat{g}(\omega)e^{-(i\omega)^3 t}.$$

We thus obtain the solution

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega) e^{i\omega^3 t + i\omega x} d\omega. \tag{5.22}$$

Writing

$$h(\omega) = i\left(\omega^3 + \frac{\omega x}{t}\right),$$

the integral (5.22) becomes

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega) e^{t h(\omega)} d\omega. \tag{5.23}$$

Note that this integral now is of the same form as (4.1). We wish to find the long-time behavior of $u(x,t)$ as $t \longrightarrow \infty$ with $\frac{x}{t}$ fixed. We first handle the cases $\frac{x}{t} < 0$ and $\frac{x}{t} > 0$.

We begin with the case $\frac{x}{t} < 0$. Examining the exponent function $h$ in integral (5.23), we find the saddle points

$$\omega_L = -\left|\frac{x}{3t}\right|^{1/2} \quad \text{and} \quad \omega_R = \left|\frac{x}{3t}\right|^{1/2}.$$

In this case we deform the real line to the union of two contours, see Figure 7. The first begins in the valley of the upper left half plane, goes through $\omega_L$ and ends close to the imaginary axis in the valley in the lower half plane. The second contour begins in the valley in the lower half plane, goes through $\omega_R$ and ends in the valley in the first quadrant. The contributions from the saddle points are of the same order, therefore both these contributions must be taken into account.

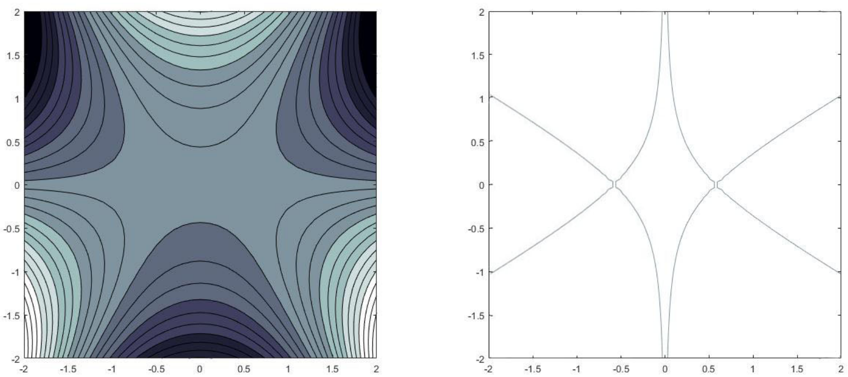

Figure 7: Left: Contour plot for $\mathrm{Re}\,(h(u))$, darker color indicate lower level curves. Right: The steepest descent paths.

We write

$$u(x,t) = u_{\omega_L}(x,t) + u_{\omega_R}(x,t). \tag{5.24}$$

Because $h''(\omega) = 6i\omega$, then

$$h''(\omega_L) = -6i\left|\frac{x}{3t}\right|^{1/2} \quad \text{and} \quad h''(\omega_R) = 6i\left|\frac{x}{3t}\right|^{1/2} \tag{5.25}$$

We begin with evaluating the contribution coming from $\omega_L$. From (5.25) we find that $\alpha = \frac{3\pi}{2}$, thus by the formula (4.6) the steepest descent directions are

$$\theta = -\frac{3\pi}{4} + \frac{n\pi}{2}.$$

From the formula (4.9) the leading order behavior of $u_{\omega_L}(x,t)$ in (5.24) is obtained:

$$\frac{1}{\sqrt{12\pi t}\left|\frac{x}{3t}\right|^{1/4}}\hat{g}\left(-\left|\frac{x}{3t}\right|^{1/2}\right)e^{2it\left|\frac{x}{3t}\right|^{3/2}-i\frac{\pi}{4}}(1+O(t^{-1})) \tag{5.26}$$

as $t \longrightarrow \infty$.
For $\omega_R$ we get $\alpha = \frac{\pi}{2}$ by using (5.25). From the formula (4.6) the steepest descent directions are

$$\theta = -\frac{\pi}{4} + \frac{n\pi}{2}.$$

From the formula (4.9) we obtain the leading order behavior of $u_{\omega_R}(x,t)$ in (5.24):

$$\frac{1}{\sqrt{12\pi t}\left|\frac{x}{3t}\right|^{1/4}}\hat{g}\left(\left|\frac{x}{3t}\right|^{1/2}\right)e^{-2it\left|\frac{x}{3t}\right|^{3/2}+i\frac{\pi}{4}}(1+O(t^{-1})) \tag{5.27}$$

as $t \longrightarrow \infty$.
Using the complex conjugate of the fourier transform $\hat{g}(-\omega) = \overline{\hat{g}(\omega)}$ and writing $\hat{g}(\left|\frac{x}{3t}\right|^{1/2}) = r(\left|\frac{x}{t}\right|)e^{ik(\frac{x}{t})}$, then by adding these formulas we obtain the leading order behavior of $u(x,t)$:

$$u(x,t) = \frac{r(\left|\frac{x}{t}\right|)}{\sqrt{\pi t}\left|\frac{3x}{t}\right|^{1/4}}\cos\left(2t\left|\frac{x}{3t}\right|^{3/2} - \frac{\pi}{4} - k\left(\frac{x}{t}\right)\right)(1+O(t^{-1})) \tag{5.28}$$

as $t \longrightarrow \infty$.

For $\frac{x}{t} > 0$ we have the saddle points

$$\omega_L = -i\left(\frac{x}{3t}\right)^{1/2} \quad \text{and} \quad \omega_R = i\left(\frac{x}{3t}\right)^{1/2}.$$

In this case the contour is deformed so it goes through $\omega_R$ and end in the valley in the first quadrant, see Figure 8. So the dominant contribution comes from $\omega_R$.

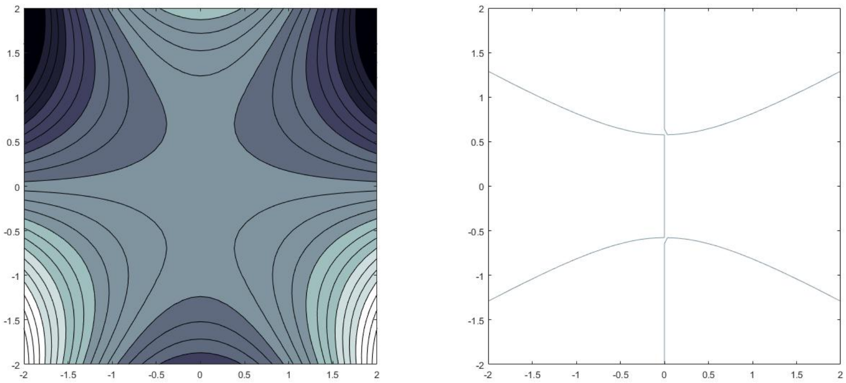

Figure 8: Left: Contour plot for $\mathrm{Re}\,(h(u))$, darker color indicate lower level curves. Right: The steepest descent paths.

Because

$$h''(\omega_R) = -6\left(\frac{x}{3t}\right)^{1/2},$$

we have $\alpha = \pi$ and the steepest descent directions

$$\theta = -\frac{\pi}{2} + \frac{n\pi}{2}.$$

The leading order behavior of $u(x,t)$ when $\frac{x}{t} > 0$ thus is

$$u(x,t) = \frac{1}{\sqrt{12\pi t}(\frac{x}{3t})^{1/4}}\hat{g}\left(i\left(\frac{x}{3t}\right)^{1/2}\right)e^{-2t(\frac{x}{3t})^{3/2}}(1+O(t^{-1})) \tag{5.29}$$

as $t \longrightarrow \infty$.
Note that the asymptotics in the two cases considered are drastically different. The error terms depend on the parameter $\eta = \frac{x}{t}$.

For $\frac{x}{t} \longrightarrow 0$, we begin by making the change of variables

$$-iz = \omega(3t)^{1/3} \tag{5.30}$$

in the integral (5.23) and write $\nu = x(3t)^{-1/3}$. The integral then becomes

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi i(3t)^{1/3}}\int_{-i\infty}^{i\infty}\hat{g}\left(\frac{-iz}{(3t)^{1/3}}\right)e^{z\nu-\frac{z^3}{3}}dz \\ &= \frac{1}{2\pi i(3t)^{1/3}}\int_{C_1}\hat{g}\left(\frac{-iz}{(3t)^{1/3}}\right)e^{z\nu-\frac{z^3}{3}}dz. \end{aligned} \tag{5.31}$$

Note the similarity between this integral and the integral (5.5). The leading order behavior of $u(x,t)$ can be calculated by expanding $\hat{g}$ in a neighborhood of $z = 0$ and using the leading order behavior of the Airy function. We thus obtain the leading order behavior:

$$u(x,t) = (3t)^{-1/3}\hat{g}(0)\mathrm{Ai}\,(\nu)(1 + O((3t)^{-1/3})). \tag{5.32}$$

as $t \longrightarrow \infty$.
We see that it connects the asymptotics in (5.28) and (5.29). Given the structure of the prefactor in equation (5.32), it is worth mentioning that there is a relation between the Airy equation and the linearized KdV equation. Indeed, if we let

$$u(x,t) = \frac{1}{(3t)^{1/3}} y\left(\frac{x}{(3t)^{1/3}}\right), \tag{5.33}$$

then a simple calculation shows that $u$ satisfies (5.23) if $y$ satisfies (5.1). We therefore expect a solution that depends on the scale $x(3t)^{-1/3}$ as $\frac{x}{t} \longrightarrow 0$. The equation (5.33) is an example of an ODE reduction of a PDE.

## 5.3 Concluding remarks

In the last sections of this paper we have discussed the method of steepest descent and some applications to linear ODE's and linear PDE's. The classical steepest descent method also has applications to other areas of mathematics, such as combinatorics. Indeed, the famous Hardy-Ramanujan-Uspensky formula for the partition function can be obtained by applying it, see [3]. In recent years a nonlinear method of steepest descent for oscillatory Riemann-Hilbert problems has been introduced by Deift and Zhou [7]. This method can be used to compute the long-time behavior of solutions of so-called integrable nonlinear PDE's, such as the KdV equation (5.20) and the nonlinear Schrödinger equation. In fact, the nonlinear method of steepest descent was introduced with the purpose of getting a better understanding of the nonlinear KdV equation. The study of the behavior of integrable nonlinear PDEs continues to be an active area of research.
In recent years so-called Painlevé transcendents has started to appear in many applications. For example, ODE reductions of integrable nonlinear PDE's lead to Painlevé equations. The Painlevé transcendents are nonlinear analogs of the classical special functions such as the Airy and Bessel functions. Also these transcendents can be studied with the nonlinear steepest descent method. Indeed, the nonlinear method of steepest descent has also been used to solve long-standing problems in e.g. combinatorics, such as Ulam's problem for the longest increasing subsequence in random permutations, see [4].

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