M341 (92150), Sample Midterm #2 Questions

Below are some sample midterm questions. Please note that the intent of these is to help prepare for the exam, and that actual exam questions will not merely be modifications of these problems. Consult your HW, lecture notes, and books for additional sources of material to review. Finally, to obtain the full effect of an exam please complete these problems under time-pressure (75 minutes or less).

1. Let $B = \begin{bmatrix} -1 & 3 & -3 \\ 0 & -6 & 5 \\ -5 & -3 & 1 \end{bmatrix}$.

   a) Show that B is invertible and compute B^{-1}.

   b) Suppose we replaced the second row $[0, -6, 5]$ of B with $[-2, 6, -6]$. Will the resulting matrix still be invertible? [Hint: There is a very quick way of finding the answer that does not require any long computations!]

2. Let $A = \begin{bmatrix} 4 & 3 & 1 & 2 \\ 1 & 9 & 0 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{bmatrix}$.

   a) Calculate the determinant of A using a cofactor expansion.

   b) Recalculate the determinant using row reduction to verify your answer to (a).

   c) What is the determinant of $-2A$? Why?

3. Prove that if A is an orthogonal matrix (i.e., $A^T = A^{-1}$) then the determinant of A is either 1 or -1.

4. Let $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$.

   a) Determine the eigenvalues of A.

   b) Find a nonsingular matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

   c) Compute the determinant of A only using your answer to part (a) (i.e., do not compute the determinant directly.

   [Hint: Recall the definition of the characteristic polynomial $p_A(\lambda)$.]

5. The parts of the following question are unrelated.

   a) Is $\mathcal{V} = \mathbb{R}$ with the usual scalar multiplication, but with addition defined as $\boldsymbol{x} \oplus \boldsymbol{y} = 3(x + y)$ a vector space? Justify your answer.

   b) Find the zero vector and the additive inverse of the vector space $\mathbb{R}^2$ with operations $[x, y] \oplus [w, z] = [x + w + 3, y + z - 4]$ and $a \odot [x, y] = [ax + 3a - 3, ay - 4a + 4]$.

   c) If $\mathcal{V}$ is a vector space with subspace $\mathcal{W}_1$ and $\mathcal{W}_2$, prove that $\mathcal{W}_1 \cap \mathcal{W}_2$ is also a subspace.

   [Hint: Do not forget to show that $\mathcal{W}_1 \cap \mathcal{W}_2$ is nonempty!]

   d) Prove that all vectors orthogonal to $[2, -3, 1]^T$ forms a subspace $\mathcal{W}$ of $\mathbb{R}^3$.