M346 (56615), Homework #11

Due: 03:30pm, Tuesday, Apr. 23

Self-adjoint and normal operators

A) Find an orthonormal basis consisting of eigenvectors of $A = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$.

B) Find an orthonormal basis consisting of eigenvectors of $A = \begin{pmatrix} 2 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 2 \end{pmatrix}$.

C) True or false? Justify your answers.

   i. If L is self-adjoint, then L^k is self-adjoint.

   ii. A normal operator with all eigenvalues real must be self-adjoint.

   iii. The sum of two normal operators is normal.

   iv. If $N \in M_{2,2}(\mathbb{R})$ is a real normal matrix then it must either be symmetric (and therefore self-adjoint) or take the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for some $a, b \in \mathbb{R}$.

Isometries

A) Recall the Frobenius inner product $\langle A|B\rangle = \mathrm{Tr}(A^*B)$ for A, $B \in M_{m,n}(\mathbb{C})$. This defines the Frobenius norm $\|A\| = \sqrt{\langle A|A\rangle}$. Note that $\|A\|^2 = \mathrm{Tr}(A^*A) = \sum_{i=1}^n \sum_{j=1}^n |A_{ij}|^2$.

   i. Show that the Frobenius norm is unitarily invariant. That is, show that if W is unitary then $\|WA\| = \|A\|$ and $\|AW\| = \|A\|$ for any A. [Hint: Use the cyclical properties of the trace: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any A, B.]

   ii. We say that A and B are unitarily equivalent if $A = UBU^*$ for some unitary U. In this case, the previous part implies that $\|A\| = \|B\|$. Use this to prove that the matrices $\begin{pmatrix} 1 & 2 \\ 2 & i \end{pmatrix}$ and $\begin{pmatrix} i & 4 \\ 1 & 1 \end{pmatrix}$ cannot be unitarily equivalent.

B) Is $A = \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix}$ unitary? If so, check that its eigenvalues all have magnitude 1. [Hint: Remember that a matrix is unitary if and only if its columns are orthonormal!]

C) Let $\boldsymbol{v} \in \mathbb{C}^n$ with $\|\boldsymbol{v}\| = 1$, and define $H_{\boldsymbol{v}} = I - 2\boldsymbol{v}\boldsymbol{v}^*$ (this is known as a *Householder transformation* and reflects the vector $\boldsymbol{v}$ to its negative while leaving its orthogonal complement invariant). Show that $H_{\boldsymbol{v}}$ is unitary.

Positive operators

A) Consider the symmetric matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. Find $\sqrt{A}$ and verify directly that $\left(\sqrt{A}\right)^2 = A$.