M346 (56615), Homework #12

Due: 03:30pm, Tuesday, Apr. 30

Singular value decomposition (SVD)

A) Show that if A is positive, its spectral decomposition $A = UDU^*$ agrees exactly with its singular value decomposition $A = U\Sigma V^*$ (i.e., show that $\Sigma = D$ and $V = U$).

B) Let $A = \begin{pmatrix} 0 & 0 & -5 \\ -9 & 12 & 0 \\ 0 & 0 & 0 \\ 8 & 6 & 0 \end{pmatrix}$

   i. Compute the SVD of A. Express your answer (i) as the sum of rank-1 terms and (ii) as $A = U\Sigma V^*$ for an appropriate U, V, and Σ.

   ii. Find the best rank-2 approximation A_2 of A (where "best" implies closest to in squared Frobenius norm). Express your answer (i) as a sum of two rank-1 terms and (ii) as $A_2 = U\Sigma_2 V^*$ for an appropriate U, V, and Σ_2.

   iii. Compute the approximation error $\|A - A_2\|$ in terms of the singular values of A.

C) Let $A = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}$.

   i. Find the SVD of A.

   ii. In $\mathbb{R}^2$, describe the image of the unit disc under the transformation A using SVD. That is, draw a picture of the region $\{A\boldsymbol{x} \colon \|\boldsymbol{x}\| \leq 1\}$.

   iii. Similarly, describe the inverse image of the unit disc by drawing a picture of the region $\{\boldsymbol{x} \colon \|A\boldsymbol{x}\| \leq 1\}$.