M346 (56615), Homework #9

Due: 3:30pm, Tuesday, Apr. 09

Instructions: Questions are from the book "Applied Linear Algebra, 2nd ed." by Sadun. Please show all your work, not only your final answer, to receive credit. Keep answers organized in the same order the problems have been assigned.

Inner products (6.1-6.2)

p. 151-152, #4, 7

Expansion in orthonormal bases (6.4)

p. 161, #1, 2, 3

In addition:

A) Consider the real vector space $V = \mathbb{R}^2$. Is $\langle \boldsymbol{x} | \boldsymbol{y} \rangle = x_1 y_2 - x_2 y_1$ an inner product? What about $\langle \boldsymbol{x} | \boldsymbol{y} \rangle = x_1 y_2 + x_2 y_1$?

B) On $\mathbb{C}^3$ with the standard inner product, compute $\langle \boldsymbol{x}|$ and $\langle \boldsymbol{y}|$, where $\boldsymbol{x} = (1, 2i, 3+i)^T$ and $\boldsymbol{y} = (-i, 2, 5-i)^T$. In addition, compute $\langle \boldsymbol{x} | \boldsymbol{y} \rangle$ and $\langle \boldsymbol{y} | \boldsymbol{x} \rangle$ by evaluating each as the product of a bra (i.e., a row vector) and a ket (i.e., a column vector).

C) It is straightforward to show that for $A, B \in M_{m,n}(\mathbb{C})$,

$$\langle A | B \rangle = \mathrm{Tr}(A^* B)$$

is an inner product (called the Frobenius inner product, which is exactly the standard inner product on $\mathbb{C}^m$ when $n = 1$). Here, $A^* = \bar{A}^T$ is known as the Hermitian adjoint of A.

Compute $\|A\|^2$ for $A = \begin{pmatrix} 2-i & 1 & 3 \\ 3 & 2i & 1+i \end{pmatrix}$. Show that this is exactly the sum of the square magnitudes of the entries of A.

D) Prove the Cauchy-Schwarz inequality $|\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle| \leq \|\boldsymbol{x}\| \|\boldsymbol{y}\|$ using the following steps:

   i. We have that $0 \leq \|\boldsymbol{x} - t\boldsymbol{y}\|^2 = \langle \boldsymbol{x} - t\boldsymbol{y} | \boldsymbol{x} - t\boldsymbol{y} \rangle$ for any $t \in \mathbb{C}$. Expand out the complex inner product using sesquilinearity.

   ii. Assuming $\boldsymbol{y} \neq \boldsymbol{0}$ (since in this case the Cauchy-Schwarz inequality is trivial), substitute $t = \langle \boldsymbol{y} | \boldsymbol{x} \rangle / \|\boldsymbol{y}\|^2$ in (i) to obtain the desired result.

E) Find all vectors in $\mathbb{R}^4$ that are orthogonal (in the standard inner product) to the subspace spanned by $\boldsymbol{b}_1 = (1, 1, 1, 1)^T$ and $\boldsymbol{b}_2 = (1, 2, 3, 4)^T$.