LECTURE 25
03/26/12

DUALITY (6.3):

- IF $V = \mathbb{C}^n$ w/ STD. INNER PROD. $\langle \underline{x} | \underline{y} \rangle = \underline{x}^* \underline{y}$, THEN $|\underline{y}\rangle = \underline{y}$ AND $\langle \underline{x} | = \underline{x}^*$.

Q: WHAT IF $V = \mathbb{C}^n$ w/ NONSTANDARD INNER PROD.?

EX. $V = \mathbb{C}^2$, $\langle \underline{x} | \underline{y} \rangle = 2\bar{x}_1 y_1 + 3\bar{x}_2 y_2$.

$$\Rightarrow |\underline{y}\rangle = (y_1, y_2)^T \in \mathbb{C}^2.$$

$$\langle \underline{x} | = (2\bar{x}_1, 3\bar{x}_2)^T \in (\mathbb{C}^2)'.$$

- IF V GENERAL INNER PROD. SPACE w/ BASIS $\mathcal{B} = \{b_i\}_{i=1}^n$,

$$\langle \underline{x} | \underline{y} \rangle_{\mathcal{B}} = \left\langle \sum_{i=1}^n a_i \underline{b}_i \,\middle|\, \sum_{j=1}^n c_j \underline{b}_j \right\rangle$$

$$= \sum_{i=1}^n \bar{a}_i \sum_{j=1}^n c_j \underbrace{\langle \underline{b}_i | \underline{b}_j \rangle}_{\substack{n \times n \text{ HERMITIAN} \\ \text{MATRIX,} \\ \text{CALL } G_{\mathcal{B}}.}}$$

$$= \overline{(a_1, \ldots, a_n)}\, G_{\mathcal{B}}\, (c_1, \ldots, c_n)^T$$

$$= \overline{[\underline{x}]_{\mathcal{B}}}^T G_{\mathcal{B}} [\underline{y}]_{\mathcal{B}}$$

WE CALL $G_{\mathcal{B}}$ THE <u>METRIC</u>. BY DEFINITION,

$$(G_{\mathcal{B}})_{ij} = \langle \underline{b}_i | \underline{b}_j \rangle$$

SO $G_{\mathcal{B}}^* = G_{\mathcal{B}}$. BY THE EXPRESSION ABOVE, WE HAVE THAT

$$|y\rangle_{\mathcal{B}} = [\underline{y}]_{\mathcal{B}} \in \mathbb{C}^n$$

$${}_{\mathcal{B}}\langle x| = \overline{[\underline{x}]_{\mathcal{B}}}^{T} G_{\mathcal{B}} \in (\mathbb{C}^n)'.$$

<u>NOTE</u>: IF $\mathcal{B} = \{b_i\}_{i=1}^{n}$ IS AN ORTHONORMAL BASIS THEN

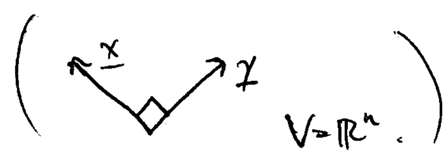
$$G_{\mathcal{B}} = I \quad \text{AND} \quad {}_{\mathcal{B}}\langle x| = \overline{[\underline{x}]_{\mathcal{B}}}^{T} = [\underline{x}]_{\mathcal{B}}^{*}.$$

IN THIS CASE, THE INNER PRODUCT IS <u>STANDARD</u>.

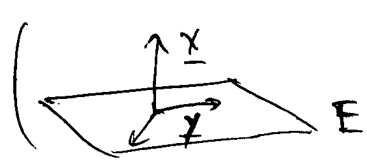
<u>Q</u>: WHAT DO WE MEAN BY AN ORTHONORMAL BASIS?

<u>ORTHOGONALITY AND ORTHONORMAL BASES (6.4):</u>

<u>DEF.</u> FOR $\underline{x}, \underline{y} \in V$, $\underline{x}$ IS <u>ORTHOGONAL</u> TO $\underline{y}$ (DENOTE $\underline{x} \perp \underline{y}$) IF $\langle \underline{x} | \underline{y} \rangle = 0$.



• IF $E \subset V$ SUBSPACE, $\underline{x} \perp E$ IF $\underline{x} \perp \underline{y}$ $\forall \underline{y} \in E$.



EX. FIND ALL VECTORS ORTHOGONAL TO $\underline{x} = (i, 2, 1+i)^T$.

$$\langle \underline{x} | \underline{y} \rangle = \underline{x}^* \underline{y} = 0 \Rightarrow (-i, 2, 1-i)(y_1, y_2, y_3)^T = 0.$$

EX. FIND ALL VECTORS ORTHOGONAL TO BOTH $\underline{x}_1 = (i, 2, 1+i)^T$ AND $\underline{x}_2 = (3, 2-i, i)^T$.

$$\left.\begin{aligned} \langle \underline{x}_1 | \underline{y} \rangle &= x_1^* \underline{y} = 0 \\ \langle \underline{x}_2 | \underline{y} \rangle &= x_2^* \underline{y} = 0 \end{aligned}\right\} \Rightarrow A^* \underline{y} = \underline{0} \quad \text{WHERE}$$

$$A = (\underline{x}_1, \underline{x}_2) \in M_{3,2}(\mathbb{C}) = \begin{pmatrix} i & 3 \\ 2 & 2-i \\ 1+i & i \end{pmatrix}.$$

- $\{\underline{b}_1, \ldots, \underline{b}_n\}$ ORTHOGONAL IF $\underline{b}_i \perp \underline{b}_j \ \forall i \neq j$.
  $\{\underline{b}_1, \ldots, \underline{b}_n\}$ ORTHONORMAL IF " " AND $\|\underline{b}_i\| = 1 \ \forall i = 1, \ldots, n$.

THM. (i) (PYTHAGOREAN THM.)

$$\{\underline{b}_i\} \text{ ORTHOGONAL} \Rightarrow \|\underline{b}_1 + \cdots + \underline{b}_n\|^2 = \|\underline{b}_1\|^2 + \cdots + \|\underline{b}_n\|^2.$$

PF. $(n=2)$

$$\begin{aligned} \|\underline{x}+\underline{y}\|^2 &= \langle \underline{x}+\underline{y} | \underline{x}+\underline{y} \rangle \\ &= \langle \underline{x}|\underline{x} \rangle + \cancelto{0}{\langle \underline{x}|\underline{y} \rangle} + \cancelto{0}{\langle \underline{y}|\underline{x} \rangle} + \langle \underline{y}|\underline{y} \rangle \\ &= \|\underline{x}\|^2 + \|\underline{y}\|^2. \end{aligned}$$

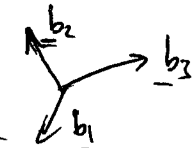
(ii) $\{\underline{b}_i\}$ ORTHOGONAL $\Rightarrow \{\underline{b}_i\}_{i=1}^n$ ARE LINEARLY INDEP.

PF.

$$\begin{aligned} c_1 \underline{b}_1 + \cdots + c_n \underline{b}_n = \underline{0} &\Rightarrow \|\underline{0}\|^2 = |c_1|^2 \|\underline{b}_1\|^2 + \cdots + |c_n|^2 \|\underline{b}_n\|^2 \\ &\Rightarrow c_i = 0 \quad \forall i = 1, \ldots, n. \\ &\Rightarrow \text{LINEARLY INDEP.} \end{aligned}$$

EXPANSION IN ORTHOGONAL BASIS:

SUPPOSE $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ IS ORTHOGONAL AND IS A BASIS OF V.



$$\underline{x} = a_1 \underline{b}_1 + \dots + a_n \underline{b}_n \iff [\underline{x}]_{\mathcal{B}} = (a_1, \dots, a_n)^T.$$

$\underline{Q}$: HOW TO FIND a_i?

$\underline{A}$:
$$\langle \underline{b}_i | \underline{x} \rangle = \langle \underline{b}_i | a_1 \underline{b}_1 + \dots + a_n \underline{b}_n \rangle$$
$$= a_1 \langle \underline{b}_i | \underline{b}_1 \rangle + \dots + a_n \langle \underline{b}_i | \underline{b}_n \rangle$$
$$= a_i \langle \underline{b}_i | \underline{b}_i \rangle = a_i \| \underline{b}_i \|^2$$

$$\Rightarrow \boxed{a_i = \frac{\langle \underline{b}_i | \underline{x} \rangle}{\| \underline{b}_i \|^2}.}$$

THEREFORE,

$$\underline{x} = \sum_{i=1}^{n} \frac{\langle \underline{b}_i | \underline{x} \rangle}{\| \underline{b}_i \|^2} \underline{b}_i. \qquad \left(\underline{x} = \sum_{i=1}^{n} \langle \underline{b}_i | \underline{x} \rangle \underline{b}_i \quad \text{IF } \{\underline{b}_i\} \text{ ORTHO}\underline{\text{NORMAL}} \right)$$

$\underline{\text{NOTE}}$: IN BRA-KET NOTATION, THIS IS

$$|\underline{x}\rangle = \sum_{i=1}^{n} \frac{\langle \underline{b}_i | \underline{x} \rangle}{\langle \underline{b}_i | \underline{b}_i \rangle} |\underline{b}_i\rangle = \left(\sum_{i=1}^{n} \frac{|\underline{b}_i\rangle \langle \underline{b}_i|}{\langle \underline{b}_i | \underline{b}_i \rangle} \right) |\underline{x}\rangle,$$

SO $$I = \sum_{i=1}^{n} \frac{|\underline{b}_i\rangle \langle \underline{b}_i|}{\langle \underline{b}_i | \underline{b}_i \rangle}$$ IS THE IDENTITY OPERATOR!

Lecture 26

03/28/12

Last time we saw that (in bra-ket notation):

$$|\underline{x}\rangle = \underbrace{\left(\sum_{i=1}^{n} \frac{|\underline{b_i}\rangle\langle\underline{b_i}|}{\langle\underline{b_i}|\underline{b_i}\rangle}\right)}_{\underline{I}} |\underline{x}\rangle$$

What is $P_{\underline{b_i}} = \dfrac{|\underline{b_i}\rangle\langle\underline{b_i}|}{\langle\underline{b_i}|\underline{b_i}\rangle}$?

Projections and Gram-Schmidt process (6.5-6.6):

Def. $\underline{v} \in V$, $\underline{v} \neq \underline{0}$. Let $P_{\underline{v}} \doteq \dfrac{|\underline{v}\rangle\langle\underline{v}|}{\langle\underline{v}|\underline{v}\rangle}$

be the projection operator in direction of $\underline{v}$.

Thm. For any $\underline{x} \in V$, can write $\underline{x} = \underline{w} + \underline{y}$

where for some given $\underline{v} \neq 0$, $\underline{w} \parallel \underline{v}$ and $\underline{y} \perp \underline{v}$.

Pr. $\underline{x} = \underline{I}\,\underline{x} = \left[P_{\underline{v}} + (\underline{I} - P_{\underline{v}})\right]\underline{x}$

$$= \underbrace{P_{\underline{v}}\,\underline{x}}_{\text{call } \underline{w}} + \underbrace{(\underline{I} - P_{\underline{v}})\,\underline{x}}_{\text{call } \underline{y}}.$$

~~LET~~ $W \subset V$ SUBSPACE WITH ORTHOGONAL BASIS $\{d_1, \dots, d_m\}$.

DEF. $W^\perp \doteq$ SPACE OF ALL VECTORS $\perp$ TO W.

- NOTE THAT

$$\underline{x} \in W^\perp \iff P_W \underline{x} = \underline{0}$$
$$\iff \underline{x} \in \mathrm{Ker}(P_W)$$

WHERE

$$P_W = \sum_{i=1}^{m} P_{\underline{d}_i} = \sum_{i=1}^{m} \frac{|\underline{d}_i\rangle\langle \underline{d}_i|}{\langle \underline{d}_i | \underline{d}_i\rangle}$$

IS THE PROJECTION ONTO SUBSPACE W.

- ANY $\underline{x} \in V$ IS $\underline{x} = \underline{w} + \underline{y}$ WHERE $\underline{w} \in W$, $\underline{y} \in W^\perp$.

SINCE

$$\underline{x} = P_W \underline{x} + \underbrace{(I - P_W)}_{P_W^\perp \doteq P_{W^\perp}} \underline{x}$$

EX. $V = \mathbb{R}^2$, $W = \mathrm{span}\{\underbrace{(1,1)^T}_{\text{CALL THIS } \underline{b}}\}$.

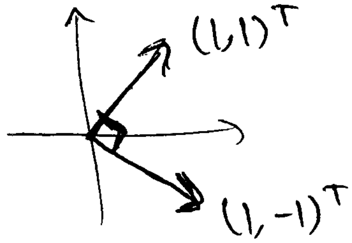
WHAT IS $W^\perp$?

$W^\perp =$ ALL $\underline{x} \in V$ S.T. $P_W \underline{x} = \underline{0}$.

$$P_W = \frac{|\underline{b}\rangle\langle \underline{b}|}{\langle \underline{b} | \underline{b}\rangle} = \frac{(1,1)^T (1,1)}{\|(1,1)^T\|^2} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

$\Rightarrow \quad Ker(P_W) = \text{SPAN}\left\{(1,-1)^T\right\}.$

THIS MAKES SENSE, AS WE SEE BY A PICTURE:



<u>EX.</u> $V = \mathbb{C}^2, \quad W = \text{SPAN}\left\{\underbrace{(1-i, 2i)}_{\text{CALL THIS } \underline{b}.}{}^T\right\}.$

$$P_W = \frac{(1-i, 2i)^T (1+i, -2i)}{\|(1-i, 2i)\|^2}$$

$$= \frac{1}{6}\begin{pmatrix} 2 & -2-2i \\ -2+2i & 4 \end{pmatrix}.$$

NOW FIND $Ker(P_W)$ TO OBTAIN $W^\perp$.

<u>Q</u>: IN ORDER TO USE THE FORMULA

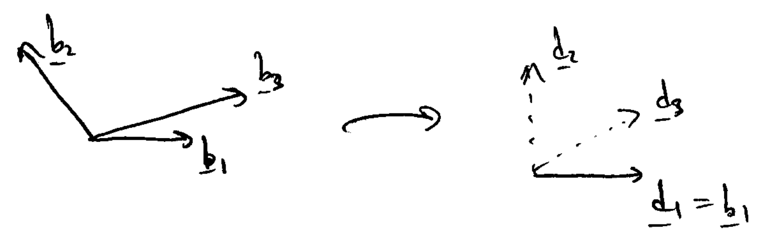
$$\underline{x} = \sum_{i=1}^{n} \frac{\langle \underline{d}_i | \underline{x} \rangle}{\|\underline{d}_i\|^2} \underline{d}_i,$$ WE HAD TO ASSUME

THAT $\{\underline{d}_i\}$ WERE ORTHOGONAL.

GIVEN SOME BASIS $\{\underline{b}_i\}$ OF V, HOW TO FIND AN ORTHOGONAL (OR ORTHONORMAL) BASIS OF V?

A: GRAM-SCHMIDT PROCEDURE.

IDEA: GIVEN BASIS $\mathcal{B} = \{\underline{b}_i\}_{i=1}^{n}$ OF V, START WITH FIRST VECTOR AND ITERATIVELY FIND "NEW" DIRECTIONS.



PROCEDURE: DEFINE NEW BASIS $\{\underline{d}_i\}_{i=1}^{n}$ BY:

1) $\underline{d}_1 = \underline{b}_1$

2) $$\underline{d}_2 = \underbrace{\left(I - P_{\underline{d}_1}\right)}_{P^{\perp}_{\underline{d}_1}} \underline{b}_2$$

3) $$\underline{d}_3 = \underbrace{\left(I - P_{\underline{d}_1} - P_{\underline{d}_2}\right)}_{P^{\perp}_{\mathrm{SPAN}\{\underline{d}_1, \underline{d}_2\}}} \underline{b}_3$$

$\vdots$

n) $$\underline{d}_n = \underbrace{\left(I - \sum_{i=1}^{n-1} P_{\underline{d}_i}\right)}_{P^{\perp}_{\mathrm{SPAN}\{\underline{d}_1, \ldots, \underline{d}_{n-1}\}}} \underline{b}_n \, .$$

$\Rightarrow \mathcal{D} = \{\underline{d}_i\}_{i=1}^{n}$ <u>ORTHOGONAL</u> BASIS OF V !

IF WE LET $\underline{e}_i = \dfrac{\underline{d}_i}{\|\underline{d}_i\|}$ FOR EACH $i = 1, \dots, n$,

THEN $\mathcal{E} = \{\underline{e}_i\}_{i=1}^{n}$ IS AN <u>ORTHONORMAL</u> BASIS OF V.

<u>EX.</u> $V = \mathbb{R}^3$, $\mathcal{B} = \{(1,1,0)^T, (3,1,1)^T, (1,1,3)^T\}$.

TO FIND AN ORTHONORMAL BASIS OF $\mathbb{R}^3$, WE USE THE GRAM-SCHMIDT PROCEDURE ON $\mathcal{B}$:

1) $\underline{d}_1 = \underline{b}_1 = (1,1,0)^T$.

2) $\underline{d}_2 = (I - P_{\underline{d}_1})\underline{b}_2 = \underline{b}_2 - P_{\underline{d}_1}\underline{b}_2$

$$= (3,1,1)^T - \frac{\langle (1,1,0)^T | (3,1,1)^T \rangle}{\|(1,1,0)^T\|}(1,1,0)^T$$

$$= (1,-1,1)^T.$$

3) $\underline{d}_3 = (I - P_{\underline{d}_1} - P_{\underline{d}_2})\underline{b}_3 = \underline{b}_3 - P_{\underline{d}_1}\underline{b}_3 - P_{\underline{d}_2}\underline{b}_3$

$$= \underline{b}_3 - \underbrace{\frac{\langle \underline{d}_1 | \underline{b}_3 \rangle}{\|\underline{d}_1\|^2}}_{=1}\underline{d}_1 - \underbrace{\frac{\langle \underline{d}_2 | \underline{b}_3 \rangle}{\|\underline{d}_2\|^2}}_{=1}\underline{d}_2$$

$$= (-1,1,2)^T.$$

$\Rightarrow \mathcal{D} = \{(1,1,0)^T, (1,-1,1)^T, (-1,1,2)^T\}$. <u>ORTHOGONAL</u> BASIS.

$$\Rightarrow \quad \mathcal{E} = \left\{ \frac{1}{\sqrt{2}}(1,1,0)^T, \ \frac{1}{\sqrt{3}}(1,-1,1)^T, \ \frac{1}{\sqrt{6}}(-1,1,2)^T \right\}$$

ORTHONORMAL BASIS.

REMARK: IN GRAM SCHMIDT PROCEDURE, REMEMBER THAT

$$\underline{d}_i = \left(I - P_{\underline{d}_1} - \cdots - P_{\underline{d}_{i-1}}\right)\underline{b}_i \qquad \checkmark$$

AND NOT

$$\cancel{\underline{d}_i = \left(I - P_{\underline{b}_1} - \cdots - P_{\underline{b}_{i-1}}\right)\underline{b}_i}.$$

THAT IS, WE MUST USE THE ITERATIVE PROCEDURE TO OBTAIN A CORRECT ANSWER.