Lecture 30
04/06/12

Last time, we saw that

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$ is **not** self-adjoint, but still

has an orthonormal basis of eigenvectors.

Note: $A^*A = \begin{pmatrix} 1 & +2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ +2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

$$\shortparallel$$

$AA^* = \begin{pmatrix} 1 & -2 \\ +2 & 1 \end{pmatrix}\begin{pmatrix} 1 & +2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$.

Normal operators.

Def. $N : V \to V$ is **normal** if $N^*N = NN^*$

(i.e., N commutes with its adjoint).

Note: L self-adjoint $\Rightarrow$ L normal

Pf: $L = L^* \Rightarrow L^*L = LL^*$.

Properties:

- N normal $\Leftrightarrow$ $\|N\underline{x}\| = \|N^*\underline{x}\|$ for all $\underline{x} \in V$.

Pf. N normal $\Leftrightarrow$ $N^*N - NN^* = 0$

$$\Leftrightarrow \langle (N^*N - NN^*)\underline{x} \mid \underline{x} \rangle = 0 \quad \text{FOR ALL } \underline{x} \in V$$

(USING THAT $N^*N - NN^*$ IS SELF-ADJOINT AND THAT FOR ANY SELF-ADJOINT L, $\langle L\underline{x} \mid \underline{x} \rangle = 0$ FOR ALL $\underline{x} \in V \Leftrightarrow L = 0$).

$$\Leftrightarrow \langle N\underline{x} \mid N\underline{x} \rangle = \langle N^*\underline{x} \mid N^*\underline{x} \rangle$$

$$\Leftrightarrow \|N\underline{x}\| = \|N^*\underline{x}\|.$$

- SUPPOSE N NORMAL. THEN λ EIGENVALUE OF N WITH E-VECTOR $\underline{v}$ $\Leftrightarrow$ $\bar{\lambda}$ EIGENVALUE OF N^* WITH E-VECTOR $\underline{v}$.

PF. $$0 = \|\underbrace{(N - \lambda I)}_{\text{NORMAL}}\underline{v}\| = \|(N - \lambda I)^* \underline{v}\| = \|(N^* - \bar{\lambda} I)\underline{v}\|$$

$$\left(\text{SINCE } \begin{aligned} (N-\lambda I)^*(N-\lambda I) &= N^*N - \bar{\lambda}N - \lambda N^* + \lambda\bar{\lambda} I \\ &= NN^* - \bar{\lambda}N - \lambda N^* + \lambda\bar{\lambda} I \\ &= (N-\lambda I)(N-\lambda I)^*. \end{aligned}\right)$$

- N NORMAL $\Rightarrow$ EIGENSPACES OF DISTINCT EIGENVALUES ARE ORTHOGONAL.

PF. SUPPOSE λ, μ DISTINCT E-VALUES WITH CORRESPONDING EIGENSPACES E_λ AND E_μ. LET $\underline{v} \in E_\lambda$ AND $\underline{w} \in E_\mu$.

$$\left.\begin{aligned} N\underline{v} &= \lambda \underline{v} \\ N\underline{w} &= \mu \underline{w} \end{aligned}\right\} \Rightarrow (\lambda - \mu)^{\neq 0} \langle \underline{v} | \underline{w} \rangle$$

$$= \langle \bar{\lambda} \underline{v} | \underline{w} \rangle - \langle \underline{v} | \mu \underline{w} \rangle$$

$$= \langle N^* \underline{v} | \underline{w} \rangle - \langle \underline{v} | N \underline{w} \rangle$$

$$= \langle \underline{v} | N \underline{w} \rangle - \langle \underline{v} | N \underline{w} \rangle$$

$$= 0.$$

$$\Rightarrow \langle \underline{v} | \underline{w} \rangle = 0 \Rightarrow \underline{v} \perp \underline{w}.$$

BEFORE WE PROVE OUR MAIN RESULT, LET US INTRODUCE THE <u>SCHUR DECOMPOSITION</u> OF A MATRIX.

<u>THM.</u> (SCHUR DECOMPOSITION) ANY $A \in M_{n,n}(\mathbb{C})$ CAN BE WRITTEN AS $A = UTU^{-1}$.

ORTHONORMAL COLUMNS. UPPER TRIANGULAR MATRIX

<u>PF.</u> (BY INDUCTION)

<u>CASE $n=1$</u>: $A \in \mathbb{C}$ ✓

<u>CASE $n-1$</u>: ASSUME TRUE.

<u>CASE n</u>: LET λ_1 BE E-VALUE OF A, W/ E-VECTOR $\underline{b}_1$, $\|\underline{b}_1\| = 1$. NOW LET $E = \underline{b}_1^{\perp}$ AND $\{\underline{e}_2, \ldots, \underline{e}_n\}$ ANY ORTHONORMAL BASIS OF E (TO BE CHOSEN LATER).

LET $\mathcal{E} = \{\underline{b}_1, \underline{e}_2, \ldots, \underline{e}_n\}$, SO $\mathcal{E}$ IS AN ORTHONORMAL BASIS OF $V = \mathbb{C}^n$.

$$\Rightarrow [A]_{\mathcal{E}} = \left(\begin{array}{c|c} \lambda_1 & ? \\ \hline 0 & \\ \vdots & A_1 \\ 0 & \end{array}\right)$$

WHERE $A_1 \in M_{n-1, n-1}(\mathbb{C})$.

BY INDUCTION HYPOTHESIS, A_1 IS TRIANGULAR, SO $[A]_{\mathcal{E}}$ IS TRIANGULAR AND $A = UTU^{-1}$ WITH $T = \left(\begin{array}{c|c} \lambda_1 & \\ \hline 0 & A_1 \end{array}\right)$ AND $U = (\underline{b}_1, \underline{e}_2, \ldots, \underline{e}_n)$.

FINALLY, WE USE THIS DECOMPOSITION TO SHOW:

<u>THM.</u> (COMPLEX SPECTRAL THM.)

$$N \text{ NORMAL} \iff N = U D U^{-1}$$

ORTHONORMAL BASIS OF E-VECTORS OF N.

DIAGONAL MATRIX OF E-VALUES OF N (POSSIBLY COMPLEX)

Pf. "$\Rightarrow$" N HAS SCHUR DECOMPOSITION

$$N = \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & & & \\ \vdots & & N_1 & \\ 0 & & & \end{array}\right)$$

IN SOME ORTHONORMAL BASIS $\hat{\mathcal{E}}$, WHERE N_1 IS AN UPPER TRIANGULAR MATRIX.

$$(N^*N)_{11} = \bar{a}_{11} a_{11} = |a_{11}|^2.$$

$$\shortparallel$$

$$(NN^*)_{11} = a_{11}\bar{a}_{11} + \cdots + a_{1n}\bar{a}_{1n} = |a_{11}|^2 + \cdots + |a_{1n}|^2$$

$$\Rightarrow \quad a_{12} = \cdots = a_{1n} = 0.$$

SO, $N = \left(\begin{array}{c|c} a_{11} & 0 \\ \hline 0 & N_1 \end{array}\right)$ AND $N^*N = NN^*$

$$\Rightarrow \quad N_1^* N_1 = N_1 N_1^*.$$

REPEATING THE SAME STEPS SHOWS THAT $N = \left(\begin{array}{cc|c} a_{11} & & 0 \\ & a_{12} & \\ \hline 0 & & N_2 \end{array}\right)$,

SO THIS FINALLY GIVES THAT $N = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$

IN ORTHONORMAL BASIS $\mathcal{E}$.

"$\Leftarrow$" $N = UDU^{-1}$ WITH $U = (\underline{e}_1, \ldots, \underline{e}_n)$

$\Rightarrow [N]_{\mathcal{E}} = D$ FOR $\mathcal{E} = \{\underline{e}_1, \ldots, \underline{e}_n\}$.

$\Rightarrow [N^*]_{\mathcal{E}} = D^*$

THEN, $D^*D = DD^* \Rightarrow [N^*N]_{\mathcal{E}} = [NN^*]_{\mathcal{E}}$

$\Rightarrow N$ NORMAL.

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RECALL THAT FOR A COMPLEX INNER PRODUCT SPACE V,

$$N \text{ normal } \iff N = UDU^{-1}$$
$$(N^*N = NN^*)$$

orthonormal e-vectors of N

diagonal matrix of e-values.

[

SPECIAL CASE:

$$L \text{ self-adjoint } \iff L = UDU^{-1}$$
$$(L^* = L)$$

REAL diagonal matrix of e-values

IF V REAL INNER PRODUCT SPACE,

$$S \text{ symmetric } \iff S = ODO^{-1}$$
$$(S^T = S)$$

real matrices.

]

Q: WHAT ARE PROPERTIES OF U?

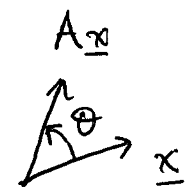
ISOMETRIES (7.4):

DEF. U IS AN ISOMETRY IF IT PRESERVES LENGTH — I.E.,

$$\|U\underline{x}\| = \|\underline{x}\| \quad \text{for all } x \in V.$$

NOTATION: IF V COMPLEX, U IS CALLED UNITARY.
" " REAL, " " " ORTHOGONAL.

Ex. $V = \mathbb{R}^2$, $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



$\underline{x} = (x_1, x_2)^T$

$$\|A\underline{x}\|^2 = \|(cx_1 - sx_2, sx_1 + cx_2)^T\|^2$$

$$= c^2 x_1^2 + s^2 x_2^2 + s^2 x_1^2 + c^2 x_2^2$$

$$= x_1^2 + x_2^2$$

$$= \|\underline{x}\|^2, \quad \text{WHERE} \quad c = \cos\theta, \ s = \sin\theta$$

$\Rightarrow$ A ISOMETRY (I.E., A ORTHOGONAL MATRIX).

REMARK: IN FACT, FOR ANY REAL SPACE V,
THERE IS AN ORTHONORMAL BASIS $\mathcal{E}$ OF V
SUCH THAT FOR AN ORTHOGONAL MATRIX A,

$$[A]_{\mathcal{E}} = \begin{pmatrix} \boxed{B_1} & & & 0 \\ & \boxed{B_2} & & \\ & & \ddots & \\ 0 & & & \boxed{B_k} \end{pmatrix},$$

WHERE EACH BLOCK $\boxed{B_i}$

IS EITHER $\boxed{1}$, $\boxed{-1}$, OR $\boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$ FOR SOME $\theta \in (0, \pi)$.

UNCHANGED — REFLECTION — ROTATION IN PLANE

THM. THESE ARE ALL EQUIVALENT (I.E., (i)-(iv) AND (i')-(iv') ALL SAME):

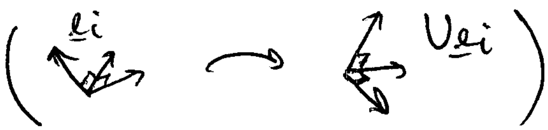
(i) U ISOMETRY	(i') U^* ISOMETRY
(ii) $\langle U\underline{x} \mid U\underline{y} \rangle = \langle \underline{x} \mid \underline{y} \rangle$ FOR ALL $\underline{x}, \underline{y} \in V$	(ii') $\langle U^*\underline{x} \mid U^*\underline{y} \rangle = \langle \underline{x} \mid \underline{y} \rangle$ FOR ALL $\underline{x}, \underline{y} \in V$

(iii) $U^*U = I$

(iv) IF $\mathcal{E} = \{\underline{e_i}\}$ ORTHONORMAL

$$[U]_{\mathcal{E}} = (U\underline{e_1}, \ldots, U\underline{e_n})$$

HAS ORTHONORMAL COLUMNS



(iii') $UU^* = I$

(iv') IF $\mathcal{E} = \{\underline{e_i}\}$ ORTHONORMAL

$$[U^*]_{\mathcal{E}} = (U^*\underline{e_1}, \ldots, U^*\underline{e_n})$$

HAS ORTHONORMAL COLUMNS.

IN ADDITION:

(a) U IS INVERTIBLE AND $U^{-1} = U^*$.

(PF. $\|U\underline{x}\| = \|\underline{x}\| \Rightarrow U\underline{x} = 0$ IFF $\underline{x} = 0$

$\Rightarrow \mathrm{Ker}(U) = \{0\} \Rightarrow U$ INVERTIBLE.

$U^*U = I \Rightarrow U^{-1} = U^*$.)

NOTE: THIS IMPLIES THAT THE SPECTRAL THM. CAN BE WRITTEN AS

N NORMAL $\Leftrightarrow N = UDU^*$ ← DIAGONAL.

(L SELF-ADJOINT $\Leftrightarrow L = UDU^*$ ← REAL DIAGONAL

S SYMMETRIC $\Leftrightarrow S = ODO^T$ ← REAL.)

(b) ALL E-VALUES OF U HAVE MAGNITUDE $|\lambda| = 1$

(SO IF U ORTHOGONAL, $\lambda = \pm 1$).

(PF. $\|U\underline{x}\| = \|\underline{x}\|$
$\|\lambda\underline{x}\| = |\lambda|\|\underline{x}\| \Rightarrow |\lambda| = 1$.)

(c) $|\det U| = 1$

(PF. $|\det U| = |\lambda_1 \cdots \lambda_n| = 1$.)

PF. OF THM.:

(i) ⇒ (ii): IF V REAL, $\langle \underline{x} | \underline{y} \rangle = \dfrac{\|\underline{x}+\underline{y}\|^2 - \|\underline{x}-\underline{y}\|^2}{4}$ PARALLELOGRAM IDENTITY

$$\Rightarrow \langle U\underline{x} | U\underline{y} \rangle = \frac{\|U(\underline{x}+\underline{y})\|^2 - \|U(\underline{x}-\underline{y})\|^2}{4}$$

$$= \frac{\|\underline{x}+\underline{y}\|^2 - \|\underline{x}-\underline{y}\|^2}{4}$$

$$= \langle \underline{x} | \underline{y} \rangle .$$

SIMILAR IF V COMPLEX..

(ii) ⇒ (iii): $\langle (U^*U - I)\underline{x} | \underline{y} \rangle = \langle U\underline{x} | U\underline{y} \rangle - \langle \underline{x} | \underline{y} \rangle$
$= 0$.

LET $\underline{y} = (U^*U - I)\underline{x} \Rightarrow \|(U^*U - I)\underline{x}\| = 0$ FOR ALL $\underline{x} \in V$

$\Rightarrow U^*U = I$.

(iii) ⇒ (iv): $\langle U\underline{e}_i | U\underline{e}_j \rangle = \langle U^*U\underline{e}_i | \underline{e}_j \rangle = \langle \underline{e}_i | \underline{e}_j \rangle$
$= 0$.

(iv) ⇒ (i): $\underline{x} = \sum_{i=1}^{n} x_i \underline{e}_i$

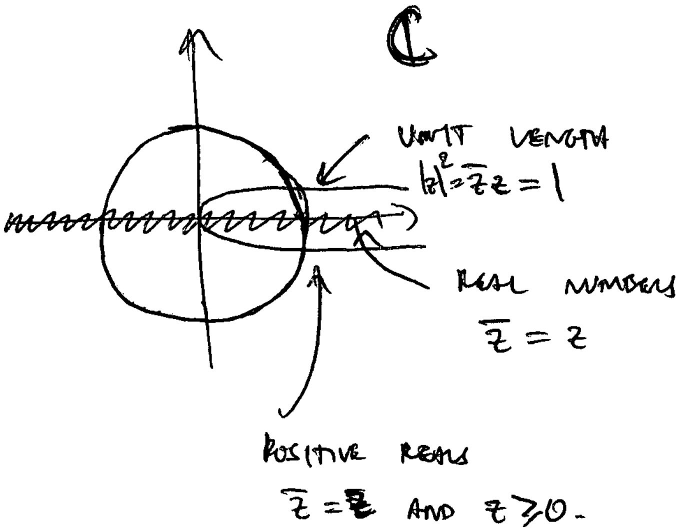
$$\Rightarrow \|U\underline{x}\|^2 = \langle U\underline{x} | U\underline{x} \rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle U\underline{e}_i | U\underline{e}_j \rangle$$

$$= x_1^2 + \cdots + x_n^2 = \|\underline{x}\|^2.$$

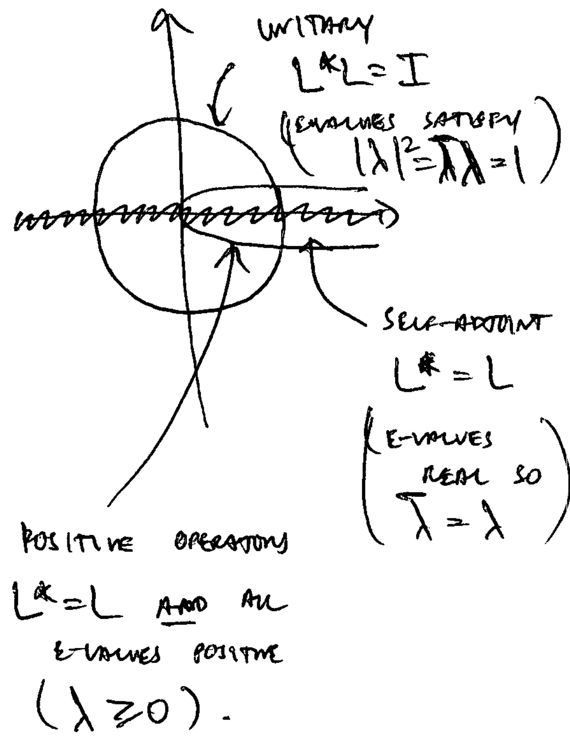
(i) $\Leftrightarrow$ (ii): EASY TO SHOW.

TO SUMMARIZE: THERE IS A NICE ANALOGY BETWEEN COMPLEX NUMBERS AND LINEAR OPERATORS!

COMPLEX NUMBERS



LINEAR OPERATORS



POLAR REPRESENTATION: FOR ANY z,

$$z = \frac{z}{|z|}|z| = \underbrace{\left(\frac{z}{|z|}\right)}_{\text{UNIT LENGTH}} \underbrace{\sqrt{\bar{z}z}}_{\text{POSITIVE REAL}}$$

POLAR DECOMPOSITION: FOR ANY L,

$$L = \underset{\uparrow \atop \text{UNITARY}}{U} \underbrace{\sqrt{L^*L}}_{\text{POSITIVE OPERATOR.}}$$

LECTURE 32
04/11/12

POSITIVE OPERATORS:

DEF. L POSITIVE IF IT IS SELF-ADJOINT AND ALL ITS EIGENVALUES ARE ≥ 0. WE DENOTE THIS BY $L \geq 0$.

(NOTE: WE ALLOW EIGENVALUES TO BE ZERO.)

REMARK: L POSITIVE $\iff$ $L^* = L$ AND $\langle L\underline{x} | \underline{x} \rangle \geq 0$ FOR ALL $\underline{x} \in V$.

THM. IF $L \geq 0$, THERE IS A UNIQUE OPERATOR $B \geq 0$ SUCH THAT $B^2 = L$. IN PARTICULAR,

$$B = U \underbrace{D^{1/2}}_{= \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}} U^*, \quad \text{WHERE} \quad L = U \underbrace{D}_{= \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}} U^*$$

PF. $$B^2 = (UD^{1/2}\underbrace{U^*)(U}_{=I}D^{1/2}U^*) = UD^{1/2}D^{1/2}U^* = UDU^* = L.$$

UNIQUENESS EASY.

EX. SUPPOSE $A \in M_{2,2}(\mathbb{C})$ HAS E-VALUES $\lambda_1 = 9$, $\lambda_2 = 4$ W/ CORRESPONDING E-VECTORS $\underline{\xi}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$, $\underline{\xi}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Q1: WHAT IS A?

A1: SINCE $\langle \underline{\xi}_1 | \underline{\xi}_2 \rangle = (-i, 1)(-i, 1)^T = 0$,

A HAS ORTHOGONAL E-VECTORS AND REAL E-VALUES

$\Rightarrow$ A SELF-ADJOINT.

$$\Rightarrow A = UDU^* = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 5i \\ -5i & 13 \end{pmatrix}.$$

Q2: WHAT IS $\sqrt{A}$?

A2: SINCE $A \geq 0$, $\sqrt{A}$ EXISTS.

$$\sqrt{A} = U\underset{\begin{pmatrix} \sqrt{9} & 0 \\ 0 & \sqrt{4} \end{pmatrix}}{\underset{\shortparallel}{D^{1/2}}}U^* = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} = \begin{pmatrix} 5 & i \\ -i & 5 \end{pmatrix}.$$

NOTE: $\sqrt{A}$ IS DETERMINED BY E-VALUES OF A, NOT BY INDIVIDUAL ENTRIES a_{ij}!

WE WILL USE POSITIVE OPERATORS TO INTRODUCE ONE OF THE MOST USEFUL MATRIX DECOMPOSITIONS, CALLED SINGULAR VALUE DECOMPOSITION (SVD).

MOTIVATION: SO FAR WE HAVE INTRODUCED SEVERAL DECOMPOSITIONS.

A DIAGONALIZABLE

$A = PDP^{-1}$

E-VECTORS

DIAGONAL OF E-VALUES

$\rightsquigarrow$

GENERAL $A \in M_{n,n}(\mathbb{C})$

$A = \tilde{P}\tilde{D}\tilde{P}^{-1}$

E-VECTORS AND GENERALIZED E-VECTORS.

BLOCK DIAGONAL

$\boxed{\text{JORDAN DECOMPOSITION}}$

A NORMAL

$A = UDU^*$

ORTHONORMAL E-VECTORS

DIAGONAL OF E-VALUES

$\rightsquigarrow$

GENERAL $A \in M_{n,n}(\mathbb{C})$

$A = UTU^*$

ORTHONORMAL BASIS

UPPER TRIANGULAR W/ E-VALUES ALONG DIAGONAL

$\boxed{\text{SCHUR DECOMPOSITION}}$

$\rightsquigarrow$

GENERAL $A \in M_{n,n}(\mathbb{C})$

$A = U \Sigma V^*$

UNITARY

DIAGONAL MATRIX OF SINGULAR VALUES.

UNITARY

$\boxed{\text{SINGULAR VALUE DECOMPOSITION}}$

THAT IS,

Q: CAN WE FIND TWO ORTHONORMAL BASES $\mathcal{E}$ AND $\mathcal{F}$ SUCH THAT $[A]_{\mathcal{E}\mathcal{F}}$ IS DIAGONAL?

A: YES, FOR ANY A. THEN U WILL HAVE COLUMNS FROM $\mathcal{E}$, V COLUMNS FROM $\mathcal{F}$, AND $\Sigma = [A]_{\mathcal{E}\mathcal{F}}$.