

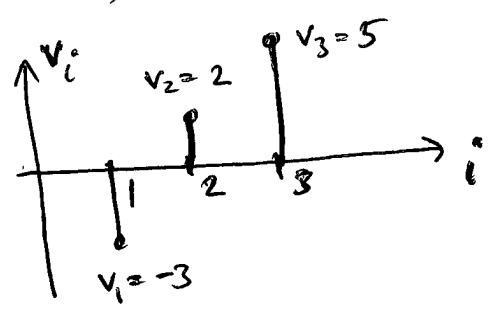
LECTURE 35
04/18/12

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INFINITE-DIM VECTOR SPACES, INNER PROD. SPACES (6.8):

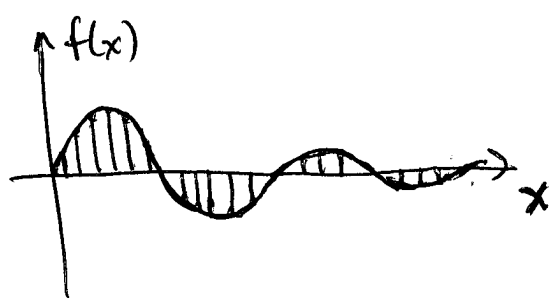
WE CAN THINK OF FUNCTIONS AS INF.-DIM. VECTORS, BY THE FOLLOWING ANALOGY:

$\underline{v} = (v_1, v_2, v_3)^T = \{v_i\}_{i=1}^3 \in \mathbb{R}^3$, FOR EX., $\underline{v} = (-3, 2, 5)^T$



ANALOGOUSLY,

$f = \{f(x)\}_{x \in \mathbb{R}} \in V$, FOR EX., $f(x) = e^{-x} \sin x$



MOTIVATION: INFINITE-DIM. SPACES ARISE IN MANY SETTINGS, BUT ARE ESPECIALLY IMPORTANT WHEN CONSIDERING PARTIAL DIFFERENTIAL EQUATIONS (PDE).

RECALL THAT FOR LINEAR, CONST-COEFF., HOMOGENEOUS ODE WE CONSIDERED

$$\begin{cases} \frac{d\underline{v}}{dt} = A \underline{v} \\ \underline{v}(0) \text{ given} \end{cases}, \text{ WHERE } \underline{v} = \underline{v}(t) \in \mathbb{R}^n \text{ FOR ALL } t \geq 0. \text{ AND } A : \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ LINEAR OPERATOR.}$$

Summary, linear, const. coeff., homogeneous PDE take the form (2)

$$\begin{cases} \frac{du}{dt} = L u \\ u(t=0) \text{ given} \end{cases}, \quad \text{where } u = u(t) \in V \text{ for all } t \geq 0 \\ \text{AND } L: V \rightarrow V \text{ linear operator.}$$

Here, V is a function space (i.e., a space in which the typical element is a function $f(x), x \in \mathbb{R}$).

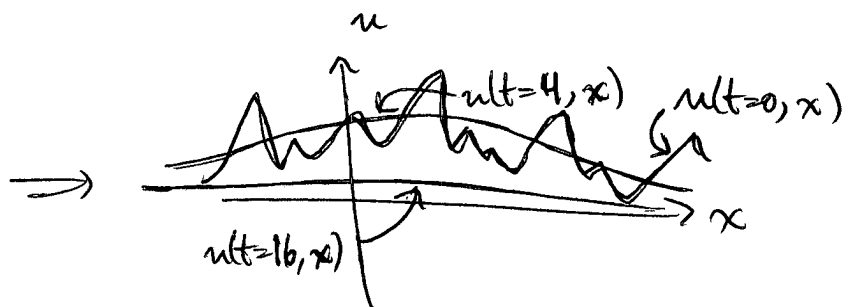
Therefore, we should think of $u(t) \equiv u(t, x)$ for $x \in \mathbb{R}$ and L as an operator on functions of x . That is,

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) = L u(x, t) \\ u(0, x) \text{ given} \end{cases}$$

For ex., if $L = \frac{\partial^2}{\partial x^2}$ this is a classical PDE called

the heat eqn.

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ u(t, x) \text{ given} \end{cases}$$



WARNING: While many of the concepts we have learned for finite-dim. spaces will carry over to ∞ -dim. spaces, many will not. Issues will typically arise due to a lack of convergence of some sum.

FOR EX., CONSIDER ANY TWO FINITE-DIM Nxn MATRICES A_n AND B_n .

(3)

THEN $\text{Tr}(A_n B_n) = \text{Tr}(B_n A_n) \Rightarrow \text{Tr}(A_n B_n - B_n A_n) = 0$

FOR ANY $A_n, B_n \in M_{n \times n}(\mathbb{C})$.

HOWEVER, IF

$A_\infty = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & \dots & & 0 \end{pmatrix}, B_\infty = A_\infty^T = \begin{pmatrix} 1 & 0 & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & \dots & & 1 \end{pmatrix}$

(INFINITE-DIM MATRICES)

THEN $A_\infty B_\infty = \begin{pmatrix} 1 & 0 & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}, B_\infty A_\infty = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & \ddots \\ 0 & & 0 \end{pmatrix}$

$\Rightarrow A_\infty B_\infty - B_\infty A_\infty = \begin{pmatrix} 1 & 0 & 0 \\ & 0 & \ddots \\ 0 & & \ddots \end{pmatrix} \Rightarrow \text{Tr}(A_\infty B_\infty - B_\infty A_\infty) = 1 \neq 0!$

WE SEE THAT $\lim_{n \rightarrow \infty} \text{Tr}(A_n B_n - B_n A_n) \neq \text{Tr}(\lim_{n \rightarrow \infty} (A_n B_n - B_n A_n))$

WHERE $A_n = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}}_n, B_n = A_n^T = \underbrace{\begin{pmatrix} 1 & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_n$

THE REASON FOR THIS FAILURE IS THAT $\text{Tr}(A_\infty B_\infty)$ AND $\text{Tr}(B_\infty A_\infty)$ ARE BOTH $\infty!$ SO WE SHOULD EXPECT SOME TROUBLE BECAUSE OF A LACK OF CONVERGENCE.

EXAMPLES OF ∞ -DIM VECTOR SPACES:

(1) $V = \ell_2(\mathbb{R})$ (SPACE OF SQUARE-SUMMABLE SEQUENCES IN \mathbb{R})

DEF. $\underline{v} = (v_1, v_2, v_3, \dots) \in \ell_2(\mathbb{R})$ IF $\sum_{i=1}^{\infty} |v_i|^2 < \infty, v_i \in \mathbb{R}$.

• $\ell_2(\mathbb{R})$ IS A VECTOR SPACE SINCE

(i) $\underline{v}, \underline{w} \in \ell_2(\mathbb{R}) \Rightarrow \underline{v} + \underline{w} \in \ell_2(\mathbb{R})$
" $(v_1 + w_1, v_2 + w_2, \dots)$

PF $\sum_{i=1}^{\infty} |v_i + w_i|^2 = \sum_{i=1}^{\infty} |v_i|^2 + |w_i|^2 + 2|w_i||v_i| < \infty$
 $\leq |v_i|^2 + |w_i|^2$

(since for any $a, b \in \mathbb{R}$, $(|a| - |b|)^2 \geq 0$)

(ii) $\underline{v} \in \ell_2(\mathbb{R}), c \text{ scalar} \Rightarrow c\underline{v} \in \ell_2(\mathbb{R})$.
PF EASY.

• $\mathcal{E} = \{\underline{e}_i\}_{i=1}^{\infty}, \underline{e}_i = (0, \dots, 0, 1, 0, \dots)$ IS THE STANDARD BASIS OF $\ell_2(\mathbb{R})$. NOTE THAT ANY $\underline{v} \in \ell_2(\mathbb{R})$ CAN BE WRITTEN AS AN INFINITE LINEAR COMBINATION OF THE $\{\underline{e}_i\}$.

• $\ell_2(\mathbb{R})$ IS AN INNER PRODUCT SPACE WITH $\langle \underline{v} | \underline{w} \rangle = \sum_{i=1}^{\infty} v_i w_i$. NOTE THAT THIS IS A GENERALIZATION OF THE INNER PRODUCT OF \mathbb{R}^n AS $n \rightarrow \infty$.

• SIMILARLY, ONE CAN DEFINE $L_2(\mathbb{C})$ AS ALL
 $\underline{v} = (v_1, v_2, v_3, \dots)$ S.T. $v_i \in \mathbb{C}$ AND $\sum_{i=1}^{\infty} |v_i|^2 < \infty$.

THE INNER PRODUCT IS THEN $\langle \underline{v} | \underline{w} \rangle = \sum_{i=1}^{\infty} \overline{v_i} w_i$.

(2) $V = L_2(U)$ (SPACE OF SQUARE-INTEGRABLE FUNCTIONS
 $f: U \rightarrow \mathbb{R}$).

DEF. $f \in L_2(U)$ IF $f: U \rightarrow \mathbb{R}$, AND $\int_U |f(x)|^2 dx < \infty$.

TYPICALLY, WE TAKE $U = [0, a]$ FOR SOME $a > 0$ OR
 $U = \mathbb{R}$.

• $L_2(U)$ IS A VECTOR SPACE, SINCE

(i) $f, g \in L_2(U) \Rightarrow f+g \in L_2(U)$

PF.

$$\int_U |f(x)+g(x)|^2 dx \leq \int_U (|f(x)|^2 + |g(x)|^2 + \underbrace{2|f(x)||g(x)|}_{\leq |f(x)|^2 + |g(x)|^2}) dx < \infty.$$

(ii) $cf \in L_2(U)$ FOR ANY SCALAR c .

PF. EASY.

• WHAT IS A BASIS FOR $L_2(U)$? WE WILL SEE LATER
 WHEN DISCUSSING FOURIER SERIES.

• $L_2(U)$ is an inner product space with

$$\langle f|g \rangle = \int_U f(x)g(x) dx.$$

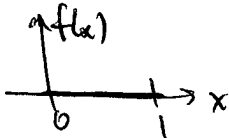
• Similarly, one can define $L_2(U; \mathbb{C})$ as all

$$f: U \rightarrow \mathbb{C} \text{ such that } \int_U |f(x)|^2 dx < \infty.$$

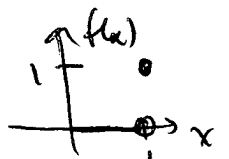
WARNING: THERE ARE MANY SUBTLETIES WHEN WORKING WITH

$L_2(U)$ WHICH WE WILL TRY TO AVOID. FOR EX.,

NOTE THAT FOR $U = [0, 1]$,

$$f = 0 \text{ for all } x \in [0, 1] \Rightarrow \|f\|^2 = \langle f|f \rangle = \int_0^1 |f(x)|^2 dx = 0,$$


AS EXPECTED. HOWEVER, WE ALSO SEE THAT

$$f(x) = \begin{cases} 0 & \text{for all } x \in (0, 1) \\ 1 & \text{for } x = 1 \end{cases} \Rightarrow \|f\|^2 = \langle f|f \rangle = \int_0^1 |f(x)|^2 dx = 0!$$


THEREFORE, POSITIVITY OF THE INNER PRODUCT DOES NOT HOLD

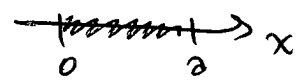
AS WE HAVE DEFINED IT SINCE THE INTEGRAL DOES NOT "SEE" INDIVIDUAL POINTS ON THE LINE. TO REMEDY THIS, WE

WILL CHANGE THE ZERO ELEMENT AS ANY f s.t. $\int_U |f| dx = 0$, AND $f = g$ IN $L_2(U)$ IF $\int_U |f-g| dx = 0$.

LECTURE 36
 04/20/12

Q: WHAT IS A BASIS OF $L^2(U)$ WHEN

(a) $U = [0, a]$, $a > 0$?



(b) $U = \mathbb{R}$?

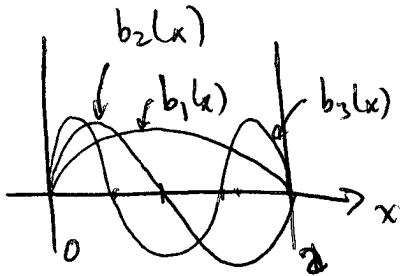


AS WE WILL SEE, THE ANSWER IN CASE (a) IS SIGNIFICANTLY SIMPLER THAN IN CASE (b).

FOURIER SERIES ON AN INTERVAL (6.9):

SUPPOSE $U = [0, a]$, $a > 0$.

LET $b_n(x) = \sin\left(\frac{n\pi x}{a}\right)$, $n=1, 2, 3, \dots$



• EASY TO CHECK THAT $b_n \in L_2([0, a])$ FOR EACH n .

• FURTHERMORE, $\{b_n\}_{n=1}^\infty$ ARE ORTHOGONAL!

Pf. $\langle b_m | b_n \rangle = \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx$

$\left(\begin{array}{l} \text{LETTING} \\ y = \frac{x}{a} \end{array} \right) \rightarrow = a \int_0^1 \sin(m\pi y) \sin(n\pi y) dy$

$\rightarrow = \frac{a}{2} \int_0^1 [\cos((n-m)\pi y) - \cos((n+m)\pi y)] dy$

$\left(\begin{array}{l} \text{USING THE IDENTITY} \\ 2 \sin(u) \sin(v) = \cos(u-v) - \cos(u+v) \end{array} \right)$

$$= \begin{cases} \frac{a}{2} \left[\frac{\sin((n-m)\pi y)}{(n-m)\pi} - \frac{\sin((n+m)\pi y)}{(n+m)\pi} \right]_{y=0}^{y=1}, & m \neq n \\ \frac{a}{2} \left[1 - \frac{\sin(2n\pi y)}{2n\pi} \right]_0^1, & m = n \end{cases} \quad \boxed{2}$$

$$= \begin{cases} 0 & \text{IF } m \neq n \\ \frac{a}{2} & \text{IF } m = n \end{cases}$$

$\Rightarrow \{b_n\}_{n=1}^{\infty}$ ORTHOGONAL (NOT ORTHONORMAL!). IN $L_2([0, a])$

• IN FACT, $\{b_n\}_{n=1}^{\infty}$ IS A BASIS OF $L_2([0, a])$.

"PF." LET $L = \frac{d^2}{dx^2}$. THEN AS WE HAVE SHOWN PREVIOUSLY, L IS SELF-ADJOINT ON $L_2([0, a])$.

NOTE THAT $\{b_n\}_{n=1}^{\infty}$ ARE "EIGENVECTORS" (WE WILL CALL THEM EIGENFUNCTIONS) OF L W/ CORRESPONDING EIGENVALUES $\lambda_n = -\frac{n^2\pi^2}{a^2}$ SINCE $L b_n(x) = \lambda_n b_n(x)$.

ASSUMING THE SPECTRAL THM. HOLDS FOR SELF-ADJOINT OPERATORS ON ∞ -DIM. SPACES, THIS IMPLIES THAT $\{b_n\}_{n=1}^{\infty}$ IS AN ORTHOGONAL BASE OF $L_2([0, a])$.

TECHNICALLY, L IS SELF-ADJOINT ON THE SPACE

$$C_0^\infty(0, a] = \{f: [0, a] \rightarrow \mathbb{R} : f(0) = f(a) = 0 \text{ AND } f \text{ INFINITELY DIFFERENTIABLE}\}$$

THIS, WITH THE FACT THAT FOR ANY $f \in L_2(0, a]$

THERE IS A SEQUENCE $\{g_m\}_{m=1}^\infty \subset C_0^\infty(0, a]$ THAT

APPROXIMATES f (I.E., $\|f - g_m\|_{L_2(0, a]} = \int_0^a |f(x) - g_m(x)|^2 dx \xrightarrow{m \rightarrow \infty} 0$.)

IMPLIES THAT $\{b_n\}_{n=1}^\infty$ IS A BASIS OF $L_2(0, a]$.

• NOTE THAT WE CAN NOW WRITE

$$f(x) = \sum_{n=1}^\infty c_n b_n(x) = \sum_{n=1}^\infty c_n \sin\left(\frac{n\pi x}{a}\right)$$

FOR ANY $f \in L_2(0, a]$. THE c_n 'S ARE CALLED

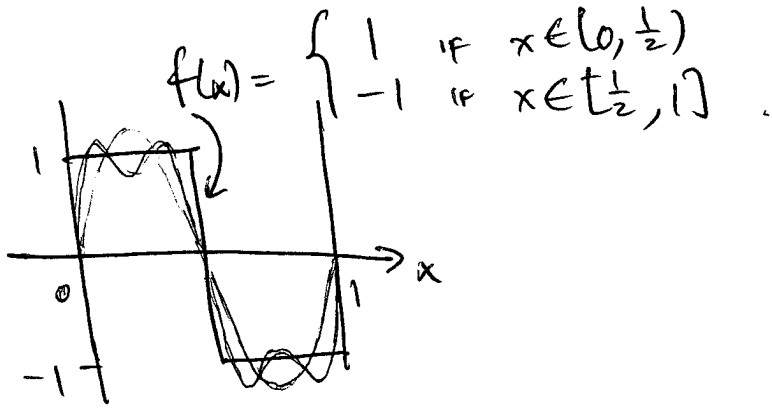
FOURIER COEFFICIENTS OF f AND THE ABOVE REPRESENTATION

IS CALLED FOURIER SERIES.

• TO FIND c_n , WE NOTE THAT

$$c_n = \frac{\langle b_n | f \rangle}{\langle b_n | b_n \rangle} = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx.$$

Ex.



$$\begin{aligned} \Rightarrow c_n &= \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx \\ &= 2 \left(\int_0^{\frac{1}{2}} \sin(n\pi x) dx + \int_{\frac{1}{2}}^1 \sin(n\pi x) dx \right) \\ &= \frac{2}{n\pi} \left(\left[-\cos(n\pi x) \right]_0^{\frac{1}{2}} + \left[\cos(n\pi x) \right]_{\frac{1}{2}}^1 \right) \\ &= \frac{2}{n\pi} \left(1 - 2\cos\left(\frac{n\pi}{2}\right) + \cos(n\pi) \right), \\ & \qquad \qquad \qquad n=1, 2, 3, \dots \end{aligned}$$

NOTE THAT $|c_n| \sim \frac{1}{n}$ AS $n \rightarrow \infty$.

WITH ONLY A FEW TERMS OF THE SERIES WE CAN GET A FAMILY GOOD APPROXIMATION TO OUR ORIGINAL FUNCTION — I.E.

$$f(x) \approx \sum_{n=1}^N c_n \sin\left(\frac{n\pi x}{a}\right)$$

LECTURE 37

04/23/12

Recall the former basis $\{b_n\}_{n=1}^{\infty}$ of $L_2([0, a])$ given

by $b_n(x) = \sin\left(\frac{n\pi x}{a}\right)$, $n=1, 2, 3, \dots$. Any function

$f \in L_2([0, a])$ can be written as $f(x) = \sum_{n=1}^{\infty} c_n b_n(x)$,

where $c_n = \frac{\langle b_n | f \rangle}{\langle b_n | b_n \rangle} = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$.

NOTE: THE FORMER COEFFICIENTS c_n TELL US "HOW MUCH" OF THE TERM $\sin\left(\frac{n\pi x}{a}\right)$ IS PRESENT IN $f(x)$ —

IN PARTICULAR, NOTE THAT $c_n \rightarrow 0$ AS $n \rightarrow \infty$

(OTHERWISE THE SERIES COULD NOT CONVERGE!) IN ADDITION,

THE RATE OF DECAY OF THE c_n 'S AS $n \rightarrow \infty$

DEPENDS ON HOW SMOOTH f IS SINCE LARGE DERIVATIVES

CAN ONLY BE CAPTURED BY HIGH FREQUENCY (I.E., LARGE n)

TERMS:

$|f'|$ LARGE \Rightarrow HIGH FREQUENCIES IMPORTANT

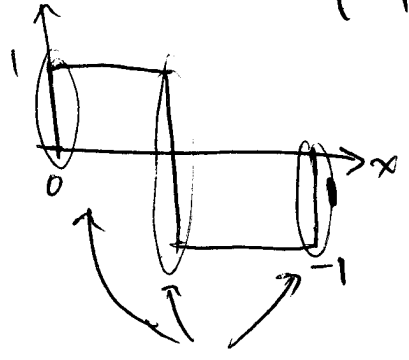
$\Rightarrow c_n \rightarrow 0$ (AS $n \rightarrow \infty$) SLOWLY

$|f'|$ SMALL \Rightarrow HIGH FREQUENCIES UNIMPORTANT

$\Rightarrow c_n \rightarrow 0$ (AS $n \rightarrow \infty$) QUICKLY.

Ex.

$$f(x) = \begin{cases} 1 & , x \in (0, \frac{1}{2}) \\ -1 & , x \in (\frac{1}{2}, 1] \end{cases}$$



$|f'| \gg 1$.

WE SAW LAST TIME THAT

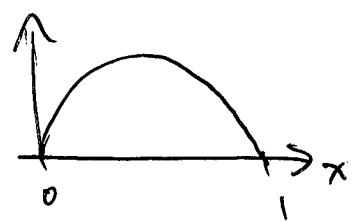
$$c_n = \frac{2}{n\pi} \left(1 - 2\cos\left(\frac{n\pi}{2}\right) + \cos(n\pi) \right)$$

$$\sim \frac{1}{n} \quad \text{AS } n \rightarrow \infty.$$

"PROPORTIONAL TO" \Rightarrow SLOW DECAY DUE TO LARGE DERIVATIVES $|f'|$ AT $x = 0, \frac{1}{2}, 1$.

Ex.

$$f(x) = x - x^2, \quad x \in (0, 1).$$



CALCULATING THE COEFFICIENTS ONE GETS

$$\begin{aligned} c_n &= 2 \int_0^1 (x - x^2) \sin(n\pi x) dx \\ &= \frac{4(1 - \cos(n\pi))}{n^2 \pi^2} \end{aligned}$$

$$\sim \frac{1}{n^3} \quad \text{AS } n \rightarrow \infty$$

\Rightarrow FAST DECAY SINCE $|f'|$ IS EVERYWHERE BOUNDED BY 2.

PROPERTIES :

(i) BASIS $\mathcal{B} = \{b_n\}_{n=1}^{\infty}$ MAKES $L_2([0,a])$ LOOK LIKE l_2 !

$$\text{IF } f, g \in L_2([0,a]), \quad f(x) = \sum_{n=1}^{\infty} c_n b_n(x) \\ g(x) = \sum_{n=1}^{\infty} d_n b_n(x)$$

$$\begin{aligned} \langle f | g \rangle_{L_2} &= \left\langle \sum_{n=1}^{\infty} c_n b_n \mid \sum_{m=1}^{\infty} d_m b_m \right\rangle_{L_2} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_n d_m \langle b_n | b_m \rangle_{L_2} \\ &= \frac{a}{2} \sum_{n=1}^{\infty} c_n d_n = \frac{a}{2} \langle [f]_{\mathcal{B}} | [g]_{\mathcal{B}} \rangle_{l_2} \end{aligned}$$

SINCE $[f]_{\mathcal{B}} = (c_1, c_2, \dots) \in l_2$.
 $[g]_{\mathcal{B}} = (d_1, d_2, \dots) \in l_2$.

$$\Rightarrow \left\| f \right\|_{L_2([0,a])}^2 = \frac{a}{2} \left\| [f]_{\mathcal{B}} \right\|_{l_2}^2$$

(PARSERVAL'S THM.)

IN PARTICULAR, THIS QUANTIFIES THE ERROR BETWEEN A FUNCTION AND ITS FOURIER SERIES (TRUNCATED AT N^{th} TERM) AS

$$\text{ERROR}^2 = \| f - f_N \|_{L_2}^2 = \left\| \sum_{n=N+1}^{\infty} c_n b_n \right\|_{L_2}^2 = \frac{a}{2} \sum_{n=N+1}^{\infty} |c_n|^2$$

(ii) FOURIER SERIES $f = \sum c_n b_n$ IS IDEAL WHEN WORKING WITH EVEN DERIVATIVES!

INDEED, RECALL THAT b_n IS THE n^{TH} EIGENFUNCTION OF $L = \frac{d^2}{dx^2}$ w/ CORRESPONDING EIGENVALUE $\lambda_n = -\frac{n^2 \pi^2}{a^2}$.

THEN,

$$\begin{aligned} f''(x) &= L f(x) = L \left(\sum_{n=1}^{\infty} c_n b_n(x) \right) \\ &= \sum_{n=1}^{\infty} c_n L b_n(x) \\ &= \sum_{n=1}^{\infty} c_n \lambda_n b_n(x). \end{aligned}$$

SO, THE n^{TH} FOURIER COEFFICIENT OF f'' IS SIMPLY

$$\lambda_n c_n = -\frac{n^2 \pi^2}{a^2} c_n \quad (\text{i.e., DERIVATIVES LEAD TO A SIMPLE MULTIPLICATION IN TERMS OF FOURIER COEFFICIENTS!})$$

EX. CONSIDER THE PARTIAL DIFFERENTIAL EQUATION (PDE)

$$\left\{ \begin{array}{l} \partial_t u = \partial_{xx} u \quad \rightsquigarrow \text{"HEAT EQN."} \\ u(0,t) = u(a,t) = 0 \quad (\text{BOUNDARY CONDITIONS}) \\ u(x,0) \text{ GIVEN.} \quad (\text{INITIAL CONDITIONS}) \end{array} \right.$$

HERE, $u(x,t)$ IS TO BE THOUGHT OF AS THE TEMPERATURE OF A THIN WIRE AT LOCATION $x \in [0, a]$ AT TIME $t \geq 0$.

THE BOUNDARY CONDITIONS $u(0,t) = u(a,t) = 0$ IMPLY THAT AT ALL TIMES THERE IS NO HEAT AT THE BOUNDARIES OF THE WIRE. SINCE HEAT SPREADS, WE SHOULD EXPECT THE SOLUTION $u(x,t) \rightarrow 0$ FOR ALL $x \in (0,a)$ AS $t \rightarrow \infty$ SINCE HEAT IS SUCKED OUT OF THE WIRE AT THE BOUNDARIES.

WE FIND THE SOLUTION BY SOLVING FOR THE EVOLUTION OF FOURIER COEFF. :

ASSUME $u(x,t) = \sum_{n=1}^{\infty} c_n(t) b_n(x)$ (SEPARATION OF VARIABLES)

↑ ↑

TIME-DEPENDENT FOURIER COEFF. FOURIER BASIS

$$\Rightarrow \partial_t u(x,t) = \sum_{n=1}^{\infty} \left(\frac{d}{dt} c_n(t) \right) b_n(x)$$

$$\partial_{xx} u(x,t) = \sum_{n=1}^{\infty} c_n(t) L b_n(x) \quad \left(L = \frac{d^2}{dx^2} \right)$$

$$= \sum_{n=1}^{\infty} (\lambda_n c_n(t)) b_n(x)$$

$$\Rightarrow \frac{d}{dt} c_n(t) = \lambda_n c_n(t) \quad \text{FOR ALL } n=1,2,3,\dots$$

$$\Rightarrow c_n(t) = c_n(0) e^{\lambda_n t} = c_n(0) e^{-\frac{(n\pi)^2}{a^2} t}, \quad n=1,2,3,\dots$$

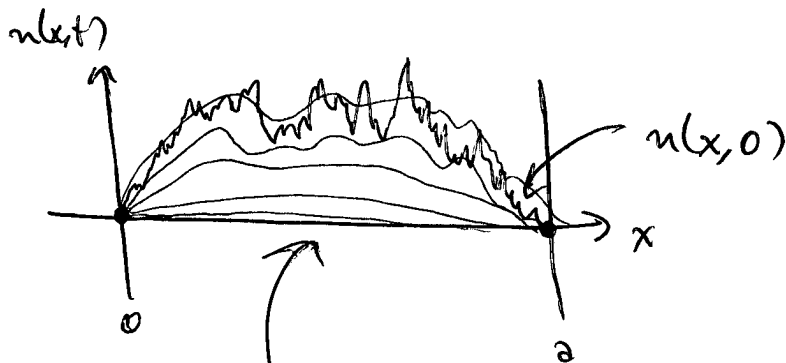
WHERE $u(x,0) = \sum_{n=1}^{\infty} c_n(0) b_n(x)$ GIVEN.

||

$$\frac{2}{a} \int_0^a u(x,0) b_n(x) dx.$$

NOTE THAT $c_n(t) \rightarrow 0$ AS $t \rightarrow \infty$ FOR EVERY n ,
 BUT THAT THE DELAY IS SIGNIFICANTLY FASTER FOR
 LARGER FREQUENCIES n !

\Rightarrow SOLUTION GETS SMOOTHER AS $t \rightarrow \infty$ AND
 GOES TO 0 EVERYWHERE.



SOLUTION GETS SMOOTHER IN x
 AS TIME PROGRESSES, GOES TO 0
 AS $t \rightarrow \infty$.

NOTE: THIS IS THE ∞ -DIM ANALOGUE OF CONTINUOUS-
 TIME EVOLUTION IN \mathbb{R}^n !

$$\begin{cases} \frac{d\underline{u}}{dt} = A \underline{u} \\ \underline{u}(0) \text{ given} \end{cases}, \quad A: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

writing $\underline{u}(t) = \sum_{i=1}^n c_i(t) \underline{b}_i$ \leftarrow i^{th} E-VECTOR OF A

WE FOUND $c_i(t) = c_i(0) e^{\lambda_i t}$ \leftarrow i^{th} E-VALUE OF A

- STABILITY OF i^{th} MODE \underline{b}_i WAS DETERMINED BY $\text{Re}(\lambda_i)$.

LECTURE 38 04/27/12

Q: WE HAVE FOUND ONE BASIS OF $L_2([0, a])$.
 ARE THERE OTHERS, AND HOW TO FIND THEM?

FOURIER SERIES (CONT'D) (8.5, 8.7)

LET US CAREFULLY REPERIVE THE BASIS WE HAVE WORKED WITH SO FAR:

① FOURIER SINE BASIS: DEFINE THE SPACE

$$L_2^{\text{DIR}}([0, a]) = \left\{ f \in L_2([0, a]) : \underbrace{f(0) = 0, f(a) = 0}_{\text{"DIRICHLET BOUNDARY CONDITIONS"}}, \left. \begin{array}{l} f \text{ INFINITELY DIFFERENTIABLE} \end{array} \right\}$$

NOTE THAT $L_2^{\text{DIR}}([0, a]) \subseteq L_2([0, a])$.

• IN FACT, THIS SUBSPACE "WELL-APPROXIMATES" $L_2([0, a])$

IN THAT FOR ANY $f \in L_2([0, a])$, THERE IS A SEQUENCE $\{f_n\}_{n=1}^{\infty} \subseteq L_2^{\text{DIR}}([0, a])$ SUCH THAT

$$\|f - f_n\|_{L_2}^2 \xrightarrow{n \rightarrow \infty} 0 \quad (\text{i.e., WE CAN APPROX. } f$$

WITHIN AN ARBITRARILY SMALL ERROR BY AN ELEMENT OF $L_2^{\text{DIR}}([0, a])$.

• $L = \frac{d^2}{dx^2}$ is SELF-ADJOINT ON $L_2^{DIR}([0, a])$.

PF.

$$\begin{aligned}
 \langle f | Lg \rangle &= \int_0^a f(x) g''(x) dx \\
 &= \underbrace{[f(x) g'(x)]_0^a}_{=0 \text{ since } f(0)=0, f(a)=0} - \int_0^a f'(x) g'(x) dx \\
 &= \underbrace{[-f'(x) g(x)]_0^a}_{=0 \text{ since } g(0)=0, g(a)=0} + \int_0^a f''(x) g(x) dx \\
 &= \langle Lf | g \rangle \quad \checkmark
 \end{aligned}$$

• WHAT ARE EIGENFUNCTIONS OF L ON $L_2^{DIR}([0, a])$?

$$\begin{cases} Lf = \lambda f \\ f \in L_2^{DIR}([0, a]) \end{cases} \Rightarrow \begin{cases} f''(x) = \lambda f(x) \\ f(0) = 0, f(a) = 0 \end{cases}$$

$$\Rightarrow f(x) = \sin\left(\frac{n\pi x}{a}\right), \quad n=1, 2, 3, \dots$$

$$\text{w/ } \lambda = -\frac{n^2\pi^2}{a^2}$$

THIS GIVES THE ORTHOGONAL BASIS (BY SELF-ADJOINTNESS OF L) $\{b_n\}_{n=1}^\infty$ ON $L_2^{DIR}([0,a])$ (AND THIS ON $L_2([0,a])$)

$$b_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad n=1,2,3,\dots$$

w/ EIGENVALUES $\lambda_n = \frac{-n^2\pi^2}{a^2}$

(FOURIER SINE BASIS)

$$\implies f(x) = \sum_{n=1}^\infty c_n b_n(x), \quad c_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx.$$

② STANDARD FOURIER BASIS (PER) : DEFINE THE SPACE "PERIODIC BOUNDARY CONDITIONS"

$$L_2^{PER}([0,a]) = \left\{ f \in L_2([0,a]) : \begin{array}{l} f(0) = f(a), f'(0) = f'(a), \\ f \text{ INFINITELY DIFFERENTIABLE} \end{array} \right\}$$

NOTE THAT $L_2^{PER}([0,a]) \subseteq L_2([0,a])$.

- IN FACT, AS WITH $L_2^{DIR}([0,a])$, WE HAVE THAT $L_2^{PER}([0,a])$ WELL-APPROXIMATES $L_2([0,a])$.
- $L_{\frac{d^2}{dx^2}}$ IS SELF-ADJOINT ON $L_2^{PER}([0,a])$.

P.F. $\langle f | Lg \rangle = \int_0^a f(x) g''(x) dx$

$f, g \in L_2^{PER}([0,a])$.

$$= \underbrace{[f(x)g'(x)]_0^a}_{=0 \text{ since } f(0)=f(a), g'(0)=g'(a)!} - \int_0^a f'(x)g'(x) dx$$

$$= \underbrace{[-f(x)g'(x)]_0^a}_{=0 \text{ since } f'(0)=f'(a), g(0)=g(a)!} + \int_0^a f''(x)g(x) dx$$

$$= \langle Lf | g \rangle . \quad \checkmark$$

• WHAT ARE EIGENFUNCTIONS OF L ON $L_2^{\text{PER}}(0, a)$?

$$\begin{cases} Lf = \lambda f \\ f \in L_2^{\text{PER}}(0, a) \end{cases} \Rightarrow \begin{cases} f''(x) = \lambda f(x) \\ f(0) = f(a), f'(0) = f'(a) \end{cases}$$

$$\Rightarrow f(x) = 1 \quad \text{w/ } \lambda = 0 \quad \underline{\text{or}}$$

$$f(x) = \sin\left(\frac{2n\pi x}{a}\right) \quad \text{w/ } \lambda = -\frac{4n^2\pi^2}{a^2} \quad \underline{\text{or}}$$

$$f(x) = \cos\left(\frac{2n\pi x}{a}\right) \quad \text{w/ } \lambda = -\frac{4n^2\pi^2}{a^2} .$$

THIS GIVES THE ORTHOGONAL BASIS FOR $L_2^{\text{PER}}(0, a)$
(AND THIS FOR $L_2(0, a)$)

$$\left\{ 1, \left\{ \sin\left(\frac{2n\pi x}{a}\right) \right\}_{n=1}^{\infty}, \left\{ \cos\left(\frac{2n\pi x}{a}\right) \right\}_{n=1}^{\infty} \right\}$$

w/ E-VALUES

$$\lambda_0 = 0, \quad \lambda_n = -\frac{4n^2\pi^2}{a^2}, \quad n = 1, 2, 3, \dots$$

(STANDARD FOURIER BASIS)

THAT IS, EIGENVALUE λ_n HAS GEOMETRIC MULTIPLICITY 2 (FOR $n=1, 2, 3, \dots$), WITH CORRESPONDING EIGENSPACE

$$E_{\lambda_n} = \text{SPAN} \left\{ \sin\left(\frac{2n\pi x}{a}\right), \cos\left(\frac{2n\pi x}{a}\right) \right\}.$$

$$\rightarrow f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\alpha_n \cos\left(\frac{2n\pi x}{a}\right) + \beta_n \sin\left(\frac{2n\pi x}{a}\right) \right],$$

"ALPHA" $\rightarrow \alpha_n = \frac{2}{a} \int_0^a f(x) \cos\left(\frac{2n\pi x}{a}\right) dx,$

"BETA" $\rightarrow \beta_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{2n\pi x}{a}\right) dx.$

③ STANDARD FOURIER BASIS (COMPLEX):

CONSIDER THE SPACE OF COMPLEX-VALUED SQUARE-INTEGRABLE FUNCTIONS $L_2([0, a]; \mathbb{C})$ w/ INNER PRODUCT

$$\langle f | g \rangle = \int_0^a \overline{f(x)} g(x) dx.$$

DEFINE THE SPACE

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$$L_2^{\text{PER}}([0, a]; \mathbb{C}) = \left\{ f \in L_2([0, a]; \mathbb{C}) : \begin{aligned} & f(0) = f(a), \text{ } f \text{ INFINITELY DIFF.} \end{aligned} \right\}$$

AS BEFORE, $L_2^{\text{PER}}([0, a]; \mathbb{C}) \subseteq L_2([0, a]; \mathbb{C})$

WELL APPROXIMATES. THEN, WE FIND THAT:

• $L = -i \frac{d}{dx}$ IS SELF-ADJOINT ON $L_2^{\text{PER}}([0, a]; \mathbb{C})$.

PF.

$$\begin{aligned} \langle f | Lg \rangle &= -i \int_0^a \overline{f(x)} g'(x) dx \\ f, g \in L_2^{\text{PER}}([0, a]; \mathbb{C}) &= \underbrace{[-i \overline{f(x)} g(x)]_0^a}_{=0 \text{ since } \overline{f(0)} = \overline{f(a)}, g(0) = g(a)} + i \int_0^a \overline{f'(x)} g(x) dx \\ &= \int_0^a \overline{(-if'(x))} g(x) dx \\ &= \langle Lf | g \rangle. \quad \checkmark \end{aligned}$$

• WHAT ARE THE EIGENFUNCTIONS OF L ON $L_2^{\text{PER}}([0, a]; \mathbb{C})$?

$$\begin{cases} Lf = \lambda f \\ f \in L_2^{\text{PER}}([0, a]; \mathbb{C}) \end{cases} \Rightarrow \begin{cases} -if'(x) = \lambda f(x) \\ f(0) = f(a) \end{cases}$$

$$\Rightarrow f(x) = \exp\left(\frac{2\pi i n x}{a}\right), \quad n \in \{\dots, -1, 0, 1, \dots\} = \mathbb{Z}.$$

$$w/ \lambda_n = \frac{2\pi n}{a}.$$

THIS GIVES THE ORTHONORMAL BASIS FOR $L_2^{\text{PER}}([0, a]) (\mathbb{C})$

(AND THIS FOR $L_2([0, a]; \mathbb{C})$)

$$\left\{ \exp\left(\frac{2\pi i n x}{a}\right) \right\}_{n=-\infty}^{\infty} \quad w/ \text{E-WAVES } \lambda_n = \frac{2\pi n}{a}$$

(STD. FORMAL BASIS (COMPLEX))

$$\rightsquigarrow f(x) = \sum_{n=-\infty}^{\infty} \gamma_n \exp\left(\frac{2\pi i n x}{a}\right),$$

"Gamma" $\rightarrow \gamma_n = \frac{1}{a} \int_0^a f(x) \exp\left(-\frac{2\pi i n x}{a}\right) dx.$

NOTE: THE STD. FORMAL BASIS IN THE REAL CASE IS A SPECIAL CASE OF WHAT WE NOW HAVE.

USING THAT

$$\exp\left(\frac{2\pi i n x}{a}\right) = \cos\left(\frac{2\pi n x}{a}\right) + i \sin\left(\frac{2\pi n x}{a}\right),$$

WE HAVE THAT FOR $n = 0, 1, 2, \dots$

$$\alpha_n = \gamma_n + \gamma_{-n}, \quad \beta_n = +i(\gamma_n - \gamma_{-n}).$$