LECTURE 14
02/20/12

MULTIPLICITY OF EIGENVALUES, DIAGONALIZABILITY (CONT'D):

TO RECAP: $A \in M_{n,n}$ HAS DISTINCT EIGENVALUES $\lambda_1, \dots, \lambda_r \in \mathbb{C}$.

$$(\lambda_1, E_{\lambda_1}), \quad \dots \quad, (\lambda_r, E_{\lambda_r})$$

$$\begin{array}{cccc} (z-\lambda_1)^{m_1} \downarrow & \downarrow \dim E_{\lambda_1} & (z-\lambda_r)^{m_r} \downarrow & \downarrow \dim E_{\lambda_r} \\ m_1 & M_1 & m_r & M_r \end{array}$$

- m_i ALGEBRAIC MULTIPLICITY OF λ_i
- M_i GEOMETRIC MULTIPLICITY OF λ_i.

LAST TIME, WE SHOWED THAT

(i) $1 \leq M_i \leq m_i$ FOR ALL $i = 1, \dots, r$

(ii) $E_{\lambda_1}, \dots, E_{\lambda_r}$ ARE DISTINCT SUBSPACES IN THAT THEY ARE LINEARLY INDEP.

THESE IMMEDIATELY IMPLY:

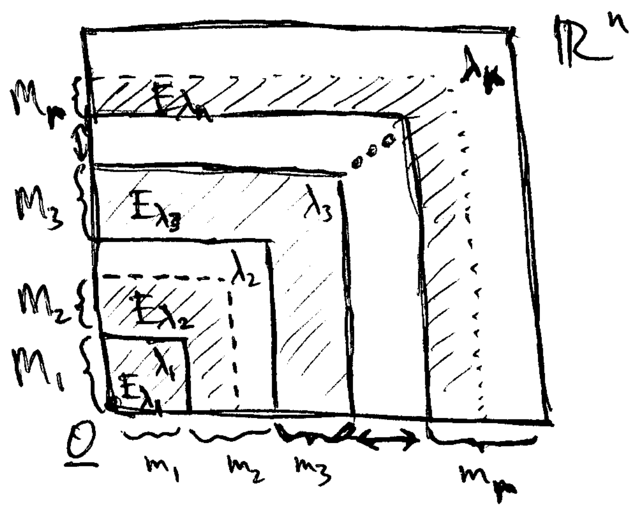
THM. (DIAGONALIZABILITY)

$A \in M_{n,n}$ DIAGONALIZABLE, I.E, CAN FIND n LINEARLY INDEP. EIGENVECTORS ($A = PDP^{-1}$ FOR SOME D DIAGONAL)

$\iff M_i = m_i$ FOR ALL $i = 1, \dots, r$.

- In particular, if A has n distinct eigenvalues (i.e, $r = n$ and $m_1 = m_2 = \cdots = m_n = 1$) then $M_1 = M_2 = \cdots = 1$ and A is diagonalizable.

To remember all of this in a picture, imagine that each distinct eigenvalue λ_i "reserves" a part of $\mathbb{R}^n$ for itself, and its eigenspace E_{λ_i} must sit inside this reserved space (and takes up all of it if the geometric mult. of λ_i matches its algebraic mult.). That is,



$\lambda_1, .., \lambda_r \in \mathbb{C}$

distinct eigenvalues.

A diagonalizable $\iff M_i = m_i$ for all i.

<u>Q</u>: what if $M_i < m_i$ for some i?

<u>A</u>: A is not diagonalizable, but is <u>almost</u> diagonalizable.

Jordan canonical form (4.9):

If A is diagonalizable, then $A = PDP^{-1}$, where

$$D = \underbrace{\begin{pmatrix} \overbrace{\lambda_1 \; \cdots \; \lambda_1}^{M_1 = m_1} & & & \\ & \overbrace{\lambda_2 \; \cdots \; \lambda_2}^{M_2 = m_2} & & \\ & & \ddots & \\ & & & \overbrace{\lambda_r \; \cdots \; \lambda_r}^{M_r = m_r} \end{pmatrix}}_{n} \Big\updownarrow n,$$

$$P = \begin{pmatrix} \overbrace{\underbrace{\text{Eigenvectors of } \lambda_1}_{M_1}}^{E_{\lambda_1}} & \Big| & \overbrace{\underbrace{\text{Eigenvectors of } \lambda_2}_{M_2}}^{E_{\lambda_2}} & \Big| & \cdots & \Big| & \overbrace{\underbrace{\text{Eigenvectors of } \lambda_r}_{M_r}}^{E_{\lambda_r}} \end{pmatrix}.$$

Q: If $M_i \neq m_i$ for all i, we are "missing" eigenvectors. Can we substitute these with a more general notion of eigenvector?

A: Yes. This leads to <u>generalized eigenvectors</u> (<u>power vectors</u>).

Suppose eigenvalue λ of A has alg. mult. m, geom. mult. M.

- $E_\lambda = \mathrm{Ker}(A - \lambda I)$ has basis $\mathcal{B}_\lambda = \{\underline{b}_1, \ldots, \underline{b}_M\}$

  ↳ eigenspace associated to λ

  and dimension $M \leq m$.

- $\tilde{E}_\lambda = \mathrm{Ker}((A - \lambda I)^m)$ has basis $\tilde{\mathcal{B}}_\lambda = \{\underline{b}_1, \ldots, \underline{b}_M, \underbrace{\underline{\xi}_1, \ldots, \xi_{m-M}}_{\text{generalized eigenvectors}}\}$

  ↳ generalized eigenspace associated to λ.

  and dimension $\underline{\underline{m}}$.

- NOTE THAT $E_\lambda \subseteq \tilde{E}_\lambda$ SINCE IF
$$(A-\lambda I)\underline{v} = \underline{0} \quad \text{THEN} \quad (A-\lambda I)^m \underline{v} = 0 .$$

- IT CAN BE SHOWN THAT IF $(A-\lambda I)^p \underline{v} = \underline{0}$ FOR <u>ANY</u> $p \in \mathbb{N}$, THEN $\underline{v} \in \tilde{E}_\lambda$.

- $\underline{v}$ IS CALLED A <u>GENERALIZED EIGENVECTOR</u> (POWER VECTOR) IF $(A-\lambda I)^p \underline{v} = 0$ FOR SOME p. THE MINIMUM VALUE OF p SUCH THAT THIS IS TRUE IS CALLED THE <u>ORDER</u> OF $\underline{v}$. NOTE THAT IF $\underline{v}$ IS OF ORDER $p \geq 1$, THEN $\underline{w} = (A-\lambda I)\underline{v}$ IS OF ORDER $p-1$.

HOW TO FIND $\tilde{\mathcal{B}}_\lambda$? <u>DEFINE</u> ξ_i BY $(A-\lambda I)\underline{\xi}_i = \underline{\xi}_{i-1}$, $i=1,\ldots$ WHERE $\underline{\xi}_0 \doteq \underline{b}_M$.

$(A-\lambda I)\underline{\xi}_1 = \underline{b}_M$ ORDER 3

$$\tilde{\mathcal{B}}_\lambda = \{\underbrace{\underline{b}_1, \ldots, \underline{b}_M}_{\substack{\text{ORDER } 1 \\ \text{(EIGENVECTORS)}}}, \underline{\xi}_1, \underline{\xi}_2, \ldots, \underline{\xi}_{m-M}\}$$

$(A-\lambda I)\underline{\xi}_2 = \underline{\xi}_1$

ORDER 2 ORDER $m-M+1$.

THEN, SINCE $(A-\lambda I)\underline{\xi}_i = \underline{\xi}_{i-1} \Rightarrow A\underline{\xi}_i = \lambda\underline{\xi}_i + \underline{\xi}_{i-1}$,

$$[A\underline{b}_i]_{\tilde{\mathcal{B}}_\lambda} = \lambda_i \underline{e}_i \quad \text{AND} \quad [A\underline{\xi}_i]_{\tilde{\mathcal{B}}_\lambda} = \lambda \underline{e}_{M+i} + \underline{e}_{M+i-1} .$$

$(i=1,\ldots,M)$ $\qquad$ $(i=1,\ldots,m-M)$

THIS IMPLIES THAT FOR <u>ANY</u> $A \in M_{n,n}$,

$$A = P\tilde{D}P^{-1} \quad \text{WHERE} \quad \tilde{D} = [A]_{\tilde{\beta}_\lambda} \quad \text{TAKES THE FORM}$$

$$\tilde{D} = \begin{pmatrix} \overbrace{\lambda_1 \cdots}^{M_1} & & & & \\ & \overbrace{\boxed{\begin{matrix} \lambda_1 & 1 & & \\ & \lambda_1 & 1 & \\ & & \lambda_1 \ddots & 1 \\ & & & \lambda_1 \end{matrix}}}^{m_1 - M_1} & & & \\ & & \overbrace{\lambda_2 \cdots}^{M_2} & & \\ & & & \overbrace{\boxed{\begin{matrix} \lambda_2 & 1 & & \\ & \lambda_2 & 1 & \\ & & \lambda_2 \ddots & 1 \\ & & & \lambda_2 \end{matrix}}}^{m_2 - M_2} & \\ & & & & \ddots \end{pmatrix},$$

"JORDAN BLOCKS"

$$P = \left(\overbrace{\underbrace{\text{EIGENVECTORS OF } \lambda_1}_{M_1} \;\Big|\; \underbrace{\text{GENERALIZED EIGENVECTORS OF } \lambda_1}_{m_1 - M_1}}^{\tilde{E}_{\lambda_1}} \;\Bigg|\; \cdots \;\Bigg|\; \overbrace{\underbrace{\text{EIGENVALUES OF } \lambda_r}_{M_r} \;\Big|\; \underbrace{\text{GENERALIZED EIGENVECTORS OF } \lambda_r}_{m_r - M_r}}^{\tilde{E}_{\lambda_r}} \right)$$

<u>EX.</u> $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. (<u>NOTE:</u> WE SAW IN A PREVIOUS LECTURE THAT THIS MATRIX IS NOT DIAGONALIZABLE.)

$$p_A(\lambda) = \lambda^2 = 0 \Rightarrow \lambda = 0.$$

$\underline{\underline{E_0}}$: $A - 0I = \begin{pmatrix} 0 & \boxed{1} \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{\xi} = \xi_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \{(1,0)^T\}$ BASIS OF E_0.

I.E., $\lambda = 0$ HAS $m = 2$, BUT $M = 1$.

WHAT TO DO?

LOOK FOR GENERALIZED EIGENVECTORS.

$\underline{\underline{\tilde{E}_0}}$: HAS BASIS $\tilde{\mathcal{B}}_0 = \{\underline{b}, \underline{\xi}\}$, WHERE

$$\underline{b} = (1,0)^T \in E_0, \quad \text{AND}$$

$$(A - 0I)\underline{\xi} = \underline{b} \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \underline{\xi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{\xi} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

GENERALIZED EIGENVALUE OF ORDER 2.

SO, $\tilde{D} = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$, $P = \begin{pmatrix} \underline{b} & \underline{\xi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

IN RETROSPECT, THIS IS OBVIOUS SINCE A IS ALREADY IN JORDAN FORM!

$\underline{\text{EX.}}$ $A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}$

$$\Rightarrow p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & -2 \\ -1 & -\lambda & 5 \\ -1 & -1 & 4-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} -\lambda & 5 \\ -1 & 4-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 1 & -2 \\ -1 & 4-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 1 & -2 \\ -\lambda & 5 \end{vmatrix}$$

$$= (3-\lambda)\left[\lambda^2 - 4\lambda + 5\right] + (2-\lambda) + (2\lambda - 5)$$

$$= (\lambda - 3)\left[(-\lambda^2 + 4\lambda - 5) + 1\right]$$

$$= -(\lambda-3)(\lambda-2)^2 = 0 \Rightarrow \lambda = 3, \lambda = 2.$$

E_3: $A-3I = \begin{pmatrix} 0 & 1 & -2 \\ -1 & -3 & 5 \\ -1 & -1 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \underline{\xi} = \xi_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \{(-1,2,1)^T\}$ BASIS OF E_3.

E_2: $A-2I = \begin{pmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \underline{\xi} = \xi_3 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \Rightarrow \{(-1,3,1)^T\}$ BASIS OF E_2.

SO, $\lambda = 3$ HAS $\begin{cases} \text{ALG. MULT. } 1 \\ \text{GEOM. MULT. } 1 \end{cases}$

$\lambda = 2$ HAS $\begin{cases} \text{ALG. MULT. } 2 \\ \text{GEOM. MULT. } 1 \end{cases} \Rightarrow A$ <u>NOT</u> DIAGONALIZABLE.

WHAT IS $\tilde{E}_2$?

$\tilde{E}_2$: HAS BASIS $\tilde{\mathcal{B}}_2 = (\underline{b}, \underline{\xi})$, WHERE $\underline{b} = (-1,3,1)^T \in E_2$.

$(A-2I)\underline{\xi} = \underline{b}$

$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & -1 \\ -1 & -2 & 5 & 3 \\ -1 & -1 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 1 \\ 0 & \boxed{1} & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{array}{l} \xi_1 = 1 - \xi_3 \\ \xi_2 = -2 + 3\xi_3 \\ \xi_3 \text{ FREE} \end{array} \Rightarrow \underline{\xi} = \begin{pmatrix} +1 \\ -2 \\ 0 \end{pmatrix} + \xi_3 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

LET $\xi_3 = 0$ (ONLY NEED <u>ONE</u> SOLN.)

$\Rightarrow \underline{\xi} = (1,-2,0)^T$. (GENERALIZED EIGENVECTOR)

THEREFORE,

$$A = P\tilde{D}P^{-1} \quad \text{WITH} \quad D = \begin{pmatrix} 3 & & \\ & 2 & 1 \\ & & 2 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 3 & -2 \\ 1 & 1 & 0 \end{pmatrix}$$