LECTURE 18
02/29/12

Q: FOR EVOLUTION PROBLEMS, WHAT IF WE HAVE COMPLEX EIGENVALUES?

A: SAME METHODS, BUT WILL SEE OSCILLATIONS IN SYSTEM ALONG W/ EXPONENTIAL DECAY / GROWTH!

EX
$$\begin{cases} \dfrac{d\underline{x}(t)}{dt} = A\underline{x}(t) \\ \underline{x}(0) \text{ given} \end{cases}, \quad A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

E-VALUES / E-VECTORS OF A:
$$\lambda = 1 \pm 2i$$
$$\underline{b} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\underline{v}} \pm i \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\underline{w}}$$

$$\Rightarrow \underline{x}(t) = e^{tA}\underline{x}(0) = \underbrace{Pe^{tD}P^{-1}}\underline{x}(0)$$
$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{(1+2i)t} & 0 \\ 0 & e^{(1-2i)t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -1 \end{pmatrix}^{-1}$$

WE USE EULER'S FORMULA $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, $\theta \in \mathbb{R}$, TO GET THAT

$$\begin{pmatrix} e^{(1+2i)t} & 0 \\ 0 & e^{(1-2i)t} \end{pmatrix} = \begin{pmatrix} e^{t}(\cos(2t) + i\sin(2t)) & 0 \\ 0 & e^{t}(\cos(2t) - i\sin(2t)) \end{pmatrix}$$

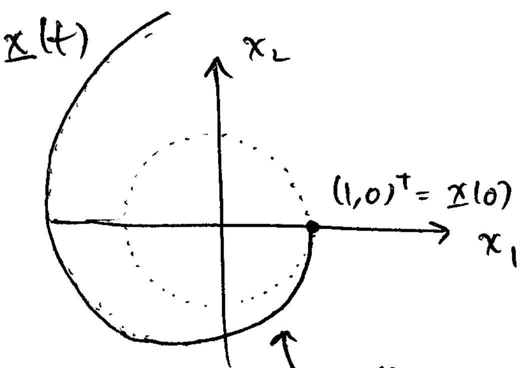
SINCE $\cos(-x) = \cos(x)$, $\sin(-x) = -\sin(x)$.

THEN, THE SOLUTION IS

$$\underline{x}(t) = e^{t}\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix}\underline{x}(0)$$

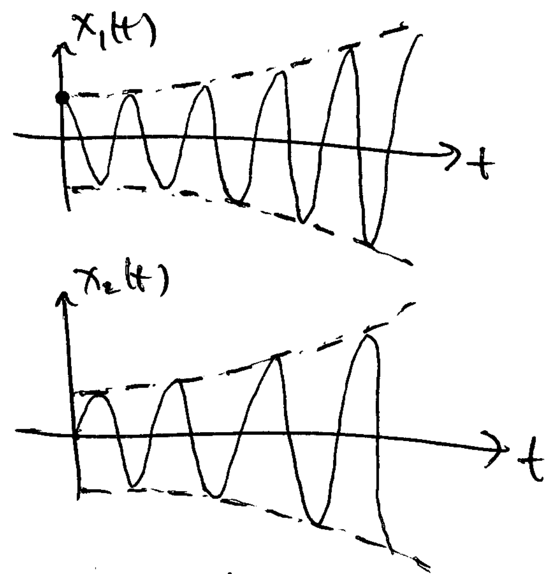
<u>REMARK</u>: REMEMBER, SINCE A IS REAL WE MUST HAVE THAT e^{tA} IS REAL AND $Pe^{tD}P^{-1}$ IS REAL AS WELL! SO THE FINAL ANSWER SHOULD CONSIST ONLY OF REAL TERMS.

WE CAN PLOT THE SOLUTION IN SEVERAL WAYS (GIVEN AN INITIAL CONDITION $\underline{x}(0)$ — FOR EX., $\underline{x}(0) = (1,0)^T$):



SPIRAL OUTWARD SINCE HAVE TERM e^t.

<u>PHASE PLANE</u>.



$$x_1(t) = e^t\cos(2t)$$
$$x_2(t) = -e^t\sin(2t).$$

- WE NOTE THAT THE REAL PART OF THE PAIR OF COMPLEX EIGENVALUES $\lambda = 1 \pm 2i$ DETERMINES THE <u>RATE OF GROWTH / DECAY</u>, WHILE THE IMAGINARY PART DETERMINES THE <u>FREQUENCY</u> OF <u>OSCILLATION</u>.

FOR EXAMPLE, IF $\lambda = -1 \pm 2i$ INSTEAD, WE WOULD HAVE HAD A SOLUTION W/ EXPONENTIAL DECAY e^{-t} INSTEAD OF GROWTH (A SPIRAL INWARD IN THE PHASE PLANE) BUT THE SAME FREQUENCY OF OSCILLATION.

REMARK: EVERY n^{th}-ORDER, HOMOGENEOUS, CONST. COEFF. SCALAR ODE

$$(\star) \quad \frac{d^n y}{dt^n} + c_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + c_0 y = 0, \quad \begin{array}{l} y(t) \in \mathbb{R} \\ t \geq 0. \\ c_i \in \mathbb{R} \text{ const.} \end{array}$$

CAN BE WRITTEN AS A SYSTEM OF 1^{st}-ORDER ODE.
TO SEE THIS, LET

$$\underline{x}(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ \vdots \\ x_{n-1}(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ \frac{dy}{dt}(t) \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}}(t) \end{pmatrix} \in \mathbb{R}^n \quad \text{FOR ALL } t \geq 0.$$

THEN,

$$\frac{d\underline{x}(t)}{dt} = \begin{pmatrix} \frac{dy}{dt}(t) \\ \frac{d^2y}{dt^2}(t) \\ \vdots \\ \frac{d^n y}{dt^n}(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ -c_{n-1}x_{n-1}(t) - \cdots - c_0 x_0(t) \end{pmatrix}$$

WHERE WE HAVE USED $(\star)$ TO REWRITE $\frac{d^n y}{dt^n}$ IN TERMS OF $y, \frac{dy}{dt}, \ldots, \frac{d^{n-1}y}{dt^{n-1}}$. THEREFORE,

$$\frac{d\underline{x}}{dt} = A\,\underline{x}, \qquad A = \underbrace{\left(\begin{array}{c|ccc} 0 & & I_{n-1} & \\ \hline -c_0 & -c_1 & \cdots & -c_{n-1} \end{array}\right)}_{n} \updownarrow n$$

w/ I_{n-1} an $(n-1)\times(n-1)$ identity matrix.

<u>Note:</u> to solve this linear evolution system, we need to be given

$$\underline{x}(0) = \left.\begin{pmatrix} y(0) \\ \frac{dy}{dt}(0) \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}}(0) \end{pmatrix}\right\} \text{ i.e., } n \text{ initial conditions.}$$

<u>Ex.</u>
$$\begin{cases} y'' + by' + y = 0 \\ \text{w/ } y(0),\ y'(0) \text{ given.} \end{cases} \qquad \text{"}{}' = \frac{d}{dt}\text{"}.$$

Let $\underline{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$. Then, $\underline{x}' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y \\ y' \end{pmatrix}}_{\underline{x}}$.

$p_A(\lambda) = \lambda^2 + b\lambda + 1$

$\rightsquigarrow$ E-values of A: $\lambda = \dfrac{-b \pm \sqrt{b^2-4}}{2}$.

So, if $|b| < 2$ we will get oscillations along w/ exponential growth/decay in solution $y(t)$.

LECTURE 19
03/02/12

Q: WHAT IS THE LONG-TIME BEHAVIOR OF A LINEAR EVOLUTION SYSTEM? HOW CAN WE DETERMINE IT WITHOUT HAVING TO SOLVE IT EXPLICITLY?

STABILITY AND LONG-TIME BEHAVIOR (5.5):

ASSUME $A = PDP^{-1}$, $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$. THEN,

DISCRETE-TIME EVOLUTION

$$\underline{x}(k) = A^k \underline{x}(0) = PD^kP^{-1}\underline{x}(0) = \lambda_1^k a_1(0)\underline{b}_1 + \dots + \lambda_n^k a_n(0)\underline{b}_n$$

WITH $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ BASIS OF E-VECTORS.

$\Rightarrow$ LONG-TIME BEHAVIOR DICTATED BY $\lambda_1^k, \dots, \lambda_n^k$.

CONT.-TIME EVOLUTION

$$\underline{x}(t) = e^{tA}\underline{x}(0) = Pe^{tD}P^{-1}\underline{x}(0) = e^{\lambda_1 t} a_1(0)\underline{b}_1 + \dots + e^{\lambda_n t} a_n(0)\underline{b}_n.$$

$\Rightarrow$ LONG-TIME BEHAVIOR DICTATED BY $e^{\lambda_1 t}, \dots, e^{\lambda_n t}$.

NOTE: IF $z \in \mathbb{C}$, $z = a + ib$ w/ $a = \mathrm{Re}(z)$, $b = \mathrm{Im}(z)$
$= re^{i\theta}$ w/ $r = \sqrt{a^2+b^2}$, $\theta = \arctan\left(\frac{b}{a}\right)$.

THE MAGNITUDE OF z IS $|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2} = r$.

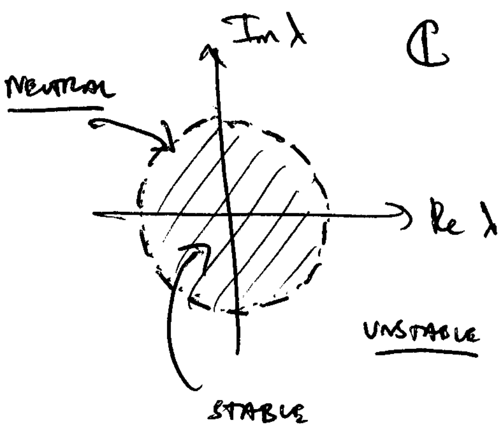
THEN,

$$\begin{cases} |z|>1 \Rightarrow z^k \to \infty \\ |z|=1 \Rightarrow |z^k|=1 \quad \text{AS } k\to\infty \\ |z|<1 \Rightarrow z^k \to 0 \end{cases}$$

$$\begin{cases} a=\mathrm{Re}(z)>0 \Rightarrow e^{zt}\to\infty \\ a=\mathrm{Re}(z)=0 \Rightarrow |e^{zt}|=|e^{ibt}|=1 \quad \text{AS } t\to\infty. \\ a=\mathrm{Re}(z)<0 \Rightarrow e^{zt}\to 0 \end{cases}$$

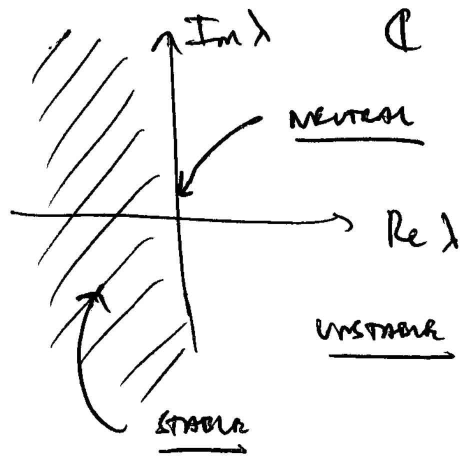
THEREFORE,

DISCRETE - TIME



- $|\lambda_i|>1 \Rightarrow \underline{b}_i$ UNSTABLE MODE
  $|\lambda_i|=1 \Rightarrow \underline{b}_i$ NEUTRAL MODE
  $|\lambda_i|<1 \Rightarrow \underline{b}_i$ STABLE MODE

CONT. - TIME.



- $\mathrm{Re}(\lambda_i)>0 \Rightarrow \underline{b}_i$ UNSTABLE MODE
  $\mathrm{Re}(\lambda_i)=0 \Rightarrow \underline{b}_i$ NEUTRAL MODE
  $\mathrm{Re}(\lambda_i)<0 \Rightarrow \underline{b}_i$ STABLE MODE.

- ASSUME $\lambda_1, \dots, \lambda_r$ ARE <u>DISTINCT</u> E-VALUES OF A,

SO $$\underline{x}(k) = \lambda_1^k \underline{d}_1 + \dots + \lambda_r^k \underline{d}_r \quad \text{(DISCRETE-TIME)}$$

$$\underline{x}(t) = e^{\lambda_1 t} \underline{d}_1 + \dots + e^{\lambda_r t} \underline{d}_r \quad \text{(CONT.-TIME)}.$$

FOR SOME VECTORS $\underline{d}_1, \dots, \underline{d}_r \in \mathbb{R}^n$.

ARRANGE E-VALUES IN ORDER OF DECREASING MAGNITUDE:

$$\rho(A) \doteq |\lambda_1| > |\lambda_2| > \dots > |\lambda_r|.$$

<u>SPECTRAL RADIUS</u> OF A

λ_1 IS THE <u>DOMINANT</u> E-VALUE OF SYSTEM.

λ_2 DETERMINES CONVERGENCE RATE TO "EQUILIBRIUM",

SINCE $$\left\| \frac{\underline{x}(k)}{\lambda_1^k} - \underline{d}_1 \right\| = \left\| \left(\frac{\lambda_2}{\lambda_1}\right)^k \underline{d}_2 + \dots \right\|$$
$$\lesssim c\left(\frac{\lambda_2}{\lambda_1}\right)^k \quad \text{AS } k \to \infty$$

(SIMILAR CALCULATION IN CONT. TIME CASE.).

- STABILITY DETERMINED BY DOMINANT E-VALUE.

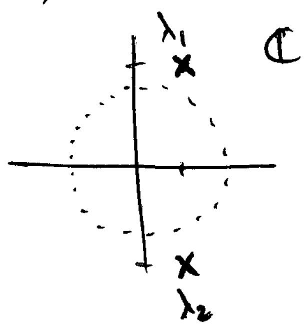
λ_1 STABLE $\Rightarrow$ SYSTEM <u>STABLE</u>

λ_1 NEUTRAL $\Rightarrow$ SYSTEM <u>NEUTRALLY STABLE</u>

λ_1 UNSTABLE $\Rightarrow$ SYSTEM <u>UNSTABLE</u>

<u>EX.</u> $$\begin{cases} \underline{x}(k) = A\,\underline{x}(k-1) \\ \underline{x}(0) \text{ GIVEN.} \end{cases}$$, A HAS EVALUES $$\lambda = \frac{1}{2} \pm \frac{4}{3} i.$$

THEN, SPECTRUM OF A IS:



SINCE $|\lambda_1| = |\lambda_2| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{4}{3}\right)^2} > 1$, THE SYSTEM IS <u>UNSTABLE</u>.

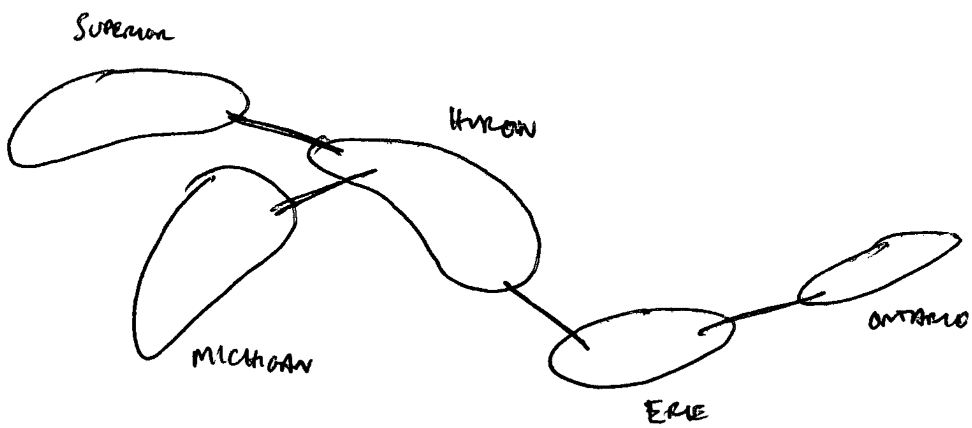
<u>EX.</u> $\begin{cases} \frac{dx}{dt} = A\underline{x} \\ \underline{x}(0) \text{ GIVEN} \end{cases}$, A HAS E-VALUES $\lambda = \frac{1}{2} \pm \frac{4}{3} i$.

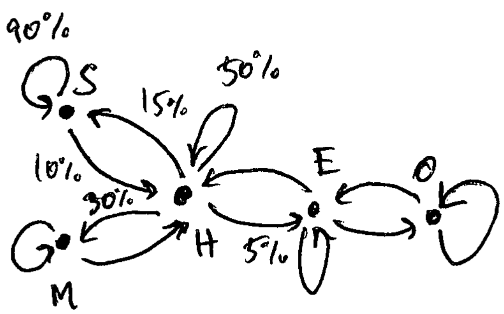
SINCE $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \frac{1}{2} > 0$, THE SYSTEM IS <u>UNSTABLE</u>.

LECTURE 20
03/05/12

MARKOV CHAINS AND STOCHASTIC MATRICES (5.7):

EX.





STATE SPACE $= \{S, M, H, E, O\}$.

MODEL: EVERY YEAR, A FIXED PROPORTION OF FISH MIGRATE FROM EACH GREAT LAKE TO ANOTHER LAKE.

WE ASSUME

(i) THESE PROPORTIONS STAY THE SAME YEAR BY YEAR

(ii) WE KNOW THE INITIAL POPULATION OF FISH IN EACH LAKE.

LET A_{ij} = PROPORTION THAT GO FROM LAKE j TO LAKE i IN ONE YEAR.

THEN,

$$A = \begin{array}{c} \begin{array}{ccccc} S & M & H & E & O \end{array} \\ \underbrace{\begin{pmatrix} .9 & \cdot & .15 & \cdot & \cdot \\ 0 & \cdot & .3 & \cdot & \cdot \\ .1 & \cdot & .5 & \cdot & \cdot \\ 0 & \cdot & .5 & \cdot & \cdot \\ 0 & \cdot & 0 & \cdot & \cdot \end{pmatrix}}_{\text{FROM}} \end{array} \left.\begin{array}{c} S \\ M \\ H \\ E \\ O \end{array}\right\} \text{TO}$$

NOTE: COLUMNS OF A SUM TO 1!

DEF. A IS A (LEFT) STOCHASTIC TRANSITION MATRIX IF $\underline{r} A = \underline{r}$, $\underline{r} = (1\ 1 \cdots 1)$ ← ROW VECTOR! AND ALL ENTRIES OF A ARE NONNEGATIVE $(A_{ij} \geq 0)$.

- THEN, $(A^k)_{ij} = \sum_{l_1} \cdots \sum_{l_{k-1}} A_{il_1} A_{l_1 l_2} \cdots A_{l_{k-1} j}$ IS THE PROPORTION THAT MOVE FROM j TO i IN EXACTLY k STEPS, GIVEN STARTED AT j.

  ($A_{il_1} \cdots A_{l_{k-1}j}$ CORRESPONDS TO PROPORTION THAT MOVE ALONG PATH $j \to l_{k-1} \to l_{k-2} \to \cdots \to l_1 \to i$.)

DEF. $\underline{v}$ IS A PROBABILITY VECTOR IF $\underline{r}\,\underline{v} = 1$ AND ALL ENTRIES OF $\underline{v}$ ARE NONNEGATIVE $(\underline{v}_i \geq 0)$.

⟶ WE DENOTE SPACE OF ALL PROBABILITY VECTORS (DISTRIBUTIONS) BY $\mathcal{P}$.

- IF INITIALLY THE PROPORTION OF FISH IN LAKE i IS $\underline{v}_i$, THEN k YEARS LATER IT IS $(A^k \underline{v})_i$.

PROBABILISTIC INTERPRETATION:

- A_{ij} IS THE TRANSITION PROBABILITY $p_{ij} = P(X_1 = i \mid X_0 = j)$ THAT AN INDIVIDUAL FISH IS AT STATE i AT TIME 1 GIVEN IT IS AT STATE j AT TIME 0 (X_k IS THE POSITION AT TIME k).

- "PROBABILITY" = PROPORTIONS OBSERVED OVER MANY INDEPENDENT TRIALS (THINK OF COIN TOSSES).
  THIS A CONSEQUENCE OF THE SO-CALLED LAW OF LARGE NUMBERS (FAIR COIN HAS 50% PROBABILITY OF H OR T ON ANY ONE FLIP SINCE OVER MANY REPEATED FLIPS WE OBSERVE 50% SHOW UP H OR T IN THE LONG-RUN).

- $\underline{x}(k) = A^k \underline{x}(0)$ IS A MARKOV CHAIN WITH INITIAL DISTRIBUTION $\underline{x}(0)$ (A PROBABILITY VECTOR).

  $\underline{x}(k)$ = PROBABILITY ANY ONE FISH IS IN A PARTICULAR STATE AT TIME $k \geq 0$
  = PROPORTION OF ENSEMBLE OF ALL FISH THAT IS IN PARTICULAR STATE AT TIME $k \geq 0$.

- MARKOV BECAUSE TRANSITION PROBABILITIES ONLY DEPEND ON CURRENT STATE, NOT ON THE PAST.
  FOR EX., IF FISH DO NOT RETURN TO THEIR PREVIOUS STATE IMMEDIATELY, THE MODEL IS NOT MARKOV.