M346 (56615), Sample Midterm #1 Questions

Below are some sample midterm questions. Please note that the intent of these is to help prepare for the exam, and that actual exam questions will not merely be modifications of these problems. Consult your HW, lecture notes, and books for additional sources of material to review. Finally, to obtain the full effect of an exam please complete these problems under time-pressure (75 minutes or less).

1. Let $V = \mathbb{R}_2[t]$ with standard basis $\mathcal{B} = \{1, t, t^2\}$ and let $W = M_{2,2}$ be the space of 2×2 real matrices with standard basis $\mathcal{D} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Consider the linear transformation $L: V \to W$ given by

$$L(\boldsymbol{p}) = \begin{pmatrix} \boldsymbol{p}(1) - \boldsymbol{p}(0) & \boldsymbol{p}(2) - \boldsymbol{p}(0) \\ \boldsymbol{p}(-1) - \boldsymbol{p}(0) & \boldsymbol{p}(-2) - \boldsymbol{p}(0) \end{pmatrix}.$$

a) Find the matrix representation $[L]_{\mathcal{DB}}$ of L relative to the bases $\mathcal{B}$ and $\mathcal{D}$.

b) What is the dimension of $\mathrm{Ker}(L)$? Find a basis for $\mathrm{Ker}(L)$.

c) What is the dimension of $\mathrm{Ran}(L)$? Find a basis for $\mathrm{Ran}(L)$.

2. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 7 \\ 5 & 10 & 13 & 18 \end{pmatrix}$.

a) Let $V = \{\boldsymbol{x} \in \mathbb{R}^4 : A\boldsymbol{x} = 0\}$. What is the dimension of V? Find a basis for V.

b) Are the vectors $(1,2,5)^T$, $(2,4,10)^T$, $(3,5,13)^T$, $(4,7,18)^T$ linearly independent? Do they span $\mathbb{R}^3$?

c) Give a basis for the span of the four vectors in part (b).

3. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8x_1 - 10x_2 \\ 3x_1 - 3x_2 \end{pmatrix}$. Define the standard basis $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and an alternate basis $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix} \right\}$. Consider a vector $\boldsymbol{v} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$.

a) Find $P_{\mathcal{EB}}$, $P_{\mathcal{BE}}$, $[\boldsymbol{v}]_{\mathcal{E}}$, and $[\boldsymbol{v}]_{\mathcal{B}}$.

b) Find $[L]_{\mathcal{E}}$ and $[L]_{\mathcal{B}}$.

4. True or false?

a) The plane $x_1 + 3x_2 - 4x_3 = 1$ is a subspace of $\mathbb{R}^3$.

b) If A is a 3×5 matrix, then the nullity of A is at least 2.

c) Let $L: \mathbb{R}_5[t] \to \mathbb{R}^3$ be a linear transformation. If L is onto, the kernel of L has dimension 2.

d) Let $\mathcal{B} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ be a basis for a vector space V. If n vectors $\{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n\}$ span V then the coordinate vectors $\left\{[\boldsymbol{d}_1]_{\mathcal{B}}, \ldots, [\boldsymbol{d}_n]_{\mathcal{B}}\right\}$ are linearly independent.

e) Every linear transformation $L: \mathbb{R}^5 \to \mathbb{R}^4$ takes the form $L(\boldsymbol{x}) = A\boldsymbol{x}$ with A a 5×4 matrix.

f) Let A_{rref} be the reduced row-echelon form of a matrix A. Then, the pivot columns of A_{rref} form a basis of the column space of A (i.e., the span of the columns of A).

g) The vectors $\boldsymbol{b}_1 = 1 + t + 2t^2$, $\boldsymbol{b}_2 = 2 + 3t + 5t^2$, $\boldsymbol{b}_3 = 3 + 7 + 9t^2$ form a basis for $\mathbb{R}_2[t]$.

h) [Harder...] The equation $p''(t) - p(t) = q(t)$ has a solution $p \in \mathbb{R}_3[t]$ for every $q \in \mathbb{R}_3[t]$.

5. Consider $A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$.

a) Write the characteristic polynomial $p_A(\lambda)$ and use this to determine the eigenvalues of A.

b) Find the eigenspaces corresponding to the eigenvalues of A.

c) Write the matrix in the form $A = PDP^{-1}$ with D a diagonal matrix.