M346 (56615), Sample Midterm #2 Solutions

1. Let $A = \begin{pmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{pmatrix}$.

a) Find the eigenvalues of A.

Solution: $p_A(\lambda) = -\lambda(1-\lambda)^2$, so the eigenvalues are 0 and 1.

b) Verify that you have obtained the correct eigenvalues by using the trace of A. Compute the determinant of A using the eigenvalues.

Solution: The sum of the eigenvalues of A is $0 + 1 + 1 = 2$ (counting multiplicities), which agrees with $\mathrm{Tr}(A) = 2$. The determinant is $0 \cdot 1 \cdot 1 = 0$.

c) Is A diagonalizable? Why or why not?

Solution: A basis for E_0 is $\{(0,-1,2)^T\}$ while a basis for E_1 is $\{(1,-1,5)^T\}$. Since the algebraic multiplicity of eigenvalue $\lambda = 1$ is 2 but its geometric multiplicity is 1, A is not diagonalizable.

d) Write A in its Jordan normal form $P\tilde{D}P^{-1}$ for an appropriate $\tilde{D}$ and P.

Solution: We need to find a generalized eigenvector (power vector) $\boldsymbol{\xi}$ for the eigenvalue $\lambda = 1$. Setting $(A - 1I)\boldsymbol{\xi} = \boldsymbol{b}$ where $\boldsymbol{b} = (1,-1,5)^T$, we find $\boldsymbol{\xi} = (0,3,-5)^T$. Therefore, $A = P\tilde{D}P^{-1}$ with $\tilde{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 3 \\ 2 & 5 & -5 \end{pmatrix}$.

2. The matrix $A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ has eigenvalues -2, 1, and 1 with corresponding eigenvectors $\boldsymbol{b}_1 = (1,1,1)^T$, $\boldsymbol{b}_2 = (1,-1,0)^T$, and $\boldsymbol{b}_3 = (1,0,-1)^T$.

a) Suppose $\boldsymbol{x}(k) = A\boldsymbol{x}(k-1)$ for all $k \geq 1$ with initial condition $\boldsymbol{x}(0) = (6,-1,-2)^T = \boldsymbol{b}_1 + 2\boldsymbol{b}_2 + 3\boldsymbol{b}_3$. Compute the solution $\boldsymbol{x}(k) = (x_1(k), x_2(k), x_3(k))^T$ explicitly.

Solution: $\boldsymbol{x}(k) = (-2)^k\boldsymbol{b}_1 + 1^k 2\boldsymbol{b}_2 + 1^k 3\boldsymbol{b}_3 = \big((-2)^k + 5, (-2)^k - 2, (-2)^k - 3\big)^T$.

b) For part (a), what are the stable, unstable, and neutrally stable modes? Determine the limit of the ratios $x_1(k)/x_2(k)$ and $x_1(k)/x_3(k)$ as $k \to \infty$.

Solution: Since $\lambda = -2$ lies outside the unit circle in the complex plane, while $\lambda = 1$ lies on the unit circle, we find that $\boldsymbol{b}_1$ is an unstable mode while $\boldsymbol{b}_2$ and $\boldsymbol{b}_3$ are neutrally stable modes. Therefore, for large k we have $\boldsymbol{x}(k) \approx (-2)^k\boldsymbol{b}_1$ and the ratios $x_1(k)/x_2(k)$ and $x_1(k)/x_3(k)$ both approach 1.

c) Now consider the continuous-time system $d\boldsymbol{x}(t)/dt = A\boldsymbol{x}(t)$ for $t \geq 0$ with initial condition $\boldsymbol{x}(0) = (6,-1,-2)^T$. Compute the solution $\boldsymbol{x}(t) = (x_1(t), x_2(t), x_3(t))^T$ explicitly.

Solution: $\boldsymbol{x}(t) = e^{-2t}\boldsymbol{b}_1 + 2e^t\boldsymbol{b}_2 + 3e^t\boldsymbol{b}_3 = \big(e^{-2t} + 5e^t, e^{-2t} - 2e^t, e^{-2t} - 3e^t\big)^T$.

d) For part (c), what are the stable, unstable, and neutrally stable modes? Determine the limit of the ratios $x_1(t)/x_2(t)$ and $x_1(t)/x_3(t)$ as $t \to \infty$.

Solution: Since $\lambda = -2$ has negative real part and $\lambda = 1$ has positive real part, $\boldsymbol{b}_1$ is a stable mode while $\boldsymbol{b}_2$ and $\boldsymbol{b}_3$ are unstable modes. For large t, $\boldsymbol{x}(t) \approx 2e^t\boldsymbol{b}_2 + 3e^t\boldsymbol{b}_3$ and the ratios $x_1(t)/x_2(t)$ and $x_1(t)/x_3(t)$ approach $-5/2$ and $-5/3$, respectively..

3. Suppose $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

a) Diagonalize the matrix by writing it as $A = PDP^{-1}$.

Solution: $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\sqrt{3} & 0 \\ 0 & 0 & -i\sqrt{3} \end{pmatrix}$, $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{pmatrix}$.

b) Write the matrix exponential e^D as $E + iF$, where E and F are real matrices.

Solution: $e^D = \begin{pmatrix} e^0 & 0 & 0 \\ 0 & e^{i\sqrt{3}} & 0 \\ 0 & 0 & e^{-i\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\sqrt{3} & 0 \\ 0 & 0 & \cos\sqrt{3} \end{pmatrix} + i \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin\sqrt{3} & 0 \\ 0 & 0 & -\sin\sqrt{3} \end{pmatrix}$.

4. Let $T = \begin{pmatrix} 0.8 & 0.4 & 0 & 0 \\ 0.2 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.2 \\ 0 & 0 & 0.4 & 0.8 \end{pmatrix}$.

a) Is T a transition matrix? If so, draw the states of the Markov chain with directed edges between states and their corresponding transition probabilities.

Solution: Yes, T is a transition matrix.

b) Is T irreducible? Is it aperiodic? Is it a regular transition matrix?

Solution: T is aperiodic, but it is not irreducible. Therefore, it is not regular.

c) Determine all possible stationary distributions $\boldsymbol{\pi}$ of the Markov chain.

Solution: $\boldsymbol{\pi} = c(2/3, 1/3, 0, 0)^T + (1-c)(0, 0, 1/3, 2/3)^T$ for any $0 \le c \le 1$.

d) Suppose $S = 0.8T + 0.2B$, where B is a matrix with all entries $1/4$. Estimate the rate of convergence of $\boldsymbol{x}(k) = S\boldsymbol{x}(0)$ to the unique stationary distribution $\boldsymbol{\pi}_S$ as $k \to \infty$.

Solution: The rate of convergence is determined by the second largest eigenvalue. Since $|\lambda_2| \le (1 - n\min_{i,j} S_{i,j}) = 1 - 4 \times \left(\frac{1}{4} \cdot 0.2\right) = 0.8$, the rate of convergence is at least as fast as 0.8^k.

5. True or false? Explain your answer by providing a complete justification if true, and a counterexample if false.

a) If A and B are similar (i.e., $B = PAP^{-1}$) then they have the same spectrum.

Solution: True. $p_B(\lambda) = \det(B - \lambda I) = \det\left(P(A - \lambda I)P^{-1}\right) = \det(A - \lambda I) = p_A(\lambda)$.

b) If A has eigenvalues $(1 \pm i)/2$, any nonzero solution to the discrete-time system $\boldsymbol{x}(k) = A\boldsymbol{x}(k-1)$ will have oscillations that grow arbitrarily large in magnitude for k large.

Solution: False. Since $|(1 \pm i)/2| = 1/\sqrt{2} < 1$ all modes are stable.

c) If A has eigenvalues $-1 \pm 4i$, any nonzero solution to the continuous-time system $d\boldsymbol{x}/dt = A\boldsymbol{x}$ will have oscillations with frequency 4 and amplitudes that decay exponentially in time.

Solution: True. The solution will be comprised of terms like $e^{(-1\pm 4i)t} = e^{-t}(\cos(4t) \pm i\sin(4t))$.

d) For regular transition matrices A, every column of A^k converges to the stationary distribution $\boldsymbol{\pi}$ as $k \to \infty$.

Solution: True. Since $\boldsymbol{x}(k) = A^k\boldsymbol{x}(0) \to \boldsymbol{\pi}$ as $k \to \infty$ for any initial probability distribution $\boldsymbol{x}(0)$, we pick $\boldsymbol{x}(0) = \boldsymbol{e}_i$ to see that the i^{th} column of A^k converges to $\boldsymbol{\pi}$.

e) The total number of cycles of length r in a directed network is $(\mathrm{Tr}(A))^r$.

Solution: False. The number of cycles of length r is $\mathrm{Tr}(A^r) = \lambda_1^r + \cdots + \lambda_n^r$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.