Lecture 01
01/18/12

Introduction (1.1):

$$
\begin{tikzcd}
{[\underline{x}(0)]_{\mathcal{B}}} \arrow[r] & {[\underline{x}(t)]_{\mathcal{B}}} \text{ Basis} \arrow[d] \\
\underline{x}(0) \arrow[u] \arrow[r, dashed, "\text{Dynamics.}"] & \underline{x}(t)
\end{tikzcd}
$$

Ex

$$
\begin{cases} x_1(n) = 2x_1(n-1) + x_2(n-1) \\ x_2(n) = x_1(n-1) + 2x_2(n-1) \end{cases} \Rightarrow \quad x_1(n) = ? \quad x_2(n) = ?
$$

$x_{1,2}(0)$ given.

Change of variables: $y_1 = \dfrac{x_1 + x_2}{2}$, $y_2 = \dfrac{x_1 - x_2}{2}$

$$
\Rightarrow \begin{cases} y_1(n) = 3y_1(n-1) \\ y_2(n) = y_2(n-1) \end{cases} \Rightarrow \quad \begin{aligned} y_1(n) &= 3^n y_1(0) \\ y_2(n) &= y_2(0) \end{aligned}
$$

Converting back to x_1, x_2 gives

$$
\left| \begin{aligned} x_1(n) &= \tfrac{1}{2}(3^n + 1)\, x_1(0) + \tfrac{1}{2}(3^n - 1)\, x_2(0) \\ x_2(n) &= \tfrac{1}{2}(3^n - 1)\, x_1(0) + \tfrac{1}{2}(3^n + 1)\, x_2(0) \end{aligned} \right| .
$$

To recap: $\underline{x} = (x_1, x_2)^T$, $\underline{y} = (y_1, y_2)^T$

$$
\underline{x}(n) = \underset{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}{\underset{\shortparallel}{A}}\, \underline{x}(n-1) \quad \xrightarrow[\text{(decoupling)}]{\text{diagonalization}} \quad \underline{y}(n) = \underset{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}{\underset{\shortparallel}{D}}\, \underline{y}(n-1).
$$

Vector spaces (2.1):

$\underline{x}$ vector, c scalar.

Def. V is a vector space if it is closed under the operations of addition and scalar multiplication (i.e., for any $\underline{x}, \underline{y} \in V$ and scalar c, $\underline{x}+\underline{y} \in V$ and $c\underline{x} \in V$), and satisfies the axioms on [p. 10–11, Sadun].

V can either be finite-dim. or infinite-dim.

$\longrightarrow$ depends on # of degrees of freedom.

V called
- real vector space if $c \in \mathbb{R}$
- complex vector space if $c \in \mathbb{C}$.

Ex.
$$V = \left\{ \underline{x} : \underbrace{\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\left(\text{i.e., } \underline{x} = (x_1, \dots, x_n)^T\right)} \right\} \doteq \mathbb{R}^n$$

with
$$\underline{x} + \underline{y} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}, \quad c\underline{x} = \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix}$$

where $x_i, c \in \mathbb{R}$.

Ex. Same as above, but with $x_i, c \in \mathbb{C}$

$$\Rightarrow V \doteq \mathbb{C}^n.$$

Ex.

$$M_{nm} \doteq \left\{ \underline{a} : \underline{a} = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}}_{m} \Big\updownarrow n \right\}$$

with $$\underline{a} + \underline{b} = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nm}+b_{nm} \end{pmatrix}$$

$$c\,\underline{a} = \begin{pmatrix} ca_{11} & \cdots & ca_{1m} \\ \vdots & \ddots & \vdots \\ ca_{n1} & \cdots & ca_{nm} \end{pmatrix}.$$

where $a_{ij}, c \in \mathbb{R}$.

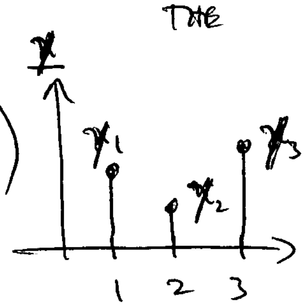
Ex.

$$C^0[0,1] \doteq \{ f : f : [0,1] \to \mathbb{R} \text{ is continuous} \}$$

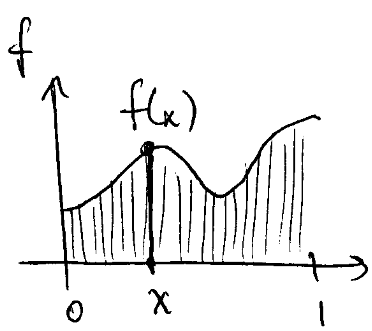
with $$(f+g)(x) = f(x) + g(x)$$
$$(cf)(x) = cf(x).$$

__Note__: $C^0[0,1]$ is an __infinite-dim.__ vector space since it takes an infinite # of pieces of information to specify an element of the vector space.

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



vs.



$\underline{x} = \underline{x}(x), \; x \in \underbrace{\{1,2,3\}}_{\text{finite set}}.$

$f = f(x), \; x \in \underbrace{[0,1]}_{\text{infinite set}}.$

EX. $\mathbb{R}_n[t] \doteq \{ p : \underbrace{p(x) = a_0 + a_1 x + \cdots + a_n x^n}_{p \text{ polynomial of degree } \leq n} \}$

WITH $(p+q)(x) = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n$.

AND $(cp)(x) = cp(x)$,

WHERE $a_i, c \in \mathbb{R}$.

NOTE: $\mathbb{R}_n[t]$ IS FINITE-DIM. SINCE WE ONLY NEED TO SPECIFY $(n+1)$ COEFFICIENTS $\{a_i\}_{i=0}^{n}$ TO DESCRIBE ANY ELEMENT.

CONTRAST THIS TO:

EX. $\mathbb{R}[t] \doteq \{ p : p$ IS A POLYNOMIAL OF ANY DEGREE $\}$.

$\mathbb{R}[t]$ IS INF.-DIM. SINCE TO SPECIFY AN ELEMENT ONE NEEDS TO KNOW AN INFINITE # OF COEFF. $\{a_i\}_{i=0}^{\infty}$ (EVEN IF ALL BUT A FINITE NO. OF THEM ARE ZERO!).

Lecture 02
01/20/12

Subspaces:

Def. $U \subset V$ is a subspace of V if it is closed under addition and scalar multiplication inherited from V.

Trivial subspaces: $\{\underline{0}\}$, V.

Ex. $V = \mathbb{R}^2$

- $U_1 = \{\underline{x} \in \mathbb{R}^2 : x_1 + x_2 = 0\}$ subspace since

$$\underline{x}+\underline{y} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix} \in U_1 \quad \left(\text{because } (x_1+y_1)+(x_2+y_2) = 0\right)$$

$$c\underline{x} = \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix} \in U_1 \quad \left(\text{because } cx_1 + cx_2 = 0\right)$$

- $U_2 = \{\underline{x} \in \mathbb{R}^2 : x_1 + x_2 = 1\}$ not a subspace since $\underline{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $y \in \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are both in U_2, but $\underline{x}+\underline{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin U_2$ since $1+1 \neq 1$.

- $\mathbb{Z}^2 \doteq \{\underline{x} \in \mathbb{R}^2 : x_1, x_2 \in \mathbb{Z}\}$ not a subspace of $\mathbb{R}^2$ since $c\underline{x} \notin \mathbb{Z}^2$ if $c \notin \mathbb{Z}$.

(Note that while $\mathbb{Z}^2$ is not a subspace of $\mathbb{R}^2$, $\mathbb{Z}^2$ is a vector space if we define scalar multiplication with $c \in \mathbb{Z}$, not $c \in \mathbb{R}$.)

EX. $\mathbb{R}[t]$ IS A SUBSPACE OF $C^0[0,1]$.

SPACE OF ALL REAL-VALUED POLYNOMIALS

$\mathbb{R}_n[t]$ IS A SUBSPACE OF $\mathbb{R}[t]$ (AND OF $C^0[0,1]$).

SPACE OF ALL REAL-VALUED POLYNOMIALS W/ DEGREE $\leq n$.

BASIS AND DIMENSION (2.2):

$\mathcal{B} \doteq \{\underline{b}_1, \dots, \underline{b}_n\}$ ORDERED SET OF VECTORS IN V.

- WE SAY $\underline{v}$ IS A LINEAR COMBINATION OF $\mathcal{B}$ IF $\underline{v} = a_1\underline{b}_1 + \dots + a_n\underline{b}_n$ FOR SOME SCALARS $\{a_i\}_{i=1}^n$.
- $\mathcal{B}$ LINEARLY INDEPENDENT IF
  $a_1\underline{b}_1 + \dots + a_n\underline{b}_n = 0 \iff a_i = 0$ FOR ALL $i=1,\dots,n$.

THM. IF $n>1$, $\mathcal{B}$ LINEARLY INDEP. $\iff$ b_i ~~IS~~ A LINEAR COMBINATION OF $\{b_j\}_{j\neq i}$ FOR SOME $i \in \{1,\dots,n\}$.
(I.E., AN ELEMENT OF $\mathcal{B}$ CAN BE WRITTEN IN TERMS OF THE OTHERS.)

PF. EASY.

NOTE: $\mathcal{B}$ LINEARLY DEP. IMPLIES ONE $\underline{b}_i$ IS LINEAR COMB. OF OTHERS, BUT DOES NOT SAY WHICH ONE.

DEPENDENT.

EX. $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$

INDEP.

DEPENDENT

INDEP.

- $\text{SPAN}(\mathcal{B}) \doteq \{\underline{v} \in V : \underline{v}$ LINEAR COMBINATION OF $\mathcal{B}\}$

THM. $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ FINITE. $\subset V$ IS A SUBSPACE OF V.

THEN, $\text{SPAN}(\mathcal{B}) \subset V$ IS A SUBSPACE OF V.

PF. EASY.

NOTE: $\text{SPAN}(\mathcal{B})$ IS THE <u>SMALLEST</u> SUBSPACE THAT CONTAINS $\mathcal{B}$. THAT IS,

$$\text{SPAN}(\mathcal{B}) = \bigcap_{\substack{\{\lambda :\, U_\lambda \text{ SUBSPACE} \\ \text{AND } \mathcal{B} \subset U_\lambda\}}} U_\lambda .$$

DEF. IF $\mathcal{B}$ SUCH THAT

(i) $\mathrm{SPAN}(\mathcal{B}) = V$

(ii) $\mathcal{B}$ LINEARLY INDEP.

WE SAY $\mathcal{B}$ IS A BASIS OF V.

EX. $V = \mathbb{R}^n$

- $\mathcal{E} = \{\underline{e}_1, \dots, \underline{e}_n\}$, $\underline{e}_j = (0, \dots, 0, 1, 0, \dots, 0)^T$ (j^{th} ENTRY)

IS THE STANDARD BASIS OF $\mathbb{R}^n$.

$V = \mathbb{C}^n$

- $\mathcal{E}$ IS BASIS IF V IS CONSIDERED A COMPLEX VECTOR SPACE (I.E, SCALAR MULTIPLICATION BY $c \in \mathbb{C}$).
- $\mathcal{E}$ NOT A BASIS IF V IS CONSIDERED A REAL VECTOR SPACE (I.E, SCALAR MULTIPLICATION BY $c \in \mathbb{R}$).

IN THIS CASE,

$$\tilde{\mathcal{E}} = \{\underline{e}_1, \dots, \underline{e}_n, \underline{f}_1, \dots, \underline{f}_n\}$$

WITH $\underline{e}_j$ AS BEFORE AND

$\underline{f}_j = (0, \dots, 0, i, 0, \dots, 0)^T$ IS A BASIS.

j^{th} ENTRY $\quad i = \sqrt{-1}$

Ex. $V = \mathbb{R}_n[t]$

- $\mathcal{B} = \{1, t, t^2, \dots, t^n\}$ IS A BASIS.
- $\mathcal{B}_1 = \{2, 3t - t^2, 4t^2, t^3, \dots, t^n\}$ IS A BASIS.
- $\mathcal{B}_2 = \{1, \underbrace{t+t^2, 2t+2t^2}, t^3, \dots, t^n\}$ IS NOT A BASIS.

(SPAN$(\mathcal{B}_2) \neq V$ AND $\mathcal{B}_2$ LINEARLY DEP.)

$\widetilde{\mathcal{B}}_2 = \{1, t, \underbrace{t+t^2, 2t+t^2}, t^3, \dots, t^n\}$ IS NOT A BASIS

($\widetilde{\mathcal{B}}_2$ LINEARLY DEP.)

Ex. $V = M_{22}$

- $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

IS STANDARD BASIS OF M_{22}.

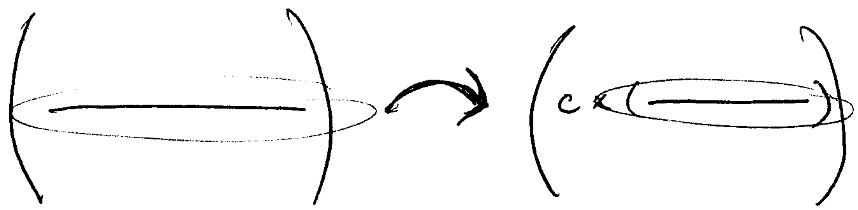
REMARK:

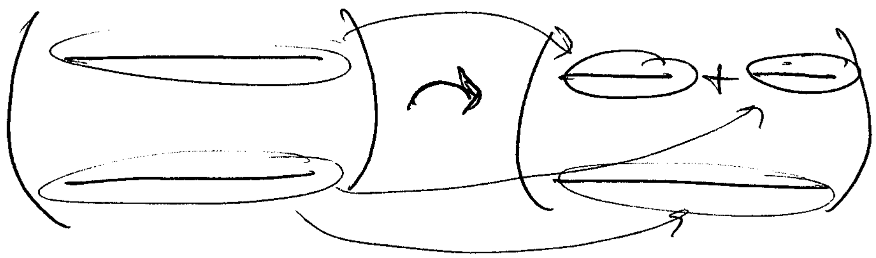
$$\underbrace{a_1 \underline{b}_1 + \dots + a_n \underline{b}_n}_{\text{LINEAR COMBINATION OF } \{\underline{b}_1, \dots, \underline{b}_n\} \subset \mathbb{R}^n} = \underbrace{\begin{pmatrix} | & & | \\ \underline{b}_1 & \cdots & \underline{b}_n \\ | & & | \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{\underline{x}}$$

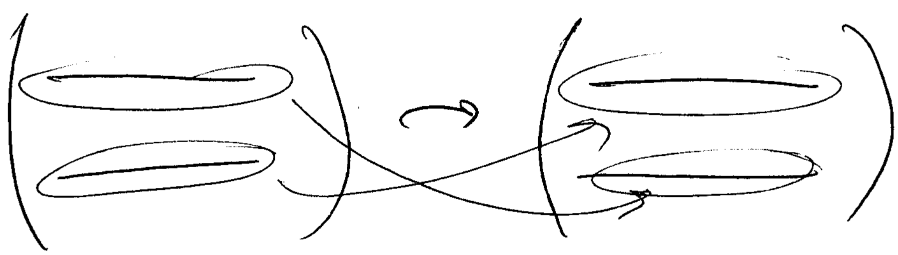
(MATRIX W/ COLUMNS $\underline{b}_1, \dots, \underline{b}_n$)

SO, SPAN$(\mathcal{B})$ = COLUMN SPACE OF A.

FOR ANY MATRIX A, DEFINE ROW OPERATIONS:

(i)  FOR SCALAR c. [MULTIPLY ROW]

(ii)  [ADD ROWS]

(iii)  [SWITCH ROWS]

LECTURE 03
01/23/12

SUPPOSE $\underline{v}_i \in \mathbb{R}^n$, $i = 1, \ldots, m$.

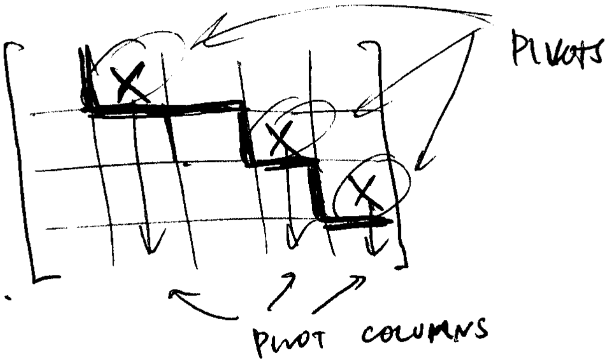
$$c_1 \underline{v}_1 + \cdots + c_m \underline{v}_m = \underbrace{\begin{pmatrix} \underline{v}_1 & \cdots & \underline{v}_m \end{pmatrix}}_{m} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$$

MATRIX $A = (\underline{v}_1, \ldots, \underline{v}_m) \in M_{n,m}$.

Q: HOW TO ANALYZE A?

ROW ECHELON FORM (ref):

- ALL 0's BELOW PIVOTS
- ALL ZERO ROWS AT BOTTOM.



REDUCED ROW ECHELON FORM (rref) IF, IN ADDITION

- ALL 0's ABOVE PIVOTS
- ALL PIVOTS ARE 1's.

CAN PUT A INTO rref USING GAUSSIAN ELIMINATION, GIVING A_{rref}. # OF PIVOTS $\doteq$ RANK(A).

Ex.

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix} \begin{matrix} \\ \\ -R_2 \end{matrix} \longrightarrow \begin{pmatrix} \boxed{1} & 0 & 0 & 2 & 3 \\ 0 & 0 & \boxed{1} & 6 & 2 \\ 0 & 0 & 0 & \boxed{-4} & 1 \end{pmatrix} \begin{matrix} +\frac{2}{4}R_3 \\ +\frac{6}{4}R_3 \\ \times -\frac{1}{4} \end{matrix}$$

NOT ref $\qquad\qquad$ ref, NOT rref.

$$\rightarrow \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & \frac{7}{2} \\ 0 & 0 & \boxed{1} & 0 & \frac{7}{2} \\ 0 & 0 & 0 & \boxed{1} & -\frac{1}{4} \end{pmatrix}$$

rref.

- SOLUTIONS TO $A\underline{x} = \underline{0}$ SAME AS SOLN'S TO $A_{rref}\underline{x} = \underline{0}$.
  (ELEMENTARY ROW OPERATIONS LEAVE $\underline{0}$ INVARIANT).
- COLUMNS W/O PIVOTS GIVE FREE VARIABLES (HERE, x_2 AND x_5 SINCE NO PIVOTS IN COLUMNS 2,5)
- ROWS W/O PIVOTS ARE $(0, \dots, 0)$ AND LEAD TO INCONSISTENCY.
- PIVOT COLUMNS OF A (NOT OF A_{rref}!) SPAN COLUMN SPACE OF A.

Q: HOW TO SOLVE $A\underline{x} = \underline{b}$ WITH $A, \underline{b}$ GIVEN?

AUGMENTED MATRIX: $\left(A \mid \underline{b} \right)$.

i) REDUCE TO rref TO GET $\left(A_{rref} \mid \underline{b}_{rref} \right)$

ii) SOLVE FOR $\underline{x}$ USING BACK SUBSTITUTION,

$$A_{rref}\,\underline{x} = \underline{b}_{rref} \Rightarrow \underline{x} \text{ SOLVES } A\underline{x} = \underline{b}.$$

EX. A AS BEFORE, $\underline{b} = (1,2,3)^T$.

$$\rightsquigarrow \underline{b}_{rref} = \left(\tfrac{3}{2}, \tfrac{7}{2}, -\tfrac{1}{4}\right)^T \Rightarrow \begin{aligned} x_5 &= \text{FREE} \\ x_4 &= -\tfrac{1}{4} + \tfrac{1}{4}x_5 \\ x_3 &= \tfrac{7}{2} - \tfrac{7}{2}x_5 \\ x_2 &= \text{FREE} \\ x_1 &= \tfrac{3}{2} - \tfrac{7}{2}x_5. \end{aligned}$$

EX. WHAT IF INSTEAD

$$\left(A_{rref} \mid \underline{b}_{rref} \right) = \left(\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & \boxed{1} & 6 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & \boxed{-\frac{1}{4}} \end{array} \right) \text{ ?}$$

$\leftarrow$ NONZERO.

NO PIVOT IN LAST ROW OF A_{rref}

$\Rightarrow$ NO SOLN. TO $A\underline{x} = \underline{b}$.

THM. [THM. 2.3–2.6 IN SADUN, SECTIONS 2.1–2.3 IN TREIL]

WITH $A = (\underline{v}_1, \ldots, \underline{v}_m) \in M_{n,m}$

(i) SOLN. TO $A\underline{x} = \underline{b}$ (IF IT EXISTS!) IS UNIQUE.

$\Leftrightarrow$ ONLY SOLN. TO $A\underline{x} = \underline{0}$ IS $\underline{x} = 0$.

$\Leftrightarrow$ PIVOT IN EVERY COLUMN OF A_{rref}

(I.E., NO FREE VARIABLES)

$\Leftrightarrow$ $\{\underline{v}_1, \ldots, \underline{v}_m\}$ LINEARLY INDEP. $(\Rightarrow m \leq n)$.

(ii) SOLN. TO $A\underline{x} = \underline{b}$ EXISTS FOR ANY $\underline{b} \in \mathbb{R}^n$

$\Leftrightarrow$ PIVOT IN EVERY ROW OF A_{rref}

$\Leftrightarrow$ $\{\underline{v}_1, \ldots, \underline{v}_m\}$ SPANS $\mathbb{R}^n$ $(\Rightarrow m \geq n)$

(iii) SOLN. TO $A\underline{x} = \underline{b}$ EXISTS AND IS UNIQUE FOR ANY $b \in \mathbb{R}^n$

$\Leftrightarrow$ PIVOT IN EVERY ROW AND COLUMN OF A_{rref}

$\Leftrightarrow$ $\{\underline{v}_1, \ldots, \underline{v}_m\}$ BASIS OF $\mathbb{R}^n$ $(\Rightarrow m = n)$.

IN ADDITION, (iii) EQUIVALENT TO

$\iff$ A IS INVERTIBLE

$\iff$ $\det(A) \neq 0$

$\iff$ $A_{rref} = I$.

<u>EX.</u> DO COLUMNS OF $\underbrace{\begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \\ 1 & -1 & 1 & 4 \end{pmatrix}}_{A}$ SPAN $\mathbb{R}^3$?

$$A \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & 1 & -1 \\ 0 & -2 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{-1} & 1 & -1 \\ 0 & 0 & \boxed{-2} & 3 \end{pmatrix}$$

<u>ref</u>. PIVOT IN EVERY ROW

$\Rightarrow$ $\boxed{\text{Yes.}}$.