LECTURE 04
01/25/12

REPRESENTATION OF VECTORS IN A BASIS (2.3):

$\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ BASIS OF V.

FOR ANY $\underline{v} \in V$, $\underline{v} = a_1 \underline{b}_1 + \cdots + a_n \underline{v}_n$ FOR SOME UNIQUE SET OF COEFFICIENTS $\{a_1, .., a_n\}$.

Q: WHY UNIQUE?

A: IF $\underline{v} = \sum a_i \underline{b}_i = \sum c_i b_i$ FOR SOME $\{a_i\}_{i=1}^{n}$, $\{c_i\}_{i=1}^{n}$,

$\sum (a_i - c_i) \underline{b}_i = \underline{0} \Rightarrow a_i = c_i$ FOR ALL $i = 1, .., n$

SINCE $\{\underline{b}_i\}$ LINEARLY INDEP.

Def. $$[\underline{v}]_{\mathcal{B}} = \underbrace{(a_1, .., a_n)}_{\text{COORDINATES OF } \underline{v} \text{ IN BASIS } \mathcal{B}.}^{T} \in \mathbb{R}^n$$

THM.

- $\underline{x}, \underline{y} \in V$, c SCALAR $\Rightarrow$ $[\underline{x} + \underline{y}]_{\mathcal{B}} = [\underline{x}]_{\mathcal{B}} + [\underline{y}]_{\mathcal{B}}$, $[c\underline{x}]_{\mathcal{B}} = c[\underline{x}]_{\mathcal{B}}$.
- $\underline{w}$ LINEAR COMBINATION OF $\{\underline{v}_1, \ldots, \underline{v}_m\}$ $\Leftrightarrow$ $[\underline{w}]_{\mathcal{B}}$ LINEAR COMBINATION OF $\{[\underline{v}_1]_{\mathcal{B}}, .., [\underline{v}_m]_{\mathcal{B}}\}$.
- $\{\underline{v}_i\}_{i=1}^{m}$ LINEARLY INDEP. $\Leftrightarrow$ $\{[v_i]_{\mathcal{B}}\}_{i=1}^{m}$ LINEARLY INDEP.

Pf. EASY.

THIS IMPLIES THAT $[\cdot]_{\mathcal{B}}$ IS AN <u>ISOMORPHISM</u> FROM V TO $\mathbb{R}^n$, AND SO, V IS <u>ISOMORPHIC</u> TO $\mathbb{R}^n$. ($V \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^n$, $\underline{b}_i \xrightarrow{[\cdot]_{\mathcal{B}}} \underline{e}_i$, $\underline{v} = \sum a_i b_i \xrightarrow{[\cdot]_{\mathcal{B}}} [\underline{v}]_{\mathcal{B}} = \sum a_i e_i$)

<u>EX.</u> CONSIDER SUBSPACE $\widetilde{\mathbb{R}}_3[t]$ OF $\mathbb{R}_3[t]$ CONSISTING OF POLYNOMIALS WITH $p(0) = 0$. ARE

$$p_1 = t - t^3, \quad p_2 = t^2 + 3t^3, \quad p_3 = 2t + 3t^2 + 4t^3$$

a) LINEARLY INDEP.?

b) SPAN $\widetilde{\mathbb{R}}_3[t]$?

c) BASIS OF $\widetilde{\mathbb{R}}_3[t]$?

<u>Ans:</u> USE A BASIS $\mathcal{B}$ OF $\widetilde{\mathbb{R}}_3[t]$ TO TRANSFORM $\widetilde{\mathbb{R}}_3[t]$ TO $\mathbb{R}^n$ FOR SOME APPROPRIATE n.

<u>STEP 1.</u> FIND A BASIS OF $\widetilde{\mathbb{R}}_3[t]$:

ANY ELEMENT IN $\widetilde{\mathbb{R}}_3[t]$ HAS FORM $p(t) = \overset{0}{\cancel{a_0}} + a_1 t + a_2 t^2 + a_3 t^3$

$\Rightarrow \mathcal{B} = \{t, t^2, t^3\}$ IS A BASIS.

<u>STEP 2.</u> WRITE p_1, p_2, p_3 IN COORDINATES OF $\mathcal{B}$:

$$[p_1]_{\mathcal{B}} = (1, 0, -1)^T, \quad [p_2]_{\mathcal{B}} = (0, 1, 3)^T, \quad [p_3]_{\mathcal{B}} = (2, 3, 4)^T.$$

<u>STEP 3.</u> ANALYZE MATRIX $\underbrace{([p_1]_{\mathcal{B}} \cdots [p_3]_{\mathcal{B}})}_{A.}$:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 3 & 4 \end{pmatrix}_{+R_1} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 3 & 6 \end{pmatrix}_{-3R_2}$$

$$\rightarrow \begin{pmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & \boxed{-3} \end{pmatrix}$$

ref. PIVOT IN EVERY ROW, COLUMN.

$\Longleftrightarrow$ $\{[p_i]_{\mathcal{B}}\}_{i=1}^{3}$ BASIS OF $\mathbb{R}^3$

$\Longleftrightarrow$ $\{p_i\}_{i=1}^{3}$ BASIS OF $\tilde{\mathbb{R}}_3[t]$ (BY THM.).

DEF. IF V ~~HAS~~ BASIS $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ OF n VECTORS, SAY $\dim(V)$ IS n (I.E., V IS n-DIMENSIONAL).

DIMENSION IS INDEPENDENT OF CHOICE OF BASIS!

WHY? IF $\mathcal{D} = \{\underline{d}_1, \dots, \underline{d}_m\} \subset V$,

$m < n \Rightarrow \mathcal{D}$ DOES NOT SPAN V

$m > n \Rightarrow \mathcal{D}$ LINEARLY DEPENDENT

SO, $\mathcal{D}$ BASIS $\Longrightarrow m = n$.

CHANGE OF BASIS (2.9):

$\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ BASES OF V

$\mathcal{D} = \{\underline{d}_1, \ldots, \underline{d}_n\}$

FIX $\underline{v} \in V$. HOW TO RELATE $[\underline{v}]_{\mathcal{B}}$ TO $[\underline{v}]_{\mathcal{D}}$?

CLAIM: THERE IS A UNIQUE $n \times n$ MATRIX $P_{\overset{\nearrow}{\mathcal{D}}\overset{\nwarrow}{\mathcal{B}}}$ TO HERE FROM HERE

SUCH THAT $[\underline{v}]_{\mathcal{D}} = P_{\mathcal{D}\mathcal{B}} [\underline{v}]_{\mathcal{B}}$

FOR ALL $\underline{v} \in V$.

Pf. $\underline{v} = \sum_{i=1}^{n} a_i \underline{b}_i \Rightarrow [\underline{v}]_{\mathcal{D}} = \left[\sum_{i=1}^{n} a_i \underline{b}_i\right]_{\mathcal{D}}$

$$= \sum_{i=1}^{n} a_i [\underline{b}_i]_{\mathcal{D}}$$

$$= \underbrace{\begin{pmatrix} [\underline{b}_1]_{\mathcal{D}} & \cdots & [\underline{b}_n]_{\mathcal{D}} \end{pmatrix}}_{n \times n \text{ MATRIX}} \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{\overset{\shortparallel}{[\underline{v}]_{\mathcal{B}}}}$$

$$= P_{\mathcal{D}\mathcal{B}} [\underline{v}]_{\mathcal{B}}.$$

TO CHECK THAT $P_{\mathcal{D}\mathcal{B}} = \begin{pmatrix} [\underline{b}_1]_{\mathcal{D}} & \cdots & [\underline{b}_n]_{\mathcal{D}} \end{pmatrix}$ IS THE ONLY MATRIX WHICH SATISFIES $[\underline{v}]_{\mathcal{D}} = P_{\mathcal{D}\mathcal{B}} [\underline{v}]_{\mathcal{B}}$, NOTE THAT IF $[\underline{v}]_{\mathcal{D}} = A [\underline{v}]_{\mathcal{B}}$ FOR ALL $\underline{v} \in V$, THEN $[\underline{b}_i]_{\mathcal{D}} = A [\underline{b}_i]_{\mathcal{B}} = A e_i$ = ith COLUMN OF A. ✓

PICTURE:

$$\begin{tikzcd}
V \arrow[r, "{[\cdot]_{\mathcal{B}}}"] \arrow[dr, "{[\cdot]_{\mathcal{D}}}"'] & \mathbb{R}^n \arrow[d, leftrightarrow, "P_{\mathcal{D}\mathcal{B}}"] \\
& \mathbb{R}^n \arrow[u, bend right=70, "P_{\mathcal{B}\mathcal{D}} = P_{\mathcal{D}\mathcal{B}}^{-1}"']
\end{tikzcd}$$

THM. $P_{\mathcal{B}\mathcal{D}} = P_{\mathcal{D}\mathcal{B}}^{-1}$.

PF. WANT $P_{\mathcal{B}\mathcal{D}} P_{\mathcal{D}\mathcal{B}} = I$. LET $\underline{y} \in \mathbb{R}^n$. THERE IS SOME $\underline{x} \in V$ S.T. $[\underline{x}]_{\mathcal{B}} = \underline{y}$. SO,

$$P_{\mathcal{B}\mathcal{D}} P_{\mathcal{D}\mathcal{B}} \underline{y} = P_{\mathcal{B}\mathcal{D}} \underbrace{P_{\mathcal{D}\mathcal{B}} [\underline{x}]_{\mathcal{B}}}_{[\underline{x}]_{\mathcal{D}}} = P_{\mathcal{B}\mathcal{D}} [\underline{x}]_{\mathcal{D}} = [\underline{x}]_{\mathcal{B}} = \underline{y}.$$

$$\Rightarrow \quad P_{\mathcal{B}\mathcal{D}} P_{\mathcal{D}\mathcal{B}} = I.$$

EX. $M_{2,2}$ = 2×2 MATRICES W/ REAL ENTRIES.

$$\mathcal{E} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

STANDARD BASIS.

SUPPOSE $\underline{v} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

a) WHAT IS $[\underline{v}]_{\mathcal{E}}$?

$$[\underline{v}]_{\mathcal{E}} = (1, 2, 4, 3)^T.$$

b) IF $\mathcal{D} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

ANOTHER BASIS OF $M_{2,2}$, WHAT IS $[\underline{v}]_{\mathcal{D}}$?

STEP. 1. COMPUTE $P_{\mathcal{E}\mathcal{D}} = \left([\underline{d}_1]_{\mathcal{E}} \cdots [\underline{d}_4]_{\mathcal{E}} \right)$

SINCE THIS IS EASY TO COMPUTE VERSUS $P_{\mathcal{D}\mathcal{E}}$.

THEN, $P_{\mathcal{E}\mathcal{D}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix}$.

STEP 2. INVERT $P_{\mathcal{E}\mathcal{D}}$ TO GET $P_{\mathcal{D}\mathcal{E}}$.

USUALLY, DO THIS BY COMPUTER. IN THIS CASE, EASY TO DO BY HAND TO GET THAT

$$P_{\mathcal{D}\mathcal{E}} = P_{\mathcal{E}\mathcal{D}}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}.$$

STEP 3.

$$[\underline{v}]_{\mathcal{D}} = P_{\mathcal{D}\mathcal{E}} [\underline{v}]_{\mathcal{E}} = (2, -1, 3, -1)^T.$$

EASY TO CHECK IN THIS EX. THAT THIS IS CORRECT!

$$2\underline{d}_1 - \underline{d}_2 + 3\underline{d}_3 - \underline{d}_4 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \underline{v} \quad ✓.$$

LECTURE 05
01/27/12

LINEAR TRANSFORMATIONS, OPERATORS (3.1):

DEF. $L : V \longrightarrow W$ LINEAR TRANSFORMATION (LINEAR OPERATOR IF $W = V$, LINEAR FUNCTIONAL IF $W = \mathbb{R}^n$)

IF FOR EVERY $\underline{v}_1, \underline{v}_2 \in V$, SCALAR c,

$$\begin{cases} L(\underline{v}_1 + \underline{v}_2) = L(\underline{v}_1) + L(\underline{v}_2) \\ L(c\underline{v}_1) = c\,L(\underline{v}_1) \end{cases}$$

EX. $V = W = \mathbb{R}$.

$L(x) = |x|$ IS NOT A LINEAR TRANSFORMATION

SINCE $L(-x) = |-x| = |x| = L(x) \neq -L(x)$.

EX. $V = \mathbb{R}^n$, $W = \mathbb{R}^m$

$L(\underline{v}) = A\underline{v}$ FOR $A \in M_{m,n}$.

IN FACT, EVERY LINEAR TRANSFORMATION FROM $\mathbb{R}^n$ TO $\mathbb{R}^m$ HAS THIS FORM. WHY?

IF $\underline{v} = (a_1, \ldots, a_n) \in \mathbb{R}^n$,

$$L(\underline{v}) = L\Big(\sum_{i=1}^{n} a_i \underline{e}_i\Big) = \sum_{i=1}^{n} a_i\, L(\underline{e}_i) = \overbrace{\Big(L(\underline{e}_1) \cdots L(\underline{e}_n)\Big)}^{m\times n \text{ MATRIX } A} \overset{``\underline{v}"}{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}$$

EX. V n-DIM. VECTOR SPACE WITH BASIS $\mathcal{B}$,

$W = \mathbb{R}^n$

$L(\underline{v}) = [\underline{v}]_{\mathcal{B}}$ LINEAR TRANSFORMATION.

EX. $V = \mathbb{R}_n[t]$, $W = \mathbb{R}_m[t]$

$L(p(t)) = \frac{dp}{dt}(t)$ IS A LINEAR TRANSFORMATION SO LONG AS $m \geq n-1$ (OTHERWISE, $L : V \not\rightarrow W$!).

EX. $V = C^\infty[0,1]$

$\nwarrow$ SPACE OF INFINITELY DIFFERENTIABLE FCN.'S. ON $[0,1]$.

$L : V \rightarrow V$ IS A LINEAR OPERATOR WHEN

$L(f(x)) = \frac{df}{dx}(x)$ (NOTATION: $L = \frac{d}{dx}$).

EX. $V = C^0[0,1]$, $W = \mathbb{R}$.

$L(f) = \int_0^1 f(s)\,ds$ IS A LINEAR FUNCTIONAL.

MATRIX REPRESENTATION OF LINEAR TRANSFORMATION (3.2)

Q: DO WE NEED TO SPECIFY $L(\underline{v})$ FOR EVERY $\underline{v} \in V$?

A: ONLY NEED $\{L(\underline{b}_i)\}_{i=1}^{n}$ FOR BASIS $\mathcal{B} = \{\underline{b}_i\}_{i=1}^{n}$ OF V, SINCE

$$L(\underline{v}) = L\left(\sum_{i=1}^{n} a_i \underline{b}_i\right) = \sum_{i=1}^{n} a_i L(\underline{b}_i).$$

Q: CAN WE WRITE THIS IN MATRIX FORM ?

A: ONLY AFTER WE'VE CHOSEN A BASIS $\mathcal{D} = \{\underline{d}_1, \dots, \underline{d}_m\}$ FOR W AND WRITTEN EVERYTHING IN COORDINATES !

$$\begin{aligned}
[L(\underline{v})]_{\mathcal{D}} &= \left[\sum_{i=1}^{n} a_i L(\underline{b}_i)\right]_{\mathcal{D}} \\
&= \sum_{i=1}^{n} a_i [L(\underline{b}_i)]_{\mathcal{D}} \\
&= \underbrace{\left([L(\underline{b}_1)]_{\mathcal{D}} \cdots [L(\underline{b}_n)]_{\mathcal{D}}\right)}_{m \times n \text{ MATRIX } [L]_{\mathcal{D}\mathcal{B}}} \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{[\underline{v}]_{\mathcal{B}}} . \\
&= [L]_{\mathcal{D}\mathcal{B}} [\underline{v}]_{\mathcal{B}} .
\end{aligned}$$

THIS MATRIX $[L]_{\mathcal{D}\mathcal{B}}$ DEPENDS ON CHOICE OF BASES $\mathcal{B}$ OF V AND $\mathcal{D}$ OF W !!

IN FACT, IT IS THE <u>UNIQUE</u> MATRIX REPRESENTATION W/ RESPECT TO THIS CHOICE OF BASES.

<u>PICTURE</u>:

$$\begin{tikzcd}
V \arrow[r, "L"] \arrow[d, "{[\cdot]_{\mathcal{B}}}"'] & W \arrow[d, "{[\cdot]_{\mathcal{D}}}"] \\
\mathbb{R}^n \arrow[r, "{[L]_{\mathcal{D}\mathcal{B}}}"'] & \mathbb{R}^m
\end{tikzcd}$$

<u>REMARKS</u>: (i) IF $V=W$, ONLY NEED ONE BASIS $\mathcal{B}$. IN THIS CASE, DENOTE $[L]_{\mathcal{B}\mathcal{B}}$ BY $[L]_{\mathcal{B}}$.

(ii) MOST OF THIS COURSE FOCUSES ON CHOICE OF $\mathcal{B}$ AND $\mathcal{D}$ SO THAT $[L]_{\mathcal{D}\mathcal{B}}$ LOOKS SIMPLE (I.E., IS DIAGONAL !).

Lecture 06
01/30/12

Last time: $[L(\underline{v})]_{\mathcal{D}} = [L]_{\mathcal{D}\mathcal{B}} [\underline{v}]_{\mathcal{B}}$

$\uparrow$ unique matrix representation of L in basis $\mathcal{B}$ of V, $\mathcal{D}$ of W.

$$[L]_{\mathcal{D}\mathcal{B}} \doteq \left([L(\underline{b}_1)]_{\mathcal{D}} \cdots [L(\underline{b}_n)]_{\mathcal{D}} \right).$$

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "{[\cdot]_{\mathcal{B}}}"'] & W \arrow[d, "{[\cdot]_{\mathcal{D}}}"] \\ \mathbb{R}^n \arrow[r, "{[L]_{\mathcal{D}\mathcal{B}}}"'] & \mathbb{R}^m \end{tikzcd}$$

- If $V=W$, only need one basis for both spaces, say $\mathcal{B}$. Will denote $[L]_{\mathcal{B}} \doteq [L]_{\mathcal{B}\mathcal{B}}$.
- Much of this course focuses on choice of basis $\mathcal{B}$ s.t. $[L]_{\mathcal{B}}$ looks as simple as possible (i.e, is diagonal!).

Ex. $V = W = \mathbb{R}_2[t]$, basis $\mathcal{B} = \{1, t, t^2\}$.

(i) $L(p) = \frac{dp}{dt}$ (i.e., $L = \frac{d}{dt}$) is linear operator.

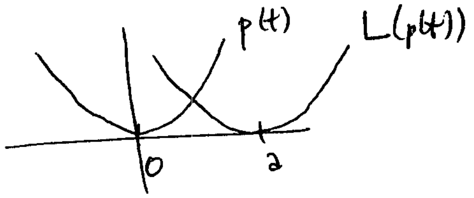
(ii) $[L]_{\mathcal{B}} = \left([L(\underline{b}_1)]_{\mathcal{B}} \; [L(\underline{b}_2)]_{\mathcal{B}} \; [L(\underline{b}_3)]_{\mathcal{B}} \right)$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

since
$L(\underline{b}_1) = 0 \Rightarrow [L(\underline{b}_1)]_{\mathcal{D}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
$L(\underline{b}_2) = 1 \Rightarrow [L(\underline{b}_2)]_{\mathcal{D}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
$L(\underline{b}_3) = 2t \Rightarrow [L(\underline{b}_3)]_{\mathcal{D}} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

EX. $V = W = \mathbb{R}_2[t]$, BASIS $\mathcal{B} = \{1, t, t^2\}$.

(i) $L(p(t)) = p(t-a)$ IS A LINEAR OPERATOR, CALLED THE SHIFT OPERATOR BY $a \in \mathbb{R}$.



(ii)
$$[L]_{\mathcal{B}} = \left([L(\underline{b}_1(t))]_{\mathcal{B}} \quad \cdots \quad [L(\underline{b}_3(t))]_{\mathcal{B}} \right)$$
$$= \begin{pmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{pmatrix} \quad \text{SINCE}$$

$$L(\underline{b}_1(t)) = 1 \Rightarrow [L(\underline{b}_1(t))]_{\mathcal{B}} = (1, 0, 0)^T$$
$$L(\underline{b}_2(t)) = t - a \Rightarrow [L(\underline{b}_2(t))]_{\mathcal{B}} = (-a, 1, 0)^T$$
$$L(\underline{b}_3(t)) = (t-a)^2 = t^2 - 2at + a^2 \Rightarrow [L(\underline{b}_3(t))]_{\mathcal{B}} = (a^2, -2a, 1)^T$$

(ii) WHAT IS $[L]_{\mathcal{D}\mathcal{B}}$, WHERE $\mathcal{D} = \{1, t-a, (t-a)^2\}$ IS ANOTHER BASIS OF $\mathbb{R}_2[t]$?

$$[L]_{\mathcal{D}\mathcal{B}} = \left([L(\underline{b}_1)]_{\mathcal{D}} \quad \cdots \quad [L(\underline{b}_3)]_{\mathcal{D}} \right)$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{SINCE} \quad \begin{aligned} [L(\underline{b}_1(t))]_{\mathcal{D}} &= (1, 0, 0)^T \\ [L(\underline{b}_2(t))]_{\mathcal{D}} &= (0, 1, 0)^T \\ [L(\underline{b}_3(t))]_{\mathcal{D}} &= (0, 0, 1)^T. \end{aligned}$$

NOTE: CHOICE OF BASIS $\mathcal{B}$ IN DOMAIN V OF L AND BASIS $\mathcal{D}$ IN IMAGE SPACE W OF L GIVES DIFFERENT REPRESENTATIONS $[L]_{\mathcal{D}\mathcal{B}}$!

EFFECT OF A CHANGE OF BASIS (3.3):

Q: $L : V \longrightarrow W$

$\mathcal{B}$ BASIS OF V
$\mathcal{B}'$ BASIS OF W. $\Rightarrow$ $[L]_{\mathcal{B}'\mathcal{B}}$ REPRESENTATION OF L.

WHAT IF WE SWITCH TO NEW BASES $\mathcal{D}$ OF V AND $\mathcal{D}'$ OF W? HOW DOES $[L]_{\mathcal{D}'\mathcal{D}}$ RELATE TO $[L]_{\mathcal{B}'\mathcal{B}}$?

A:

$$\begin{tikzcd}
& V \arrow[r, "L"] \arrow[dl, bend right, "{[\cdot]_{\mathcal{D}}}"'] \arrow[d, "{[\cdot]_{\mathcal{B}}}"'] & W \arrow[d, "{[\cdot]_{\mathcal{B}'}}"] \arrow[dr, bend left, "{[\cdot]_{\mathcal{D}'}}"] & \\
\mathbb{R}^n \arrow[r, "P_{\mathcal{B}\mathcal{D}}"'] & \mathbb{R}^n \arrow[r, "{[L]_{\mathcal{B}'\mathcal{B}}}"'] & \mathbb{R}^m \arrow[r, "P_{\mathcal{D}'\mathcal{B}'}"'] & \mathbb{R}^m
\end{tikzcd}$$

THEREFORE (SINCE WE MULTIPLY MATRICES FROM RIGHT TO LEFT),

$$[L]_{\mathcal{D}'\mathcal{D}} = P_{\mathcal{D}'\mathcal{B}'}[L]_{\mathcal{B}'\mathcal{B}}P_{\mathcal{B}\mathcal{D}}$$

WHERE $P_{\mathcal{B}\mathcal{D}}$ AND $P_{\mathcal{D}'\mathcal{B}'}$ ARE CHANGE OF BASIS MATRICES.

- IF $V=W$ AND WE USE COMMON BASIS $\mathcal{B}$, MATRIX REPRESENTATION OF $L : V \longrightarrow V$ IN NEW BASIS $\mathcal{D}$ IS

$$[L]_{\mathcal{D}} = P_{\mathcal{D}\mathcal{B}}[L]_{\mathcal{B}}P_{\mathcal{B}\mathcal{D}}$$

INVERSES OF EACH OTHER SINCE $P_{\mathcal{B}\mathcal{D}} = P_{\mathcal{D}\mathcal{B}}^{-1}$.

EX. SUPPOSE $V = W$ HAS BASIS $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$.

LET $L : V \to V$ S.T. $L(\underline{b}_1) = 2\underline{b}_1 + \underline{b}_2$

$L(\underline{b}_2) = \underline{b}_1 + 2\underline{b}_2$.

(i) $[L]_{\mathcal{B}} = \left([L(\underline{b}_1)]_{\mathcal{B}} \quad [L(\underline{b}_2)]_{\mathcal{B}} \right) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(ii) IF $\mathcal{D} = \{ \underset{\underline{d}_1}{\underset{\shortparallel}{\underline{b}_1 + \underline{b}_2}}, \underset{\underline{d}_2}{\underset{\shortparallel}{\underline{b}_1 - \underline{b}_2}} \}$, WHAT IS $[L]_{\mathcal{D}}$?

$$P_{\mathcal{B}\mathcal{D}} = \left([\underline{d}_1]_{\mathcal{B}} \quad [\underline{d}_2]_{\mathcal{B}} \right) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$P_{\mathcal{D}\mathcal{B}} = P_{\mathcal{B}\mathcal{D}}^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

$$\Rightarrow \quad [L]_{\mathcal{D}} = P_{\mathcal{D}\mathcal{B}} [L]_{\mathcal{B}} P_{\mathcal{B}\mathcal{D}} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

NOTE: THIS IS EXACTLY THE DECOUPLING OF SECTION 1!

FOR EX., $\frac{d\underline{x}}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \underline{x}$ BECAME $\frac{d\underline{y}}{dt} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \underline{y}$

AFTER MAKING THE CHANGE OF VARIABLES $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$.

LECTURE 07
02/01/12

FUNDAMENTAL SUBSPACES: KERNEL AND RANGE (3.5)

$L: V \to W$ LINEAR TRANSFORMATION

DEF.
- $\mathrm{Ker}(L) = \mathrm{Null}(L) := \{\underline{v} \in V : L(\underline{v}) = \underline{0}\}$.
- $\mathrm{Ran}(L) = \{\underline{w} \in W : L(\underline{v}) = \underline{w}$ FOR SOME $v \in V\}$.

NOTE:
- $\mathrm{Ker}(L)$ IS SOLUTION SET OF HOMOGENEOUS EQN.

$$L(\underline{v}) = \underline{0}.$$

- $\mathrm{Ran}(L)$ IS SET OF $\underline{w} \in W$ S.T. NONHOMOGENEOUS EQN. $L(\underline{v}) = \underline{w}$ HAS A SOLN. (I.E, $\underline{w}$ IS IN SPAN OF $\{L(\underline{b}_1), \dots, L(\underline{b}_n)\}$, WHERE $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ IS A BASIS OF V.)

CLAIM: $\mathrm{Ker}(L)$ IS A SUBSPACE OF V AND $\mathrm{Ran}(L)$ IS A SUBSPACE OF W.

WHY? (i) $\underline{x}, \underline{y} \in \mathrm{Ker}(L)$, c SCALAR $\Rightarrow$

$$L(\underline{x}+\underline{y}) = \underset{\underline{0}}{\underset{\shortparallel}{L(\underline{x})}} + \underset{\underline{0}}{\underset{\shortparallel}{L(\underline{y})}} = \underline{0}$$

SO, $\underline{x+y} \in \mathrm{Ker}(L)$

$$L(c\underline{x}) = cL(\underline{x}) = \underline{0}$$

SO, $c\underline{x} \in \mathrm{Ker}(L)$.

(ii) $x, y \in \mathrm{Ran}(L)$, c SCALAR $\Rightarrow$ THERE ARE $\underline{v}_1, \underline{v}_2 \in V$ S.T.

$$L(\underline{v}_1) = \underline{x}, \quad L(\underline{v}_2) = \underline{y}.$$

$\Rightarrow$ $L(\underline{v}_1+\underline{v}_2) = L(\underline{v}_1)+L(\underline{v}_2) = \underline{x}+\underline{y}$

SINCE $\underline{v}_1+\underline{v}_2 \in V$ SINCE V VECTOR SPACE,

$\underline{x}+\underline{y} \in Ran(L)$.

SIMILARLY, $c\underline{x} \in Ran(L)$.

$Ker(L)$ AND $Ran(L)$ CLOSED UNDER VECTOR ADDITION AND ~~SCALAR~~ MULTIPLICATION $\Rightarrow$ SUBSPACES.

THM. $Ker(L) = \{0\} \iff L$ IS 1-1 (I.E., $L(\underline{x}) = L(\underline{y}) \Rightarrow \underline{x}=\underline{y}$).

PF. "$\Rightarrow$" SUPPOSE $Ker(L) = \{0\}$. THEN, IF $L(\underline{x}) = L(\underline{y})$,

$L(\underline{x}-\underline{y}) = \underline{0} \Rightarrow \underline{x}-\underline{y} \in Ker(L) \Rightarrow \underline{x}-\underline{y} = \underline{0}$

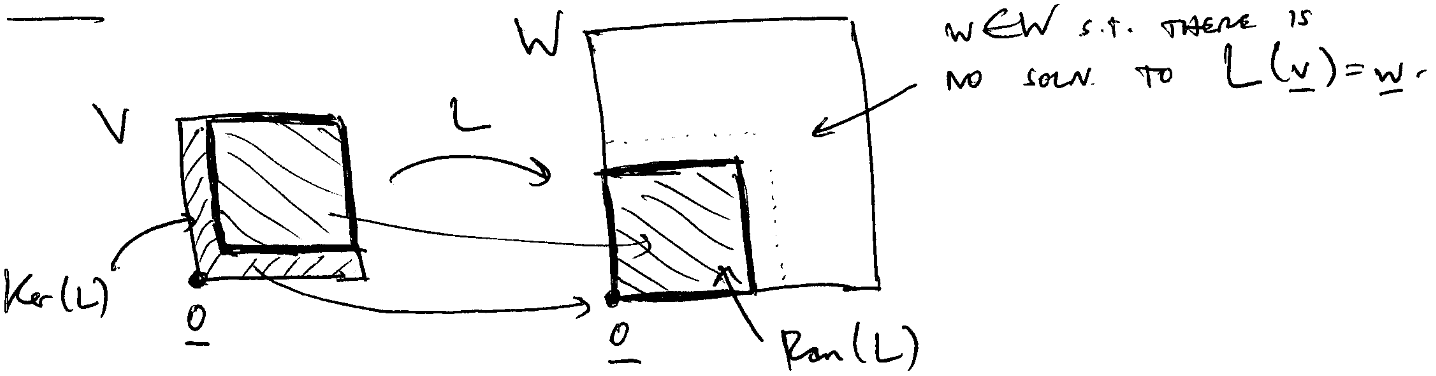
$\Rightarrow \underline{x}=\underline{y}$.

$\Rightarrow L$ IS 1-1.

"$\Leftarrow$" SUPPOSE L IS 1-1. SINCE $L(\underline{0}) = \underline{0}$, IF $L(\underline{v}) = \underline{0}$ FOR SOME $\underline{v} \in V$, THEN $L(\underline{v}) = L(\underline{0})$

$\Rightarrow \underline{v} = \underline{0} \Rightarrow Ker(L) = \{0\}$.

PICTURE:



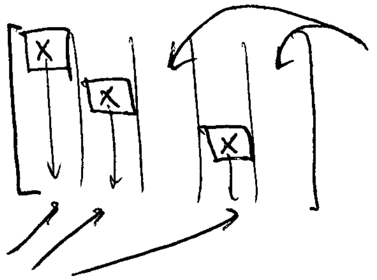
Q: HOW TO FIND BASIS OF $\operatorname{Ker}(L)$, $\operatorname{Ran}(L)$?

A: SUPPOSE $V = \mathbb{R}^n$, $W = \mathbb{R}^m$. THEN, ANY LINEAR TRANSFORMATION $L : V \to W$ TAKES THE FORM $L(\underline{v}) = A\underline{v}$ FOR SOME $A \in M_{m,n}$.

① $\operatorname{Ker}(A) = \{ x \in \mathbb{R}^n : A\underline{x} = \underline{0} \}$.

$$A\underline{x} = \underline{0} \iff A_{rref}\,\underline{x} = \underline{0}$$

FOR EXAMPLE, $A_{rref} =$



NO PIVOTS ⇒ FREE VARIABLES. x_3 AND x_5.

$$\Rightarrow \underline{x} = x_3\begin{pmatrix} \\ \\ \end{pmatrix} + x_5\begin{pmatrix} \\ \\ \end{pmatrix}$$

PIVOT COLUMNS

FORMS BASIS OF $\operatorname{Ker}(L)$.

LET $\{x_{i_j}\}$ BE FREE VARIABLES — THAT IS, $\{x_{i_1}, \dots, x_{i_k}\}$ CORRESPOND TO COLUMNS WITHOUT PIVOTS.

THEN SOLVING BY BACK SUBSTITUTION GIVES

$$\underline{x} = x_{i_1}\underline{v}_{i_1} + \dots + x_{i_k}\underline{v}_{i_k}.$$

$\Rightarrow \{\underline{v}_{i_1}, \dots, \underline{v}_{i_k}\}$ FORMS A BASIS OF $\operatorname{Ker}(L)$ AFTER DROPPING ANY LINEARLY DEPENDENT VECTORS.

② $\operatorname{Ran}(A)$ [redacted] $= \{ \underline{b} \in \mathbb{R}^m : \underline{b} \in \operatorname{span}(\text{COLUMNS OF } A) \}$.

THE COLUMN SPACE OF A HAS A BASIS GIVEN BY THE PIVOT COLUMNS OF THE ORIGINAL MATRIX A (NOT A_{rref}).

EX. $$A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 3 \end{pmatrix}$$

SINCE $$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 3 \end{pmatrix} \begin{matrix} \\ -2R_1 \\ -R_1-R_2 \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} +\frac{2}{3}R_2 \\ \times -\frac{1}{3} \\ \\ \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{rref} = \begin{pmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow x_2, x_4 \text{ FREE VARIABLES.}$$

PIVOT COLUMNS OF A_{rref}

① $$A\underline{x} = \underline{0} \iff A_{rref}\,\underline{x} = \underline{0}$$

$$\iff \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{matrix} x_1 = -x_2 \\ x_2 \text{ free} \\ x_3 = -x_4 \\ x_4 \text{ FREE} \end{matrix} \iff \underline{x} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

SO, $\{(-1,1,0,0)^T, (0,0,-1,1)^T\}$ IS A BASIS OF $Ker(A)$.

② $A\underline{x} = \underline{b} \iff \underline{b} \in$ span (columns of A)

SINCE THE PIVOT COLUMNS OF A ARE THE FIRST AND THIRD COLUMNS OF A,

$\{(1,2,3)^T, (2,1,3)^T\}$ IS A BASIS OF $Ran(A)$.

NOTE: IN GENERAL, IF $L : V \to W$

① $$\underline{L(\underline{v}) = \underline{0}} \iff [L(\underline{v})]_{\mathcal{D}} = \underline{0}$$

$$\iff \underbrace{[L]_{\mathcal{D}\mathcal{B}}}_{\text{MATRIX}} [\underline{v}]_{\mathcal{B}} = \underline{0}$$

THEREFORE, IF WE FIND A BASIS OF $\mathrm{Ker}([L]_{\mathcal{D}\mathcal{B}})$ THIS GIVES A BASIS FOR $\mathrm{Ker}(L)$ BY MAPPING BACK USING $[\cdot]_{\mathcal{B}}^{-1} : \mathbb{R}^n \to V$.

② $$L(\underline{v}) = \underline{w} \iff [L(\underline{v})]_{\mathcal{D}} = [\underline{w}]_{\mathcal{D}}$$

$$\iff \underbrace{[L]_{\mathcal{D}\mathcal{B}}}_{\text{MATRIX.}} [\underline{v}]_{\mathcal{B}} = [\underline{w}]_{\mathcal{D}}$$

SO, IF WE FIND A BASIS OF $\mathrm{Ran}([L]_{\mathcal{D}\mathcal{B}})$ THIS GIVES A BASIS FOR $\mathrm{Ran}(L)$ BY MAPPING BACK USING $[\cdot]_{\mathcal{D}}^{-1} : \mathbb{R}^m \to W$.

__EX.__ $V = M_{2,2}$, $W = \mathbb{R}_2[t]$. SUPPOSE $L : V \to W$ IS GIVEN BY

$$L\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = 1 + 2t + 3t^2$$
$$L\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = 1 + 2t + 3t^2$$
$$L\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = 2 + t + 3t^2$$
$$L\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = 2 + t + 3t^2$$

$$\Rightarrow [L]_{\mathcal{D}\mathcal{B}} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 3 \end{pmatrix}$$

WHERE $\mathcal{B} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$.
$\mathcal{D} = \{1, t, t^2\}$.

SINCE A BASIS OF $\mathrm{Ker}([L]_{\mathcal{D}\mathcal{B}})$ IS $\{(-1,1,0,0)^T, (0,0,-1,1)^T\}$,

$$\downarrow \; [\cdot]_{\mathcal{B}}^{-1} \; \downarrow$$

A BASIS OF $\mathrm{Ker}(L)$ IS $\left\{\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}\right\}$.

SINCE A BASIS OF $\mathrm{Ran}\left([L]_{\mathcal{D}\mathcal{B}}\right)$ IS $\left\{(1,2,3)^T, (2,1,3)^T\right\}$,

$$\swarrow \quad [\cdot]_{\mathcal{D}}^{-1} \quad \downarrow$$

A BASIS OF $\mathrm{Ran}(L)$ IS $\left\{1+2t+3t^2,\ 2+t+3t^2\right\}$.

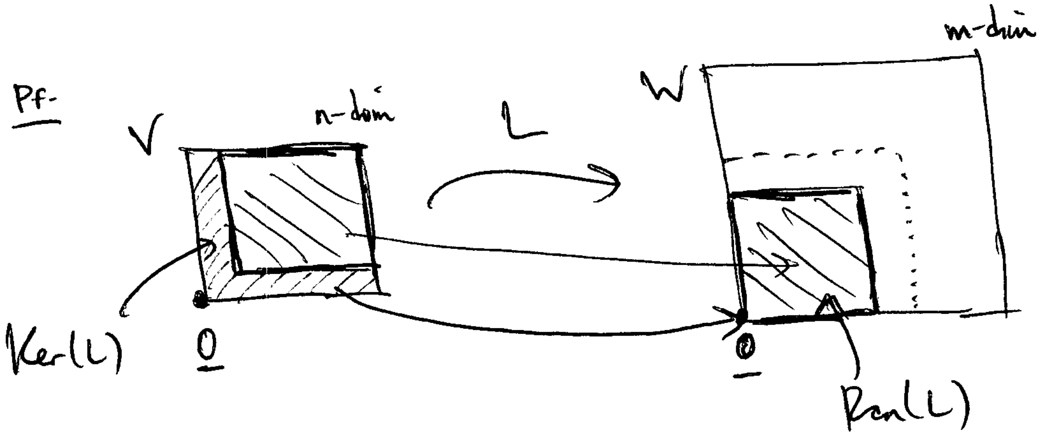
LECTURE 08
02/03/12

$L : V \to W$, $\mathcal{B}$ BASIS OF $V \simeq \mathbb{R}^n$
$\mathcal{D}$ BASIS OF $W \simeq \mathbb{R}^m$.

DEF. $\text{nullity}(L) \doteq \dim(\text{Ker}(L))$;
$\text{rank}(L) \doteq \dim(\text{Ran}(L))$.

THM. (RANK-NULLITY THM.)
$\text{rank}(L) + \text{nullity}(L) = \dim(V)$.

Pf.



$[L]_{\mathcal{D}\mathcal{B}} = m \times n$ MATRIX

$\Rightarrow$ rref OF $[L]_{\mathcal{D}\mathcal{B}}$ HAS k PIVOT COLUMNS AND $n-k$ NON-PIVOT COLUMNS.

$\Rightarrow$ $\text{rank}(L) = k$
$\text{nullity}(L) = n-k$
$\Rightarrow$ $\text{rank} + \text{nullity} = n = \dim(V)$.

#.

CONSEQUENCES:

$$L \text{ IS } 1\text{-}1 \iff \text{nullity}(L) = 0 \overset{\text{(RANK-NULLITY THM.)}}{\iff} \text{rank}(L) = n.$$

$$L \text{ IS ONTO} \iff \text{rank}(L) = m.$$

THEN SINCE

$$\text{rank}(L) = \#\text{ pivots IN } \text{rref}([L]_{\mathcal{D}\mathcal{B}}) \leq \min\{m, n\},$$

THM.

- $n < m \Rightarrow \text{rank}(L) < m \Rightarrow L$ IS NOT ONTO
- $n > m \Rightarrow \text{rank}(L) < n \Rightarrow L$ IS NOT 1-1
- $n = m \Rightarrow$ EITHER: (i) $\text{rank}(L) < n$ OR (ii) $\text{rank}(L) = n$ $\Rightarrow$ EITHER: (i) L NOT 1-1 NOR ONTO. OR (ii) L BOTH 1-1 AND ONTO.

THE LAST STATEMENT GIVES

THM. (FREDHOLM ALTERNATIVE)

$L : V \to V$ LINEAR OPERATOR. THEN, EITHER:

(i) $L(\underline{v}) = \underline{0}$ HAS NONTRIVIAL SOLN'S $\underline{v} \neq \underline{0}$.

OR (ii) $L(\underline{v}) = \underline{b}$ HAS SOLN. FOR ANY $\underline{b} \in V$.

EX. ON $\mathbb{R}_{n-1}[t]$, LET L BE THE OPERATOR

$$L(p(t)) = t\,p'(t) - p(t) \qquad \left(\text{I.E, } L = t\frac{d}{dt} - \text{Id.}\right)$$

Q: IS $t\,p'(t) - p(t) = q(t)$ SOLVABLE IN $\mathbb{R}_{n-1}[t]$ FOR ANY $q \in \mathbb{R}_{n-1}[t]$?

A: L IS A LINEAR OPERATOR SUCH THAT $L(t) = t\frac{dt}{dt} - t = \underline{0}$.

THEREFORE, SINCE $t \neq \underline{0}$ SOLVES $L(p(t)) = \underline{0}$ WE HAVE THAT $L(p(t)) = q(t)$ DOES NOT HAVE A SOLN. FOR EVERY q.

REMARK: REMEMBER, WE ARE LOOKING FOR SOLUTIONS $p \in \mathbb{R}_{n-1}[t]$, NOT SIMPLY ANY SOLN'S $p(t)$ TO THE DIFFERENTIAL EQN. $tp'(t) - p(t) = q(t)$! THAT IS, WE CAN'T LOOK FOR SOLUTIONS LIKE e^t, $\sin(kt)$, ETC.

LECTURE 09
02/06/12

EIGENVALUES AND EIGENVECTORS: DEFINITIONS AND NOTATION (4.1-4.2):

$L : V \longrightarrow V$ LINEAR OPERATOR.

DEF. A SCALAR λ IS AN EIGENVALUE OF L IF THERE IS SOME NONZERO $\underline{\xi} \in V$ S.T. $L(\underline{\xi}) = \lambda \underline{\xi}$. THEN, $\underline{\xi}$ IS CALLED AN EIGENVECTOR CORRESPONDING TO λ.

- SET OF ALL EIGENVALUES OF L IS CALLED THE SPECTRUM OF L, AND IS DENOTED $\sigma(L)$.
- EIGENSPACE CORRESPONDING TO λ IS $E_\lambda \doteq Ker(L - \lambda I)$.

$\uparrow$ IDENTITY OPERATOR

MOTIVATION: LET V VECTOR SPACE, WITH BASIS $\mathcal{B}$.

DISCRETE-TIME LINEAR EVOLUTION

$$\begin{cases} \underline{x}(k) = L\,\underline{x}(k-1), \quad k = 1, 2, \dots \\ \underline{x}(0) \text{ GIVEN} \end{cases}$$

$$\Downarrow$$

$$\underline{x}(k) = L(L(\cdots L(\underline{x}(0))))$$
$$= \underbrace{(L \circ L \circ \cdots \circ L)}_{k \text{ times}} \underline{x}(0)$$

$$\Downarrow$$

$$[\underline{x}(k)]_{\mathcal{B}} = [L \circ \cdots \circ L]_{\mathcal{B}} [\underline{x}(0)]_{\mathcal{B}}$$
$$= ([L]_{\mathcal{B}})^k [\underline{x}(0)]_{\mathcal{B}}$$

(CONTINUOUS-TIME EVOLUTION

$$\begin{cases} \dfrac{dx}{dt} = L\,\underline{x}(t) \\ \underline{x}(0) \text{ GIVEN} \end{cases}$$
)

$$\Downarrow$$

$$\left(\underline{x}(t) = e^{tL}\, \underline{x}(0) \right)$$

$\uparrow$ WE'LL EXPLAIN WHAT THIS MEANS LATER.

$$\leadsto$$

(IF $[L]_{\mathcal{B}}$ IS DIAGONAL,

$$[\underline{x}(t)]_{\mathcal{B}} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} [\underline{x}(0)]_{\mathcal{B}}$$
)

$$[L]_{\mathcal{B}} = \left([L(\underline{b}_1)]_{\mathcal{B}} \cdots [L(\underline{b}_n)]_{\mathcal{B}} \right)$$ SQUARE MATRIX.

NOTE: IF $A \in M_{n,n}$, WHAT IS A^k? IF k LARGE?

SIMPLEST WHEN $A = (d_1 \underline{e}_1 \cdots d_n \underline{e}_n) = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

SINCE THEN, $A^k = (d_1^k \underline{e}_1 \cdots d_n^k \underline{e}_n) = \begin{pmatrix} d_1^k & & 0 \\ & \ddots & \\ 0 & & d_n^k \end{pmatrix}$.

IDEA: IF $L(\underline{b}_i) = \lambda_i \underline{b}_i$ (I.E, λ_i EIGENVALUE w/ EIGENVECTOR $\underline{b}_i$)

THEN $[L(\underline{b}_i)]_{\mathcal{B}} = [\lambda_i \underline{b}_i]_{\mathcal{B}} = \lambda_i \underline{e}_i$, AND

$$[L]_{\mathcal{B}} = (\lambda_1 \underline{e}_1 \cdots \lambda_n \underline{e}_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

$$\Rightarrow [\underline{x}(k)]_{\mathcal{B}} = \left([L]_{\mathcal{B}} \right)^k [\underline{x}(0)]_{\mathcal{B}} = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix} [\underline{x}(0)]_{\mathcal{B}}$$

I.E., $$\underbrace{\begin{aligned} x_1(k) &= \lambda_1^k x_1(0) \\ &\vdots \\ x_n(k) &= \lambda_n^k x_n(0) \end{aligned}}_{\text{SIMPLE EXPRESSION FOR COORDINATES OF } \underline{x}(k) \text{ IN } \mathcal{B}!}$$ WHERE $[\underline{x}(k)]_{\mathcal{B}} = (x_1(k), \ldots, x_n(k))^T$.

Q: WHAT IF $L(\underline{d}_i) = \lambda_i \underline{d}_i$ ONLY IN SOME OTHER BASIS $\mathcal{D} = (\underline{d}_1, \ldots, \underline{d}_n)$?

$$[\underline{x}(k)]_{\mathcal{B}} = \left([L]_{\mathcal{B}} \right)^k [\underline{x}(0)]_{\mathcal{B}} = \left(P_{\mathcal{B}\mathcal{D}} [L]_{\mathcal{D}} P_{\mathcal{D}\mathcal{B}} \right)^k [\underline{x}(0)]_{\mathcal{B}}$$

CHANGE OF BASIS.

EX. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

EIGENVALUES: $3, \ 1$

$$\begin{tikzcd} 3 & 1 \\ (1,1)^T \arrow[u] & (1,-1)^T \arrow[u, bend left] \end{tikzcd}$$

CHECK: $A\underline{\xi} = \lambda \underline{\xi}$.

$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓

$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. ✓

EX. (∞-DIM) $V = \mathbb{R}[t]$, $L(p) = t\frac{dp}{dt}$.

EIGENVALUES: $\{k\}_{k=0}^{\infty}$

$$\begin{tikzcd} \{t^k\}_{k=0}^{\infty} \arrow[r, bend right] & \{k\}_{k=0}^{\infty} \end{tikzcd}$$

CHECK:

$L(t^k) = t\frac{d}{dt}(t^k) = t k t^{k-1}$
$= k t^k$. ✓

EX. (∞-DIM.) $V = C^{\infty}(\mathbb{R})$, $L(f) = \frac{d^2 f}{dx^2}$.

EIGENVALUES: $\{-k^2\}_{k=0}^{\infty}$, $\{+k^2\}_{k=0}^{\infty}$

$$\begin{tikzcd} \{-k^2\}_{k=0}^{\infty} & \{+k^2\}_{k=0}^{\infty} \\ \{\sin(kx)\}_{k=0}^{\infty}, \{\cos(kx)\}_{k=0}^{\infty} \arrow[u] & \{e^{-kx}\}_{k=0}^{\infty}, \{e^{+kx}\}_{k=0}^{\infty} \arrow[u, bend left] \end{tikzcd}$$

CHECK:

$L(\sin(kx)) = \frac{d^2}{dx^2}(\sin(kx)) = -k^2 \sin(kx)$ ✓

$L(\cos(kx)) = \frac{d^2}{dx^2}(\cos(kx)) = -k^2 \cos(kx)$ ✓

$L(e^{-kx}) = \frac{d^2}{dx^2}(e^{-kx}) = +k^2 e^{-kx}$ ✓

$L(e^{+kx}) = \frac{d^2}{dx^2}(e^{+kx}) = +k^2 e^{+kx}$ ✓.

CHARACTERISTIC POLYNOMIAL (4.3):

Q: HOW TO FIND EIGENVALUES OF MATRIX $A \in M_{n\times n}$?

A: $A\underline{x} = \lambda \underline{x} \iff (A-\lambda I)\underline{x} = 0$

$$\iff \operatorname{Ker}(A-\lambda I) \neq \{\underline{0}\}$$

$$\iff \det(A-\lambda I) = 0$$

DEF. $p_A(\lambda) \doteq \det(A-\lambda I)$ POLYNOMIAL OF DEGREE n. (WHY?)

- EIGENVALUES OF A $\iff$ ROOTS OF p_A.
  $(A\underline{\xi} = \lambda\underline{\xi})$ $\quad$ $(p_A(\lambda) = 0)$.

EX. FIND EIGENVALUES AND EIGENVECTORS OF $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

A: $A - \lambda I = \begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix}$

① $p_A(\lambda) = \det(A-\lambda I) = (3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix}$

$$= (3-\lambda)(\lambda^2 - 3\lambda - 4) + 2(2\lambda + 2) + 4(4\lambda + 4)$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8$$

$$= (\lambda - 8)(\lambda + 1)^2 = (\lambda - 8)(\lambda + 1)(\lambda + 1).$$

$\Rightarrow$ EIGENVALUES ARE -1 AND 8.

② SOLVE $(A-\lambda I)\underline{\xi} = \underline{0}$ TO GET EIGENVECTOR(S) CORRESPONDING TO λ.

- FOR $\lambda = -1$, $A - (-1)I = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} \boxed{1} & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow$ $\xi_1 = -\frac{1}{2}\xi_2 - \xi_3$
$\xi_2 =$ FREE
$\xi_3 =$ FREE

$\Rightarrow$ $\underline{\xi} = \xi_2 \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + \xi_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

THEREFORE, THE TWO EIGENVECTORS CORRESPONDING TO EIGENVALUE -1 ARE $(-\frac{1}{2}, 1, 0)^T$ AND $(-1, 0, 1)^T$. THE EIGENSPACE E_{-1} CONSISTS OF ALL LINEAR COMBINATIONS OF THESE VECTORS.

- FOR $\lambda = 8$, $A - 8I = \begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$

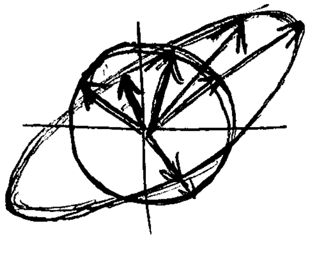
$$\Rightarrow \begin{array}{l} \xi_1 = \xi_3 \\ \xi_2 = \frac{1}{2}\xi_3 \\ \xi_3 = \text{FREE} \end{array} \Rightarrow \underline{\xi} = \xi_3 \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$

THEREFORE, EIGENVECTOR ASSOCIATED TO EIGENVALUE 8 IS $(1, \frac{1}{2}, 1)^T$. E_8 CONSISTS OF THE SPAN OF THIS VECTOR.

<u>NOTE</u>: IN THIS EXAMPLE, EIGENVALUE -1 HAD <u>TWO</u> LINEARLY INDEP. EIGENVECTORS.

<u>Q</u>: WHAT IS GEOMETRIC MEANING OF EIGENVALUES / EIGENVECTORS?

<u>A</u>: FOR EX, CONSIDER $A \in M_{2,2}$ S.T. EIGENVALUES ARE IN $\mathbb{R}$. THEN



- EIGENVECTORS CORRESPOND TO <u>DIRECTIONS</u> OF STRETCHING / COMPRESSION,
- EIGENVALUES MEASURE DEGREE OF STRETCHING $(+)$ OR COMPRESSION $(-)$.

- LATER, WE WILL SEE WHAT IMAGINARY EIGENVALUES MEAN.

LECTURE 10
02/08/12

Q: HOW TO FIND EIGENVALUES / EIGENVECTORS OF $L : V \to V$?

A: USE MATRIX REPRESENTATION. IF V HAS BASIS $\mathcal{D}$,

$$\begin{array}{c}\lambda \text{ EIGENVALUE OF } L \\ \text{w/ CORRESPONDING EIGENVECTOR } \underline{v}\end{array} \iff \begin{array}{c}\lambda \text{ EIGENVALUE OF } [L]_{\mathcal{D}} \\ \text{w/ CORRESPONDING EIGENVECTOR } [\underline{v}]_{\mathcal{D}}.\end{array}$$

PF.

$$L\underline{v} = \lambda\underline{v} \iff [L\underline{v}]_{\mathcal{D}} = [\lambda\underline{v}]_{\mathcal{D}} \iff [L]_{\mathcal{D}}[\underline{v}]_{\mathcal{D}} = \lambda[\underline{v}]_{\mathcal{D}}.$$

- THEREFORE, SPECTRUM $\sigma(L)$ OF L IS INDEPENDENT OF CHOICE OF BASIS ON V. THAT IS, IF $\mathcal{B}, \mathcal{D}$ BASES ON V,

$$[L]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{D}}\,[L]_{\mathcal{D}}\,P_{\mathcal{B}\mathcal{D}}^{-1}$$

HAVE SAME EIGENVALUES
(BUT DIFFERENT EIGENVECTORS).

THIS MOTIVATES THE FOLLOWING RESULT:

DEF. WE SAY A IS CONJUGATE (OR SIMILAR) TO B IF $A = PBP^{-1}$ FOR SOME P.

THM. IF A IS CONJUGATE TO B (I.E, $A = PBP^{-1}$),

$$\begin{array}{c}\lambda \text{ EIGENVALUE OF } B \\ \text{w/ EIGENVECTOR } \underline{v}\end{array} \iff \begin{array}{c}\lambda \text{ EIGENVALUE OF } A \\ \text{w/ EIGENVECTOR } P\underline{v}.\end{array}$$

PF. "$\Rightarrow$" $B\underline{v} = \lambda\underline{v} \Rightarrow A(P\underline{v}) = PB\cancel{P^{-1}P}\underline{v} = P\lambda\underline{v} = \lambda(P\underline{v})$. ✓

"$\Leftarrow$" $A(P\underline{v}) = \lambda(P\underline{v}) \Rightarrow B\underline{v} = P^{-1}AP\underline{v} = P^{-1}\lambda P\underline{v} = \lambda\underline{v}$. ✓

WE CAN USE THIS TO PROVE ONE OF THE MAIN RESULTS OF THE COURSE:

THM. (DIAGONALIZATION)

$A \in M_{n,n}$ HAS n LINEARLY INDEP. EIGENVECTORS $\mathcal{B} = \{\underline{\xi}_1, \dots, \underline{\xi}_n\}$ CORRESPONDING TO EIGENVALUES $\lambda_1, \dots, \lambda_n$

$$\iff \quad A = PDP^{-1} \quad \text{WITH} \quad D = \mathrm{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$P = \left(\underline{\xi}_1 \ \cdots \ \underline{\xi}_n \right)$$

I.E, ith COLUMN IS ith EIGENVECTOR

- MAIN CONSEQUENCE: $L : V \to V$ LINEAR OPERATOR IS DIAGONALIZABLE (I.E, CAN BE REPRESENTED AS A DIAGONAL MATRIX FOR SOME BASIS) $\iff$ EIGENVECTORS OF L FORM A BASIS OF V.

PF. OF THM.

„$\Rightarrow$" LET $\mathcal{E} = \{\underline{e}_1, \dots, \underline{e}_n\}$ BE THE STD. BASIS IN $\mathbb{R}^n$.

$$A = [A]_{\mathcal{E}} = P_{\mathcal{E}\mathcal{B}} \, [A]_{\mathcal{B}} \, P_{\mathcal{E}\mathcal{B}}^{-1} \quad \text{WHERE}$$

$$[A]_{\mathcal{B}} = \left([\underset{\overset{\shortparallel}{\lambda_1 \underline{\xi}_1}}{A\underline{\xi}_1}]_{\mathcal{B}} \ \cdots \ [\underset{\overset{\shortparallel}{\lambda_n \underline{\xi}_n}}{A\underline{\xi}_n}]_{\mathcal{B}} \right) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = D$$

$$P_{\mathcal{E}\mathcal{B}} = \left([\underline{\xi}_1]_{\mathcal{E}} \ \cdots \ [\underline{\xi}_n]_{\mathcal{E}} \right) = \left(\underline{\xi}_1 \ \cdots \ \underline{\xi}_n \right) = P.$$

"$\Leftarrow$" $D = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$ HAS EIGENVALUES $\lambda_1, \dots, \lambda_n$

W/ CORRESPONDING EIGENVECTORS $\{\underline{e}_i\}_{i=1}^{n}$. THEN BY PREVIOUS THM, $A = PDP^{-1}$ HAS SAME EIGENVALUES, BUT WITH CORRESPONDING EIGENVECTORS $\{P\underline{e}_i\}_{i=1}^{n}$. $P\underline{e}_i$ IS SIMPLY THE i^{th} COLUMN OF P. SINCE P IS INVERTIBLE BY HYPOTHESIS, $\{P\underline{e}_i\}_{i=1}^{n}$ ARE LINEARLY INDEPENDENT SINCE COLUMNS OF AN INVERTIBLE MATRIX MUST BE LINEARLY INDEP.

<u>Ex.</u> $A = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$.

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{vmatrix} = (2-\lambda)^2 - 1$$

$$= \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1)$$

$p_A(\lambda) = 0 \implies$ EIGENVALUES $\lambda = 3$, $\lambda = 1$.

<u>FOR $\lambda = 3$:</u> $A - \lambda I = \begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$

$\implies$ BASIS OF $\operatorname{Ker}(A - 3I)$ IS $\underline{x} = x_2(1,1)^T$.

<u>FOR $\lambda = 1$:</u> $A - \lambda I = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$

$\implies$ BASIS OF $\operatorname{Ker}(A - I)$ IS $\underline{x} = x_2(1,-1)^T$.

SINCE $(1,1)^T$ AND $(1,-1)^T$ ARE LINEARLY INDEP., WE MUST HAVE THAT

$$\underset{\underset{A}{\shortparallel}}{\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}} = \underset{\underset{P}{\shortparallel}}{\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}} \underset{\underset{D}{\shortparallel}}{\begin{pmatrix}\textcircled{3} & 0\\ 0 & \textcircled{1}\end{pmatrix}} \underset{\underset{P^{-1}}{\shortparallel}}{\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}^{-1}}.$$

LECTURE 11
02/10/12

LAST TIME WE SAW THAT $A \in M_{n,n}$ IS DIAGONALIZABLE (I.E, $A = PDP^{-1}$ FOR SOME DIAGONAL MATRIX D AND INVERTIBLE MATRIX P) $\Longleftrightarrow$ A HAS n LINEARLY INDEP. EIGENVECTORS.

- DOES THIS MEAN THAT ALL MATRICES ARE DIAGONALIZABLE, AND WHAT ISSUES MUST WE DEAL WITH IF WE WANT TO DIAGONALIZE A MATRIX ?

A FEW EXAMPLES:

EX. $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 2^2$$

$\Rightarrow$ $p_A(\lambda)$ HAS NO REAL ROOTS! $p_A(\lambda) = 0 \Rightarrow$

$\lambda = 1+2i$, $\lambda = 1-2i$ WITH $i = \sqrt{-1} \in \mathbb{C}$.

SO, NEED TO CONSIDER COMPLEX EIGENVALUES AND EIGENVECTORS TO HAVE ANY HOPE OF FINDING ENOUGH LINEARLY INDEP. EIGENVECTORS OF A $\Rightarrow$ $\boxed{\text{COMPLEXIFICATION}}$.

EX. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 \Rightarrow \lambda = 0$$

ONLY EIGENVALUE.

$A - 0I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\Rightarrow$ BASIS OF $\operatorname{Ker}(A - 0I)$ IS $(1, 0)^T$.

SO WE ONLY HAVE ONE EIGENVECTOR FOR A AND

$$A \neq P D P^{-1} \quad \text{FOR } D \text{ DIAGONAL} \; !$$

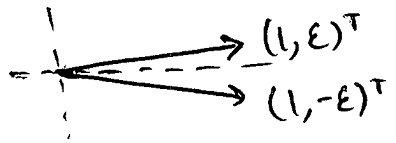
HOWEVER, NOTE THAT THIS PATHOLOGICAL CASE CAN BE RESOLVED IF WE SLIGHTLY PERTURB A:

$$A_\varepsilon \doteq \begin{pmatrix} 0 & 1 \\ \varepsilon^2 & 0 \end{pmatrix} \quad \text{WITH} \quad \varepsilon > 0 \quad \text{SMALL AS DESIRED.}$$

$$p_{A_\varepsilon}(\lambda) = \det(A_\varepsilon - \lambda I) = \lambda^2 - \varepsilon^2 = 0 \Rightarrow \lambda = \varepsilon, \; \lambda = -\varepsilon$$

(I.E, THE ZERO EIGENVALUE SPLIT INTO TWO DISTINCT EIGENVALUES)

$$\left\{ \begin{array}{l} \mathrm{Ker}(A_\varepsilon - \varepsilon I) \text{ HAS BASIS } (1, -\varepsilon)^T \\ \mathrm{Ker}(A_\varepsilon + \varepsilon I) \text{ HAS BASIS } (1, \varepsilon)^T \end{array} \right.$$



(I.E, THE EIGENVECTOR $(1,0)^T$ SPLIT INTO TWO DISTINCT EIGENVECTORS)

THEREFORE, $A_\varepsilon = P_\varepsilon D_\varepsilon P_\varepsilon^{-1}$ WITH

$$D_\varepsilon = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}$$

$$P_\varepsilon = \begin{pmatrix} 1 & 1 \\ -\varepsilon & \varepsilon \end{pmatrix}$$

AND THE ORIGINAL MATRIX A IS ALMOST DIAGONALIZABLE.

TO RECAP, IN THIS EXAMPLE THE EIGENVALUE $\lambda = 0$ WAS A DOUBLE ROOT OF $p_A(\lambda)$, BUT ITS ASSOCIATED EIGENSPACE E_0 HAD ONLY ONE DIMENSION. THIS WILL LEAD US TO A DISCUSSION ON MULTIPLICITY OF EIGENVALUES AND $\boxed{\text{JORDAN CANONICAL FORM}}$.