LECTURE 12
02/15/12

COMPLEXIFICATION (4.4):

$A \in M_{n,n}(\mathbb{R})$

$p_A(z) = \det(A - zI)$ DEGREE n POLYNOMIAL IN z
(USE COFACTOR EXPANSION TO SEE THIS)

PROBLEM: MAY NOT HAVE ENOUGH REAL ROOTS OF $p_A(z)$ TO FIND n EIGENVALUES OF A! TO REMEDY THIS, LOOK FOR COMPLEX ROOTS OF p_A.

THM. (FUNDAMENTAL THM. OF ALGEBRA)
ANY n DEGREE POLYNOMIAL (WITH COMPLEX COEFF.) HAS n (POSSIBLY REPEATED) ROOTS IN $\mathbb{C}$.

$$\Rightarrow \quad p_A(z) = c_0 + c_1 z + \dots + c_n z^n \qquad \left(\begin{array}{ll} c_i \in \mathbb{R} \text{ IF } A \in M_{n,n}(\mathbb{R}) \\ c_i \in \mathbb{C} \text{ IF } A \in M_{n,n}(\mathbb{C}) \end{array} \right).$$
$$= c_n(z-\lambda_1)(z-\lambda_2)\dots(z-\lambda_n)$$

WITH $\lambda_i \in \mathbb{C}$ POSSIBLY REPEATED. THESE ARE THE EIGENVALUES OF A. WHAT ARE THE CORRESPONDING EIGENVECTORS?

BY ALLOWING FOR COMPLEX EIGENVALUES, WE HAVE IMPLICITY MOVED FROM CONSIDERING THE ACTION OF THE OPERATOR $A \in M_{n,n}(\mathbb{R})$ ON THE REAL VECTOR SPACE $\mathbb{R}^n$ TO THE COMPLEX VECTOR SPACE $\mathbb{C}^n$. THIS IS KNOWN AS COMPLEXIFICATION.

REAL VECTOR SPACE		COMPLEX VECTOR SPACE
V	$\xrightarrow{\text{COMPLEXIFICATION}}$	$V_c \doteq \{\underline{v}+i\underline{w} : \underline{v}, \underline{w} \in V\}$
SCALARS $c \in \mathbb{R}$		SCALARS $c \in \mathbb{C}$.
BASIS $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$		BASIS $\mathcal{B}_c = \{\underline{b}_1 + i\underline{0}, \ldots, \underline{b}_n + i\underline{0}\}$.
$V \simeq \mathbb{R}^n$ (I.E, V ISOMORPHIC TO n-DIM <u>REAL</u> SPACE)		$V_c \simeq \mathbb{C}^n$ (I.E, V_c ISOMORPHIC TO n-DIM. <u>COMPLEX</u> SPACE) <u>NOTE:</u> $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ (I.E, $\mathbb{C}^n$ ISOMORPHIC TO n-DIM <u>REAL</u> SPACE).
$L : V \to V$ LINEAR	$\longrightarrow$	$L : V_c \to V_c$ GIVEN BY $L(\underline{v}+i\underline{w}) \doteq L(\underline{v}) + i\,L(\underline{w})$.

WITH THIS, WE CAN CONSIDER COMPLEX EIGENVALUES / EIGENVECTORS OF ANY $A \in M_{n,n}(\mathbb{C})$.

<u>DEF.</u> $\lambda \in \mathbb{C}$ EIGENVALUE W/ CORRESPONDING EIGENVECTOR $\underline{\xi} \in \mathbb{C}^n$ IF
$$A\underline{\xi} = \lambda \underline{\xi} \quad (\text{AND } \underline{\xi} \neq \underline{0}).$$

<u>THM.</u> IF $A \in M_{n,n}(\mathbb{R})$ (I.E, IF A IS A <u>REAL</u> MATRIX) COMPLEX EIGENVALUES / EIGENVECTORS ALWAYS COME IN COMPLEX CONJUGATE <u>PAIRS</u>. THAT IS,

$$\underbrace{\lambda = a+ib}_{\text{E-VALUE}} \text{ w/ } \underbrace{\underline{\xi} = \underline{v}+i\underline{w}}_{\text{E-VECTOR}} \iff \underbrace{\overline{\lambda} = a-ib}_{\text{E-VALUE}} \text{ w/ } \underbrace{\overline{\underline{\xi}} = \underline{v}-i\underline{w}}_{\text{E-VECTOR}}.$$

PF. $A\underline{\xi} = \lambda\underline{\xi} \iff \overline{A\underline{\xi}} = \overline{\lambda\underline{\xi}}$

$$\iff A\overline{\underline{\xi}} = \overline{\lambda}\,\overline{\underline{\xi}}$$

SINCE $\overline{A} = A$ (ALL ENTRIES OF A ARE REAL).

REMARK: IF A IS AN $n \times n$ REAL MATRIX AND $n \in \mathbb{N}$ IS ODD, A MUST HAVE AT LEAST ONE REAL EIGENVALUE!

WE CAN NOW EASILY FIND n EIGENVALUES AND THEIR CORRESPONDING EIGENSPACES AS BEFORE, EXCEPT WE CAN NOW USE MULTIPLICATION WITH COMPLEX-VALUED SCALARS WHEN PERFORMING ANY ROW REDUCTION OPERATIONS.

NOTE: IF $z = a + ib \in \mathbb{C}$, WHERE $a = \mathrm{Re}(z) \in \mathbb{R}$ AND $b = \mathrm{Im}(z) \in \mathbb{R}$ ARE THE REAL AND IMAGINARY PARTS OF z, THE MAGNITUDE OF z IS DEFINED BY

$$|z|^2 = z\overline{z} = (a+ib)(a-ib) = a^2 + b^2.$$

THIS IS USEFUL WHEN PERFORMING ROW OPERATIONS. INSTEAD OF MULTIPLYING BY $\frac{1}{z}$, SIMPLY MULTIPLY BY $\frac{\overline{z}}{|z|^2}$ IN ORDER TO GET THE SAME RESULT. FOR EXAMPLE,

$$\frac{1}{3+4i} = \frac{1}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{3-4i}{3^2+4^2} = \frac{1}{25}(3-4i).$$

EX. FIND EIGENVALUES AND CORRESPONDING EIGENSPACES OF

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}.$$

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & -2 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (3-\lambda)\Big[\underbrace{(1-\lambda)^2 + 2^2}\Big] = 0$$

IF THIS IS ZERO, THEN

$$(1-\lambda)^2 = -2^2$$

$$\Rightarrow 1-\lambda = \pm\sqrt{-2^2} = \pm 2i$$

$$\Rightarrow \lambda = 1 \pm 2i.$$

SO, THE EIGENVALUES OF A ARE $\lambda = 3$, $\underbrace{\lambda = 1+2i, \ \lambda = 1-2i}_{\text{COMPLEX CONJUGATE PAIR.}}$

$\underline{\underline{E_3}}$:
$$A - 3I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \underline{x} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \{(1,0,0)^T\} \text{ BASIS OF } E_3.$$

$\underline{\underline{E_{1+2i}}}$:
$$A - (1+2i)I = \begin{pmatrix} 2-2i & 0 & 0 \\ 0 & -2i & 2 \\ 0 & -2 & -2i \end{pmatrix} \begin{matrix} \times \frac{2+2i}{8} \\ \times \frac{2i}{4} \\ \times \frac{-1}{2} \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 1 & i \end{pmatrix}_{-R_2} \xrightarrow{\text{rref}} \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & i \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\Rightarrow \underline{\xi} = \xi_3 \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} = -i\xi_3 \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}.$$

$\Rightarrow \quad \{(0,1,i)^T\}$ BASIS FOR E_{1+2i}.

$\underline{\underline{E_{1-2i}}}$: NO NEED TO DO EXTRA WORK! SINCE COMPLEX EIGENVALUES / EIGENVECTORS OF __REAL__ MATRICES COME IN PAIRS, WE HAVE THAT $\overline{(0,1,i)^T} = (0,1,-i)^T$, SO

$\Rightarrow \quad \{(0,1,-i)^T\}$ BASIS FOR E_{1-2i}.

LECTURE 13
02/17/12

Q: WHAT IS THE MEANING OF COMPLEX EIGENVALUE/EIGENVECTORS PAIRS OBTAINED FOR A <u>REAL</u> MATRIX A?

A: <u>ROTATION</u> AND <u>SCALING</u> IN THE PLANE SPANNED BY THE REAL AND IMAGINARY PARTS OF THE COMPLEX EIGENVECTOR PAIR.

TO SEE THIS, SUPPOSE $\lambda = a + ib$, $b \neq 0$, IS AN EIGEN~~VALUE~~ OF A. THEN SO IS $\bar{\lambda} = a - ib$. THEIR CORRESPONDING EIGENVECTORS ARE $\underline{\xi} = \underline{v} + i\underline{w}$ AND $\overline{\underline{\xi}} = \underline{v} - i\underline{w}$ (NOTE THAT SINCE $b \neq 0$, $\underline{w} \neq \underline{0}$).

HOW DOES A ACT ON THE REAL VECTORS $\underline{v}$ AND $\underline{w}$?

$$\underline{v} = \frac{\underline{\xi} + \overline{\underline{\xi}}}{2} \Rightarrow A\underline{v} = A\left(\frac{\underline{\xi} + \overline{\underline{\xi}}}{2}\right) = \frac{\lambda\underline{\xi} + \bar{\lambda}\overline{\underline{\xi}}}{2}$$

$$= \left(\frac{\lambda + \bar{\lambda}}{2}\right)\underline{v} + i\left(\frac{\lambda - \bar{\lambda}}{2}\right)\underline{w}$$

$$= a\underline{v} - b\underline{w}.$$

$$\underline{w} = \frac{\underline{\xi} - \overline{\underline{\xi}}}{2i} \Rightarrow A\underline{w} = b\underline{v} + a\underline{w}$$

BY SIMILAR COMPUTATION.

ASSUMING $\underline{z}$ AND $\overline{\underline{z}}$ ARE LINEARLY INDEPENDENT SINCE THEY CORRESPOND TO DIFFERENT EIGENVALUES (WE WILL PROVE THIS LATER), $\underline{v}$ AND $\underline{w}$ ARE LINEARLY INDEPENDENT AND SPAN A PLANE

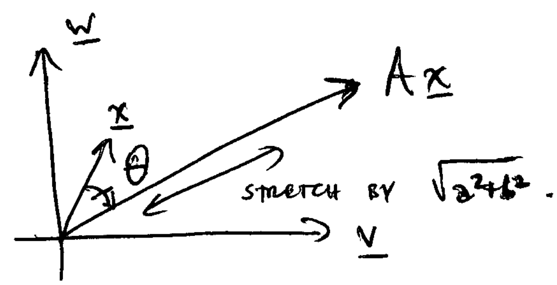
$$P = \text{span}\{\underline{v}, \underline{w}\} \quad \text{IN} \quad \mathbb{R}^n.$$

WE HAVE JUST SHOWN THAT $A : P \to P$, AND IS REPRESENTED IN THE BASIS $\mathcal{D} = \{\underline{v}, \underline{w}\}$ OF P BY

$$[A]_{\mathcal{D}} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \leftarrow \text{THIS IS A ROTATION MATRIX IN } \mathbb{R}^2 \text{ !}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = [A\underline{v}]_{\mathcal{D}}, \quad \begin{pmatrix} -b \\ a \end{pmatrix} = [A\underline{w}]_{\mathcal{D}}$$

THAT IS, IN PLANE P, A ROTATES VECTORS BY ANGLE $\theta = \arctan\left(-\frac{b}{a}\right)$ AND STRETCHES THEM BY A FACTOR OF $|\lambda| = \sqrt{a^2+b^2}$:



EX. IN PREVIOUS LECTURE, WE FOUND THAT

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}$$

HAS EIGENVALUES / EIGENSPACES GIVEN BY

$$\lambda = 3, \quad \lambda = 1+2i, \quad \lambda = 1-2i$$

$$\{(1,0,0)^T\} \quad \underbrace{\{(0,1,i)^T\} \quad \{(0,1,-i)^T\}}_{\text{COMPLEX CONJUGATE PAIR.}}$$

- THE REAL EIGENVALUE 3 W/ EIGENVECTOR $(1,0,0)^T$ IMPLIES THAT THE ACTION OF A STRETCHES ALL VECTORS IN $\mathbb{R}^3$ IN THE x-DIRECTION BY A FACTOR OF 3.

- $\lambda = 1 \pm 2i$ W/ CORRESPONDING EIGENVECTORS

$$\underline{\xi} = (0,1,\pm i)^T \implies a = 1,\ b = 2$$

$$= (0,1,0)^T \qquad \underline{v} = (0,1,0)^T,\ \underline{w} = (0,0,1)^T.$$

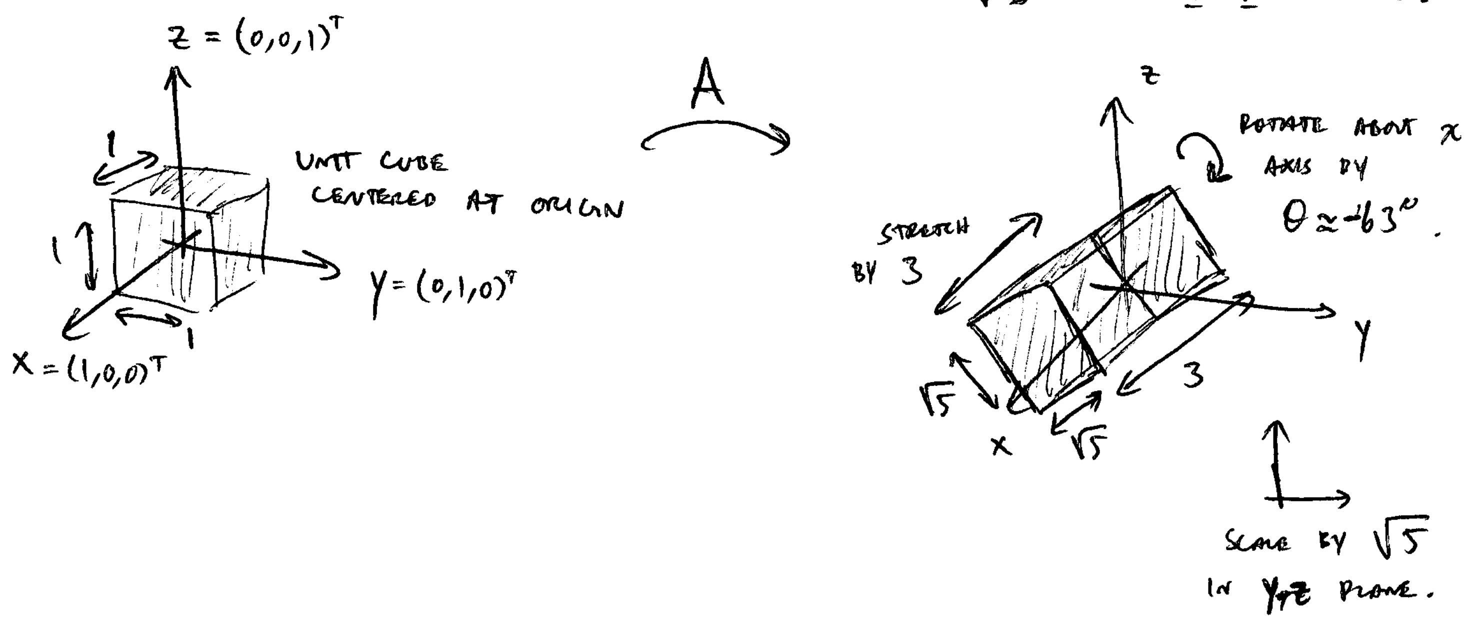
$$\pm i(0,0,1)^T$$

$$\implies \theta = \arctan\left(-\frac{b}{a}\right) = \arctan(-2)$$

$$\approx -63^\circ.$$

$$|\lambda| = \sqrt{a^2+b^2} = \sqrt{5}.$$

$\implies$ ROTATE BY $\theta \approx -63^\circ$ IN $\underline{v}$-$\underline{w}$ PLANE (I.E., y-z PLANE) AND SCALE BY $\sqrt{5}$ IN $\underline{v}$-$\underline{w}$ PLANE.



BY ALLOWING FOR COMPLEX EIGENVALUES, WE HAVE NOW GIVEN OURSELVES A CHANCE OF FINDING ENOUGH EIGENVECTORS TO FORM A BASIS OF $\mathbb{R}^n$ (IN ORDER TO DIAGONALIZE A). BUT WHAT IF WE STILL CAN'T FIND ENOUGH LINEARLY INDEP. EIGENVECTORS ?

MULTIPLICITY OF EIGENVALUES, DIAGONALIZABILITY (4.5):

$$p_A(z) = c_n (z-\tilde{\lambda}_1)(z-\tilde{\lambda}_2)\cdots(z-\tilde{\lambda}_n), \quad \tilde{\lambda}_i \text{ POSSIBLY REPEATED}$$

$$= c_n (z-\lambda_1)^{m_1}(z-\lambda_2)^{m_2}\cdots(z-\lambda_r)^{m_r}, \quad \lambda_1,\dots,\lambda_r \in \mathbb{C}$$

NOT REPEATED.

NOTE THAT $m_1+\cdots+m_r = n$.

EX. IF $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

$$p_A(z) = (\lambda-1)(\lambda-1) \qquad (\text{I.E., } \tilde{\lambda}_1 = 1, \tilde{\lambda}_2 = 1)$$

$$= (\lambda-1)^2 \qquad (\text{I.E., } \lambda_1 = 1, \text{ WITH } m_1 = 2).$$

DEF • $m_i \geq 1$, $i=1,\dots,r$, ARE THE ALGEBRAIC MULTIPLICITIES OF THE EIGENVALUES λ_i, $i=1,\dots,r$.

• $M_i \doteq \dim(\mathrm{Ker}(A-\lambda_i I)) \geq 1$, $i=1,\dots,r$, ARE THE GEOMETRIC MULTIPLICITIES OF THE λ_i, $i=1,\dots,r$.

NOTE: IN SADUN, m_i IS DENOTED $m_a(\lambda_i)$ AND M_i IS DENOTED $m_g(\lambda_i)$.

NOTE THAT M_i IS SIMPLY THE DIMENSION OF THE EIGENSPACE E_{λ_i} AND TELLS US HOW MANY LINEARLY INDEP. EIGENVECTORS WE CAN GET FROM EIGENVALUE λ_i.

THM. $1 \leq M_i \leq m_i$ FOR ALL $i=1,\dots,r$

(I.E., GEOMETRIC MULTIPLICITY $\leq$ ALGEBRAIC MULTIPLICITY).

PF. LET λ BE AN EIGENVALUE OF $A \in M_{n,n}$ w/ ALG. MULT. m AND GEOM. MULT. M. SUPPOSE E_λ HAS BASIS $\{\underline{b}_1,\dots,\underline{b}_M\}$.

THEN, WE CAN FIND A BASIS OF $\mathbb{R}^n$ GIVEN BY

$$\mathcal{B} = \{\underbrace{\underline{b}_1,\dots,\underline{b}_M}_{\text{IN } E_\lambda}, \underline{b}_{M+1},\dots,\underline{b}_n\}.$$

WHAT IS $[A]_{\mathcal{B}}$?

$$[A]_{\mathcal{B}} = \left([A\underline{b}_1]_{\mathcal{B}} \cdots [A\underline{b}_M]_{\mathcal{B}}\ [A\underline{b}_{M+1}]_{\mathcal{B}} \cdots [A\underline{b}_n]_{\mathcal{B}}\right)$$

$$= \left(\lambda\underline{e}_1 \cdots \lambda\underline{e}_M\ [A\underline{b}_{M+1}]_{\mathcal{B}} \cdots [A\underline{b}_n]_{\mathcal{B}}\right)$$

$$= \begin{array}{c} \\ M \\ n-M \end{array} \begin{array}{c} \overbrace{\qquad}^{M} \quad \overbrace{\qquad}^{n-M} \\ \left(\begin{array}{c|c} \lambda I & \tilde{B} \\ \hline O & B \end{array}\right) \end{array}$$

FOR SOME MATRICES

$B \in M_{n-M,n-M}$

$\tilde{B} \in M_{M,n-M}$.

THEREFORE,

$$p_A(z) = p_{[A]_\beta}(z) = \det\left([A]_\beta - zI\right)$$

E-VALUES INDEP. OF BASIS

$$= \begin{vmatrix} \begin{matrix} \lambda - z & & \\ & \ddots & \\ & & \lambda - z \end{matrix} & \tilde{B} \\ 0 & B - zI \end{vmatrix}$$

$$= \underbrace{(\lambda - z)^M}_{(-1)^M (z-\lambda)^M} \underbrace{\det(B - zI)}_{n-M \text{ DEGREE POLYNOMIAL}}$$

$\Rightarrow$ m HAS TO BE AT LEAST M!

(SINCE OTHER FACTORS OF $(z-\lambda)$ MAY BY IN $\det(B - zI)$.)

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FURTHERMORE, WE HAVE:

THM. IF $\lambda_1, \ldots, \lambda_r$ ARE A DISTINCT EIGENVALUES W/ CORRESPONDING EIGENVECTORS $\underline{\xi}_1, \ldots, \underline{\xi}_r$, THEN $\{\underline{\xi}_1, \ldots, \underline{\xi}_r\}$ ARE LINEARLY INDEPENDENT.

PF. (BY INDUCTION).

SUPPOSE $\{\underline{\xi}_1, \ldots, \underline{\xi}_{k-1}\}$ LIN. INDEP. FOR SOME $k \le r$.

THEN $a_1 \underline{\xi}_1 + \cdots + a_{k-1} \underline{\xi}_{k-1} = \underline{0} \Rightarrow a_1 = \cdots = a_{k-1} = 0$.

NOW SUPPOSE $c_1 \underline{\xi}_1 + \cdots + c_{k-1}\underline{\xi}_{k-1} + c_k \underline{\xi}_k = \underline{0}$

FOR SOME $\{c_i\}_{i=1}^k$. WE WOULD LIKE TO SHOW

THAT $c_i = 0$ FOR ALL i. TO DO THIS, APPLY

$A - \lambda_k I$ TO BOTH SIDES TO GET

$$\begin{aligned}
\underline{0} &= (A - \lambda_k I)(c_1 \underline{\xi}_1 + \cdots + c_{k-1}\underline{\xi}_{k-1} + c_k \underline{\xi}_k) \\
&= c_1(\lambda_1 - \lambda_k)\xi_1 + \cdots + c_{k-1}(\lambda_1 - \lambda_k)\xi_{k-1} + \underline{0}. \\
&= \sum_{i=1}^{k-1} \underbrace{c_i(\lambda_i - \lambda_k)}_{=0 \text{ BY LINEAR INDEP. OF } \xi_1,\ldots,\xi_{k-1}} \xi_i .
\end{aligned}$$

$\Rightarrow \; c_i(\lambda_i - \lambda_k) = 0$ FOR $i = 1, \ldots, k-1$

$\Rightarrow \; c_i = 0$ FOR $i = 1, \ldots, k-1$ (SINCE $\lambda_i \neq \lambda_k$)

$\Rightarrow \; c_k = 0$ (SINCE $c_k \underline{\xi}_k = \underline{0} \Rightarrow c_k = 0$).

SO, $\{\xi_i\}_{i=1}^k$ ARE LINEARLY INDEP.

SINCE TRIVIALLY TRUE FOR $k=1$, WE ARE DONE.

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<u>COROLLARY</u>: IF $\lambda_1, \ldots, \lambda_r$ ARE <u>DISTINCT</u> EIGENVALUES, THEIR CORRESPONDING EIGENSPACES $E_{\lambda_1}, \ldots, E_{\lambda_r}$ ARE LINEARLY INDEP.

<u>PF.</u>

$$\underline{0} = \underbrace{\left(a_1^{(1)} \underline{b}_1^{(1)} + \cdots + a_{M_1}^{(1)} \underline{b}_{M_1}^{(1)}\right)}_{\doteq d_1 \in E_{\lambda_1}} + \cdots + \underbrace{\left(a_1^{(r)} \underline{b}_1^{(r)} + \cdots + a_{M_r}^{(r)} \underline{b}_{M_r}^{(r)}\right)}_{\doteq d_r \in E_{\lambda_r}}$$

$\Rightarrow \; \underline{0} = \underline{d}_1 + \cdots + \underline{d}_r \Rightarrow \underline{d}_i = \underline{0}$ FOR ALL $i = 1, \ldots, r$ (BY PREVIOUS THM.)

$\Rightarrow \; a_1^{(i)} = \cdots = a_{M_i}^{(i)} = 0$ FOR ALL $i = 1, \ldots, r$.

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LECTURE 14
02/20/12

MULTIPLICITY OF EIGENVALUES, DIAGONALIZABILITY (CONT'D):

TO RECAP: $A \in M_{n,n}$ HAS DISTINCT EIGENVALUES $\lambda_1, \dots, \lambda_r \in \mathbb{C}$.

$$(\lambda_1, E_{\lambda_1}), \quad \dots, \quad (\lambda_r, E_{\lambda_r})$$

$$\begin{array}{cccc} (z-\lambda_1)^{m_1} \downarrow & \downarrow \dim E_{\lambda_1} & (z-\lambda_r)^{m_r} \downarrow & \downarrow \dim E_{\lambda_r} \\ m_1 & M_1 & m_r & M_r \end{array}$$

- m_i ALGEBRAIC MULTIPLICITY OF λ_i
- M_i GEOMETRIC MULTIPLICITY OF λ_i.

LAST TIME, WE SHOWED THAT

(i) $1 \leq M_i \leq m_i$ FOR ALL $i = 1, \dots, r$

(ii) $E_{\lambda_1}, \dots, E_{\lambda_r}$ ARE DISTINCT SUBSPACES IN THAT THEY ARE LINEARLY INDEP.

THESE IMMEDIATELY IMPLY:

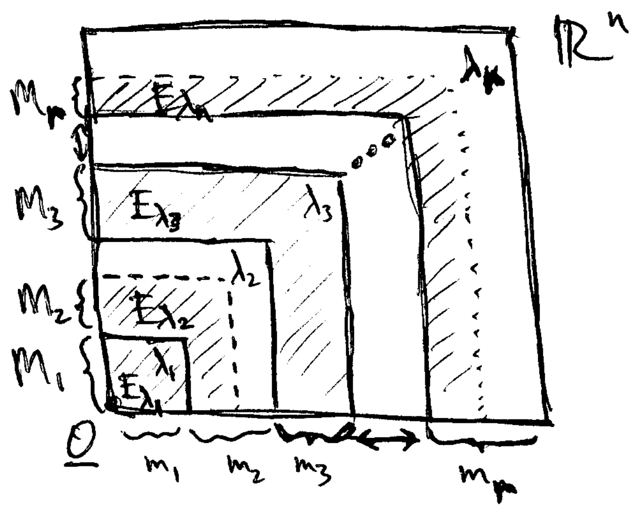
THM. (DIAGONALIZABILITY)

$A \in M_{n,n}$ DIAGONALIZABLE, I.E, CAN FIND n LINEARLY INDEP. EIGENVECTORS
$(A = PDP^{-1}$ FOR SOME D DIAGONAL$)$

$\iff M_i = m_i$ FOR ALL $i = 1, \dots, r$.

- IN PARTICULAR, IF A HAS n DISTINCT EIGENVALUES (I.E, $r=n$ AND $m_1 = m_2 = \cdots = m_n = 1$) THEN $M_1 = M_2 = \cdots = 1$ AND A IS DIAGONALIZABLE.

TO REMEMBER ALL OF THIS IN A PICTURE, IMAGINE THAT EACH DISTINCT EIGENVALUE λ_i "RESERVES" A PART OF $\mathbb{R}^n$ FOR ITSELF, AND ITS EIGENSPACE E_{λ_i} MUST SIT INSIDE THIS RESERVED SPACE (AND TAKES UP ALL OF IT IF THE GEOMETRIC MULT. OF λ_i MATCHES ITS ALGEBRAIC MULT.). THAT IS,



$\lambda_1, .., \lambda_r \in \mathbb{C}$ DISTINCT EIGENVALUES.

A DIAGONALIZABLE $\Longleftrightarrow$ $M_i = m_i$ FOR ALL i.

Q: WHAT IF $M_i < m_i$ FOR SOME i?

A: A IS NOT DIAGONALIZABLE, BUT IS <u>ALMOST</u> DIAGONALIZABLE.

Jordan canonical form (4.9):

If A is diagonalizable, then $A = PDP^{-1}$, where

$$D = \begin{pmatrix} \underbrace{\lambda_1 \cdots \lambda_1}_{M_1 = m_1} & & & \\ & \underbrace{\lambda_2 \cdots \lambda_2}_{M_2 = m_2} & & \\ & & \ddots & \\ & & & \underbrace{\lambda_r \cdots \lambda_r}_{M_r = m_r} \end{pmatrix} \; n \times n,$$

$$P = \left(\overbrace{\underbrace{\text{Eigenvectors of } \lambda_1}_{M_1}}^{E_{\lambda_1}} \;\middle|\; \overbrace{\underbrace{\text{Eigenvectors of } \lambda_2}_{M_2}}^{E_{\lambda_2}} \;\middle|\; \cdots \;\middle|\; \overbrace{\underbrace{\text{Eigenvectors of } \lambda_r}_{M_r}}^{E_{\lambda_r}} \right).$$

<u>Q</u>: If $M_i \neq m_i$ for all i, we are "missing" eigenvectors. Can we substitute these with a more general notion of eigenvector?

<u>A</u>: Yes. This leads to <u>generalized eigenvectors</u> (power vectors).

Suppose eigenvalue λ of A has alg. mult. m, geom. mult. M.

- $E_\lambda = \mathrm{Ker}(A - \lambda I)$ has basis $\mathcal{B}_\lambda = \{\underline{b}_1, \dots, \underline{b}_M\}$ and dimension $M \leq m$.

  ↳ eigenspace associated to λ

- $\tilde{E}_\lambda = \mathrm{Ker}((A - \lambda I)^m)$ has basis $\tilde{\mathcal{B}}_\lambda = \{\underline{b}_1, \dots, \underline{b}_M, \underbrace{\underline{\xi}_1, \dots, \xi_{m-M}}_{\text{generalized eigenvectors}}\}$ and dimension $\underline{\underline{m}}$.

  ↳ generalized eigenspace associated to λ.

- NOTE THAT $E_\lambda \subseteq \tilde{E}_\lambda$ SINCE IF
$$(A-\lambda I)\underline{v} = \underline{0} \quad \text{THEN} \quad (A-\lambda I)^m \underline{v} = 0 .$$

- IT CAN BE SHOWN THAT IF $(A-\lambda I)^p \underline{v} = \underline{0}$ FOR <u>ANY</u> $p \in \mathbb{N}$, THEN $\underline{v} \in \tilde{E}_\lambda$.

- $\underline{v}$ IS CALLED A <u>GENERALIZED EIGENVECTOR</u> (POWER VECTOR) IF $(A-\lambda I)^p \underline{v} = 0$ FOR SOME p. THE MINIMUM VALUE OF p SUCH THAT THIS IS TRUE IS CALLED THE <u>ORDER</u> OF $\underline{v}$. NOTE THAT IF $\underline{v}$ IS OF ORDER $p \geq 1$, THEN $\underline{w} = (A-\lambda I)\underline{v}$ IS OF ORDER $p-1$.

HOW TO FIND $\tilde{\mathcal{B}}_\lambda$? <u>DEFINE</u> ξ_i BY $(A-\lambda I)\underline{\xi}_i = \underline{\xi}_{i-1}$, $i=1,\dots$ WHERE $\underline{\xi}_0 \doteq \underline{b}_M$.

$(A-\lambda I)\xi_1 = b_M$ ORDER 3

$$\tilde{\mathcal{B}}_\lambda = \{ \underbrace{\underline{b}_1, \dots, \underline{b}_M}_{\substack{\text{ORDER } 1 \\ \text{(EIGENVECTORS)}}}, \underline{\xi}_1, \underline{\xi}_2, \dots, \underline{\xi}_{m-M} \}$$

$(A-\lambda I)\underline{\xi}_2 = \underline{\xi}_1$

ORDER 2 ORDER $m-M+1$.

THEN, SINCE $(A-\lambda I)\underline{\xi}_i = \underline{\xi}_{i-1} \Rightarrow A\underline{\xi}_i = \lambda \underline{\xi}_i + \underline{\xi}_{i-1}$,

$$[A\underline{b}_i]_{\tilde{\mathcal{B}}_\lambda} = \lambda_i \underline{e}_i \quad \text{AND} \quad [A\underline{\xi}_i]_{\tilde{\mathcal{B}}_\lambda} = \lambda \underline{e}_{M+i} + \underline{e}_{M+i-1} .$$
$$(i = 1, \dots, M) \qquad\qquad (i = 1, \dots, m-M)$$

THIS IMPLIES THAT FOR ANY $A \in M_{n,n}$,

$$A = P\tilde{D}P^{-1} \quad \text{WHERE} \quad \tilde{D} = [A]_{\tilde{\beta}_\lambda} \quad \text{TAKES THE FORM}$$

$$\tilde{D} = \begin{pmatrix} \overbrace{\lambda_1 \ \ddots}^{M_1} & & & & & \\ & \overbrace{\boxed{\begin{matrix} \lambda_1 & 1 & & \\ & \lambda_1 & 1 & \\ & & \lambda_1 & \ddots \ 1 \\ & & & \lambda_1 \end{matrix}}}^{m_1 - M_1} & & & & \\ & & \overbrace{\lambda_2 \ \ddots}^{M_2} & & & \\ & & & \overbrace{\boxed{\begin{matrix} \lambda_2 & 1 & & \\ & \lambda_2 & 1 & \\ & & \lambda_2 & \ddots \ 1 \\ & & & \lambda_2 \end{matrix}}}^{m_2 - M_2} & & \\ & & & & & \ddots \end{pmatrix},$$

"JORDAN BLOCKS"

$$P = \left(\overbrace{\underbrace{\begin{matrix}\text{EIGENVECTORS OF} \\ \lambda_1\end{matrix}}_{M_1} \ \Big\vdots \ \underbrace{\begin{matrix}\text{GENERALIZED EIGENVECTORS OF} \\ \lambda_1\end{matrix}}_{m_1 - M_1}}^{\tilde{E}_{\lambda_1}} \ \Bigg| \ \cdots \ \Bigg| \ \overbrace{\underbrace{\begin{matrix}\text{EIGENVALUES OF} \\ \lambda_r\end{matrix}}_{M_r} \ \Big\vdots \ \underbrace{\begin{matrix}\text{GENERALIZED EIGENVECTORS OF} \\ \lambda_r\end{matrix}}_{m_r - M_r}}^{\tilde{E}_{\lambda_r}} \right)$$

Ex. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. (NOTE: WE SAW IN A PREVIOUS LECTURE THAT THIS MATRIX IS NOT DIAGONALIZABLE.)

$$p_A(\lambda) = \lambda^2 = 0 \Rightarrow \lambda = 0.$$

E_0: $A - 0I = \begin{pmatrix} 0 & \boxed{1} \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{\xi} = \xi_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \{(1,0)^T\}$ BASIS OF E_0.

I.E, $\lambda = 0$ HAS $m = 2$, BUT $M = 1$.

WHAT TO DO?

LOOK FOR GENERALIZED EIGENVECTORS.

$\underline{\underline{\tilde{E}_0}}$: HAS BASIS $\tilde{\mathcal{B}}_0 = \{\underline{b}, \underline{\xi}\}$, WHERE

$\underline{b} = (1,0)^T \in E_0$, AND

$$(A - 0I)\underline{\xi} = \underline{b} \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\underline{\xi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{\xi} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

GENERALIZED EIGENVALUE OF ORDER 2.

SO, $\tilde{D} = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$, $P = \begin{pmatrix} \underline{b} & \underline{\xi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

IN RETROSPECT, THIS IS OBVIOUS SINCE A IS ALREADY IN JORDAN FORM!

$\underline{\text{EX.}}$ $A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}$

$$\Rightarrow p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & -2 \\ -1 & -\lambda & 5 \\ -1 & -1 & 4-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} -\lambda & 5 \\ -1 & 4-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 1 & -2 \\ -1 & 4-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 1 & -2 \\ -\lambda & 5 \end{vmatrix}$$

$$= (3-\lambda)\left[\lambda^2 - 4\lambda + 5\right] + (2-\lambda) + (2\lambda - 5)$$

$$= (\lambda - 3)\left[(-\lambda^2 + 4\lambda - 5) + 1\right]$$

$$= -(\lambda-3)(\lambda-2)^2 = 0 \Rightarrow \lambda = 3,\ \lambda = 2.$$

$\underline{\underline{E_3}}$: $A-3I = \begin{pmatrix} 0 & 1 & -2 \\ -1 & -3 & 5 \\ -1 & -1 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \underline{\xi} = \xi_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \{(-1,2,1)^T\}$ BASIS OF E_3.

$\underline{\underline{E_2}}$: $A-2I = \begin{pmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \underline{\xi} = \xi_3 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \Rightarrow \{(-1,3,1)^T\}$ BASIS OF E_2.

SO, $\lambda = 3$ HAS $\begin{cases} \text{ALG. MULT. } 1 \\ \text{GEOM. MULT. } 1 \end{cases}$

$\lambda = 2$ HAS $\begin{cases} \text{ALG. MULT. } 2 \\ \text{GEOM MULT. } 1 \end{cases} \Rightarrow A$ $\underline{\text{NOT}}$ DIAGONALIZABLE.

WHAT IS $\tilde{E}_2$?

$\underline{\underline{\tilde{E}_2}}$: HAS BASIS $\tilde{\mathcal{B}}_2 = (\underline{b}, \underline{\xi})$, WHERE $\underline{b} = (-1,3,1)^T \in E_2$.

$(A-2I)\underline{\xi} = \underline{b}$

$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & -1 \\ -1 & -2 & 5 & 3 \\ -1 & -1 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 1 \\ 0 & \boxed{1} & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{array}{l} \xi_1 = 1 - \xi_3 \\ \xi_2 = -2 + 3\xi_3 \\ \xi_3 \text{ FREE} \end{array} \Rightarrow \underline{\xi} = \begin{pmatrix} +1 \\ -2 \\ 0 \end{pmatrix} + \xi_3 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

LET $\underline{\xi_3} = 0$ (ONLY NEED ONE SOLN.)

$\Rightarrow \underline{\xi} = (1,-2,0)^T$. (GENERALIZED EIGENVECTOR)

THEREFORE,

$$A = P\tilde{D}P^{-1} \quad \text{WITH} \quad D = \begin{pmatrix} 3 & & \\ & 2 & 1 \\ & & 2 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 3 & -2 \\ 1 & 1 & 0 \end{pmatrix}$$

LECTURE 15
02/22/12

TRICKS FOR FINDING / VERIFYING EIGENVALUES (4.6):

- IF $A \in M_{n,n}$ TRIANGULAR (UPPER- OR LOWER-TRIANGULAR), EIGENVALUES OF A ARE DIAGONAL ENTRIES $\{A_{ii}\}_{i=1}^{n}$.
- IF $A \in M_{n,n}(\mathbb{R})$ (I.E., A REAL MATRIX), $\lambda = a+ib$ EIGENVALUE $\Longleftrightarrow$ $\overline{\lambda} = a-ib$ EIGENVALUE.
- DEFINE <u>TRACE</u> $Tr(A) \doteq \sum_{i=1}^{n} A_{ii}$. (SUM OF DIAGONAL ENTRIES). THEN, $Tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ (SUM OF EIGENVALUES, COUNTING MULTIPLICITIES).

  <u>PF.</u> EVEN THOUGH $AB \neq BA$ GENERALLY (I.E., MATRICES TYPICALLY DON'T COMMUTE), WE STILL HAVE THAT $Tr(AB) = Tr(BA)$. THEN, SINCE A HAS JORDAN FORM $A = P\tilde{D}P^{-1}$,

$$Tr(A) = Tr(\underbrace{P\tilde{D}}_{\text{CALL } B_1}\underbrace{P^{-1}}_{\text{CALL } B_2}) = Tr(\underbrace{P^{-1}}_{B_2}\underbrace{P\tilde{D}}_{B_1}) = Tr(\tilde{D})$$

$$= \lambda_1 + \cdots + \lambda_n.$$

- $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$ (DETERMINANT IS PRODUCT OF EIGENVALUES)

  <u>PF.</u> SINCE $\det(AB) = \det(A)\det(B)$ FOR ANY $A, B \in M_{n,n}$,

$$\det(A) = \det(P\tilde{D}P^{-1}) = \cancel{\det(P)}\det(\tilde{D})\cancel{\det(P^{-1})} = \det(\tilde{D})$$

$$\det(P^{-1}) = \frac{1}{\det(P)}$$

$$= \lambda_1 \cdots \lambda_n.$$

Ex. What are eigenvalues of $A = \begin{pmatrix} -6 & 7 \\ 4 & 6 \end{pmatrix}$?

$$\left.\begin{aligned} \mathrm{Tr}(A) &= 0 = \lambda_1 + \lambda_2 \\ \det(A) &= -36 - 28 = -64 = \lambda_1 \lambda_2 \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda_1 &= 8 \\ \lambda_2 &= -8 \end{aligned} .$$

Note: For most problems, use these tricks to <u>verify</u> your answers are correct.

Evolution problems (5.1, 5.2):

① **Discrete-time evolution:**

$$\begin{cases} \underline{x}(k) = A\,\underline{x}(k-1) \\ \underline{x}(0) \text{ known.} \end{cases} ,$$

given.

$$\underbrace{\underline{x}}_{\text{state of system at time } k \in \mathbb{N}}(k) \in \mathbb{R}^n \quad \text{for all } k = 0, 1, 2, \ldots$$

$$\underline{x}(k) = \underbrace{(x_1(k), x_2(k), \ldots, x_n(k))^T}_{\text{components of state vector } \underline{x}(k)}.$$

$\boxed{\text{Sol'n:}}$ $\underline{x}(k) = A\,\underline{x}(k-1) = A(A\,x(k-2)) = \ldots = \underbrace{A^k}_{\text{needs to be determined}} \underline{x}(0)$.

② **Continuous-time evolution:**

$$\begin{cases} \dfrac{d\underline{x}(t)}{dt} = A\,\underline{x}(t) \\ \underline{x}(0) \text{ known.} \end{cases} ,$$

given.

$$\underbrace{\underline{x}}_{\text{state of system at time } t \in [0,\infty)}(t) \in \mathbb{R}^n \quad \text{for all } t \geq 0.$$

$$\underline{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T.$$

This is known as ~~a~~ <u>system</u> of first-order ordinary differential equations (ODE).

SOL'N: IF $n=1$, I.E., IF A IS A SCALAR, SAY a, THEN THE EVOLUTION PROBLEM IS EASILY SOLVED.

$$\begin{cases} \frac{dx(t)}{dt} = a x(t) \\ x(0) \text{ known} \end{cases} \Rightarrow \int \frac{dx(t)}{x(t)} = \int a\,dt \Rightarrow \ln(x(t)) - \ln(x(0)) = ta$$

$$\Rightarrow x(t) = e^{ta} x(0).$$

(TO CHECK THIS IS CORRECT, SUBSTITUTE INTO THE ODE TO FIND THAT $\frac{d}{dt}\big(\underbrace{e^{ta}x(0)}_{x(t)}\big) = a e^{ta} x(0) = a x(t)$. ✓)

BY ANALOGY, IF $n > 1$ THEN $A \in M_{n,n}$ IS A MATRIX AND THE SOLN. IS

$$\underline{x}(t) = \underbrace{e^{tA}} \underline{x}(0).$$

WE WILL DEFINE THIS LATER USING MATRIX EXPONENTIALS, WHICH WILL DETERMINE THIS TO BE

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{A^k}.$$

NEEDS TO BE DETERMINED.

NOTE: FOR BOTH DISCRETE- AND CONTINUOUS-TIME EVOLUTIONS WE NEED TO FIND A^k, FOR SOME GIVEN $k \in \mathbb{N}$. FOR LARGE k, THIS CAN BE DIFFICULT — UNLESS WE DIAGONALIZE A! IF A IS DIAGONALIZABLE, THEN

$$A^k = (PDP^{-1})^k = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\cdots(\cancel{P}DP^{-1})$$

$$= PD^kP^{-1}, \text{ WHERE } D^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}.$$

Lecture 16
02/24/12

Discrete-time Evolution:

Ex (Population growth, Fibonacci sequence)

Let $x_1(t)$, $x_2(t)$ be the number of juvenile and adult rabbits that live in a certain region. These populations evolve according to the following rules:

- Every month, each adult gives birth to one juvenile.
- Juveniles grow into adults in one month
- Adults / juveniles do not die.

Then, for each month $k = 0, 1, 2, \ldots$,

$$\Delta x_1(k) \doteq x_1(k) - x_1(k-1) = \overbrace{x_2(k-1)}^{\text{birth of new juveniles}} - \overbrace{x_1(k-1)}^{\text{juveniles become adults}}$$

$$\Delta x_2(k) \doteq x_2(k) - x_2(k-1) = \overbrace{x_1(k-1)}^{\text{juveniles become adults}}$$

Letting $\underline{x}(k) = (x_1(k), x_2(k))^T$, we have that.

$$\underline{x}(k) = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}_{A} \underline{x}(k-1).$$

$$\Rightarrow p_A(\lambda) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow \lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \underbrace{\frac{1-\sqrt{5}}{2}}_{\approx -0.618}.$$

$$\lambda_1 \approx \underbrace{1.618}_{\text{"golden ratio"}}$$

THE CORRESPONDING EIGENSPACES ARE

$$\underline{\underline{E_{\frac{1+\sqrt{5}}{2}}}} : \begin{pmatrix} -\left(\frac{1+\sqrt{5}}{2}\right) & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \times \frac{1-\sqrt{5}}{2} \rightarrow \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix} -R_1$$

$$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ 0 & 0 \end{pmatrix} \Rightarrow \left\{ \left(-(1-\sqrt{5}), 2\right)^T \right\} \text{ BASIS.}$$

$$\underline{\underline{E_{\frac{1-\sqrt{5}}{2}}}} : \begin{pmatrix} -\left(\frac{1-\sqrt{5}}{2}\right) & 1 \\ 1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} \times \frac{1+\sqrt{5}}{2} \rightarrow \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} -R_1$$

$$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 0 \end{pmatrix} \Rightarrow \left\{ \left(-(1+\sqrt{5}), 2\right)^T \right\} \text{ BASIS.}$$

LET $\begin{cases} \underline{b}_1 = \left(-(1-\sqrt{5}), 2\right)^T \\ \underline{b}_2 = \left(-(1+\sqrt{5}), 2\right)^T \end{cases}$ BE THE BASIS $\mathcal{B}$ OF EIGENVECTORS OF A.

THEN,

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(a_1(0), a_2(0))^T = [\underline{x}(0)]_{\mathcal{B}} \arrow[r, "D^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}"] & {[\underline{x}(k)]_{\mathcal{B}} = (a_1(k), a_2(k))^T} \arrow[d, "P = (\underline{b}_1 \; \underline{b}_2)"] \\
(x_1(0), x_2(0))^T = \underline{x}(0) \arrow[u, "P^{-1}"] \arrow[r, dashed, "A^k"'] & \underline{x}(k) = (x_1(k), x_2(k))^T.
\end{tikzcd}$$

THAT IS, $\underline{x}(k) = \underset{\lambda_1^k a_1(0)}{\underset{\shortparallel}{a_1(k)}} \, \underline{b}_1 + \underset{\lambda_2^k a_2(0)}{\underset{\shortparallel}{a_2(k)}} \, \underline{b}_2$, WITH $\begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} = [\underline{x}(0)]_{\mathcal{B}}$

FOR EXAMPLE, SUPPOSE WE START WITH ONE JUVENILE AND NO ADULTS. THEN,

$$\underline{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow [\underline{x}(0)]_{\mathcal{B}} = \Big(\underbrace{\tfrac{1}{2\sqrt{5}}}_{a_1(0)}, \; \underbrace{-\tfrac{1}{2\sqrt{5}}}_{a_2(0)} \Big)^T.$$

$$\Rightarrow \underline{x}(k) = \underbrace{\left(\frac{1+\sqrt{5}}{2}\right)^k}_{\to \infty \text{ AS } k \to \infty.} \frac{1}{2\sqrt{5}} \underline{b}_1 + \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^k}_{\to 0 \text{ AS } k \to \infty} \left(-\frac{1}{2\sqrt{5}}\right) \underline{b}_2$$

$\Rightarrow \underline{b}_1$ "UNSTABLE MODE" $\qquad \Rightarrow \underline{b}_2$ "STABLE MODE".

FOR k LARGE

$$\approx \left(\frac{1+\sqrt{5}}{2}\right)^k \frac{1}{2\sqrt{5}} \underline{b}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{k} \end{pmatrix}.$$

CONSEQUENCES (FOR LARGE k):

- POP. OF ADULTS WILL BE APPROX. $\frac{1+\sqrt{5}}{2} \approx 1.618$ TIMES MORE THAN THAT OF JUVENILES.
- BOTH POPULATIONS GROW EXPONENTIALLY, APPROX. BY 1.618 TIMES EACH MONTH.
- TOTAL POPULATION OF RABBITS IS APPROX.

$$p(k) = x_1(k) + x_2(k) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{k} \right)$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}.$$

CONTINUOUS-TIME EVOLUTION:

LAST TIME, WE SAW THAT THE SOLN. TO

$$\begin{cases} \dfrac{d\underline{x}}{dt} = A\underline{x} \\ \underline{x}(0) \text{ given} \end{cases} \quad \text{is} \quad \underline{x}(t) = e^{tA}\,\underline{x}(0).$$

WHAT IS THIS?

MATRIX EXPONENTIAL (4.8):

FOR SCALAR $a \in \mathbb{R}$, THE POWER SERIES OF e^a IS

$$e^a = 1 + \frac{a}{1!} + \frac{a^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{a^k}{k!}.$$

FOR $A \in M_{n,n}$, DEFINE THE MATRIX EXPONENTIAL

$$e^A \doteq I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

(FOR $L: V \to V$, CAN DEFINE $L^n \doteq L \circ L \circ \cdots \circ L$ (n TIMES)
AND $e^L = I + \frac{L}{1!} + \frac{L^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{L^k}{k!}$.)

- IF A IS DIAGONALIZABLE, $A = PDP^{-1}$ AND

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} (PD^kP^{-1})$$

$$= P\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} D^k\right)P^{-1} = Pe^{tD}P^{-1}.$$

NOTE THAT FOR D DIAGONAL,

$$e^{tD} = \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{pmatrix}.$$

THEREFORE, IF A IS DIAGONALIZABLE,

$$\underline{x}(t) = e^{tA}\,\underline{x}(0)$$
$$= P e^{tD} P^{-1}\,\underline{x}(0)$$

$$\begin{tikzcd}
{[\underline{x}(0)]_{\mathcal{B}}} \arrow[rr, "{e^{tD} = \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{pmatrix}}"'] & & {[\underline{x}(t)]_{\mathcal{B}}} \arrow[d, "{P = (\underline{b}_1 \cdots \underline{b}_n)}"] \\
\underline{x}(0) \arrow[u, "P^{-1}"] \arrow[rr, dashed, "e^{tA}"'] & & \underline{x}(t)
\end{tikzcd}$$

WHERE $\{\underline{b}_1, \ldots, \underline{b}_n\} = \mathcal{B}$ IS THE BASIS OF EIGENVECTORS.

LECTURE 17
02/27/12

RECALL :

① DISCRETE-TIME EVOLUTION

$$\begin{cases} \underline{x}(k) = A\,\underline{x}(k-1) \\ \underline{x}(0) \text{ KNOWN} \end{cases} \implies \begin{aligned} \underline{x}(k) &= A^k\,\underline{x}(0) \\ &= P D^k P^{-1}\,\underline{x}(0) \end{aligned}$$

IF $A = PDP^{-1}$.

EX. $(n=2)$

$$\begin{tikzcd} (a_1(0), a_2(0))^T = [\underline{x}(0)]_{\mathcal{B}} \arrow[r, "D^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}"] & {[\underline{x}(k)]_{\mathcal{B}}} \arrow[d, "P"] \\ (x_1(0), x_2(0))^T = \underline{x}(0) \arrow[u, "P^{-1}"] \arrow[r, "A^k"'] & \underline{x}(k) \end{tikzcd}$$

$$[\underline{x}(k)]_{\mathcal{B}} = (a_1(k), a_2(k))^T = (\lambda_1^k a_1(0),\ \lambda_2^k a_2(0))^T.$$

WITH $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ BASIS OF E-VECTORS OF A.

② CONT. TIME EVOLUTION

$$\begin{cases} \dfrac{dx(t)}{dt} = A\,x(t) \\ x(0) \text{ KNOWN} \end{cases} \implies \begin{aligned} \underline{x}(t) &= e^{tA}\,\underline{x}(0) \\ &= P e^{tD} P^{-1}\,\underline{x}(0) \end{aligned}$$

IF $A = PDP^{-1}$.

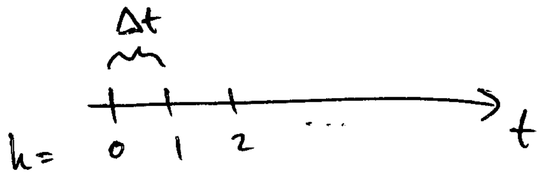
EX $(n=2)$

$$\begin{tikzcd} (a_1(0), a_2(0))^T = [\underline{x}(0)]_{\mathcal{B}} \arrow[r, "e^{tD} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}"] & {[\underline{x}(t)]_{\mathcal{B}}} \arrow[d, "P"] \\ (x_1(0), x_2(0))^T = \underline{x}(0) \arrow[u, "P^{-1}"] \arrow[r, "e^{tA}"'] & \underline{x}(t) \end{tikzcd}$$

$$[\underline{x}(t)]_{\mathcal{B}} = (a_1(t), a_2(t))^T = (e^{t\lambda_1} a_1(0),\ e^{t\lambda_2} a_2(0))^T$$

WITH $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ BASIS OF E-VECTORS OF A

NOTE: (2) IS A LIMIT OF (1).



RESTRICT TIMES TO $t = k\Delta t,\ k \in \mathbb{N}$

$$\Rightarrow \quad \frac{d\underline{x}(t)}{dt} \approx \frac{\underline{x}((k+1)\Delta t) - \underline{x}(k\Delta t)}{\Delta t}$$

$$A\underline{x}(t) \approx A\,\underline{x}(k\Delta t).$$

$$\Rightarrow \begin{cases} \underline{x}((k+1)\Delta t) = (I + (\Delta t)A)\,\underline{x}(k\Delta t) \\ \underline{x}(0) \text{ known} \end{cases}$$

$$\Rightarrow \quad \underline{x}(k\Delta t) = (I + (\Delta t)A)^k\,\underline{x}(0).$$

NOTE NOW THAT AS $\Delta t \to 0$,

$$\begin{aligned}(I + (\Delta t)A)^k &= (I + (\Delta t)A)^{\frac{k\Delta t}{\Delta t}} \\ &= (I + (\Delta t)A)^{\frac{t}{\Delta t}} \\ &\xrightarrow[\Delta t \to 0]{} e^{tA}\end{aligned}$$

(THIS IS THE ANALOGUE OF THE DEFINITION OF THE EXPONENTIAL:

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$
)

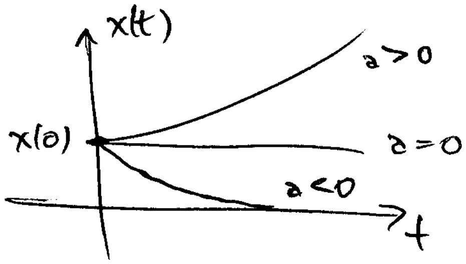
THAT IS, WE CAN THINK OF CONTINUOUS-TIME SYSTEMS (SYSTEMS OF LINEAR ODE) AS LIMITS OF DISCRETE-TIME SYSTEMS.

Ex. (SCALAR EXPONENTIAL GROWTH / DECAY WITH RATE a)

$$\begin{cases} \dfrac{dx}{dt} = ax \\ x(0) \text{ GIVEN} \end{cases} \Rightarrow \quad \underline{x}(t) = e^{ta}\, x(0)$$

$$\xrightarrow{t \to \infty} \begin{cases} \pm\infty, & a > 0 \\ 0, & a < 0 \\ x(0), & a = 0 \end{cases}$$

IF $x(0) \neq 0$.



Ex. (RADIOACTIVE DECAY W/ MULTIPLE POPULATIONS)

$$X \xrightarrow{r_1} Y, \quad Y \xrightarrow{r_2} Z \qquad \text{WITH} \quad r_1, r_2 \geq 0, \; r_1 \neq r_2.$$

$$\Rightarrow \begin{cases} \dot{x}_1 = -r_1 x_1 \\ \dot{x}_2 = r_1 x_1 - r_2 x_2 \end{cases}$$

WHERE $x_1(t), x_2(t)$ ARE POPULATIONS OF X, Y AT TIME $t \geq 0$, AND $\bullet = \frac{d}{dt}$.

$$\Rightarrow \quad \frac{d\underline{x}(t)}{dt} = A\, \underline{x}(t) \qquad \text{WITH} \quad A = \begin{pmatrix} -r_1 & 0 \\ r_1 & -r_2 \end{pmatrix}$$

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

E-VALUES / E-VECTORS ARE

$$\lambda_1 = -r_1, \quad E_{-r_1} = \{\overbrace{((r_2 - r_1), r_1)^T}^{\underline{b}_1}\}$$

$$\lambda_2 = -r_2, \quad E_{-r_2} = \{\underbrace{(0, 1)^T}_{\underline{b}_2}\}.$$

SUPPOSE $\underline{x}(0) = (1, 0)^T$. THEN,

$$\underline{x}(0) = \frac{1}{r_2 - r_1} \underline{b}_1 - \frac{r_1}{r_2 - r_1} \underline{b}_2$$

AND
$$\underline{x}(t) = \frac{e^{-r_1 t}}{r_2 - r_1} \underline{b}_1 - \frac{r_1 e^{-r_2 t}}{r_2 - r_1} \underline{b}_2$$

$$= \Big(\underbrace{e^{-r_1 t}}_{x_1(t)}, \underbrace{\frac{r_1}{r_2 - r_1} \left(e^{-r_1 t} - e^{-r_2 t} \right)}_{x_2(t)} \Big)^T .$$

NOTE THAT AS $r_1 \to r_2$ WE CAN STILL RECOVER A SOLUTION FOR $\underline{x}(t)$:

$$\lim_{r_1, r_2 \to r} \underline{x}(t) = \left(e^{-rt}, rt\, e^{-rt} \right)^T .$$

IN THIS CASE, A IS NOT DIAGONALIZABLE WHEN $r_1 = r_2 = r$, BUT IT IS ALMOST DIAGONALIZABLE.

LECTURE 18
02/29/12

Q: FOR EVOLUTION PROBLEMS, WHAT IF WE HAVE COMPLEX EIGENVALUES?

A: SAME METHODS, BUT WILL SEE OSCILLATIONS IN SYSTEM ALONG w/ EXPONENTIAL DECAY / GROWTH!

EX
$$\begin{cases} \dfrac{d\underline{x}(t)}{dt} = A\underline{x}(t) \\ \underline{x}(0) \text{ GIVEN} \end{cases}, \quad A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

E-VALUES / E-VECTORS OF A: $\lambda = 1 \pm 2i$

$$\underline{b} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\underline{v}} \pm i \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\underline{w}}$$

$$\Rightarrow \underline{x}(t) = e^{tA}\underline{x}(0) = \underbrace{Pe^{tD}P^{-1}}\underline{x}(0)$$

$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{(1+2i)t} & 0 \\ 0 & e^{(1-2i)t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -1 \end{pmatrix}^{-1}$$

WE USE EULER'S FORMULA $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, $\theta \in \mathbb{R}$, TO GET THAT

$$\begin{pmatrix} e^{(1+2i)t} & 0 \\ 0 & e^{(1-2i)t} \end{pmatrix} = \begin{pmatrix} e^{t}(\cos(2t) + i\sin(2t)) & 0 \\ 0 & e^{t}(\cos(2t) - i\sin(2t)) \end{pmatrix}$$

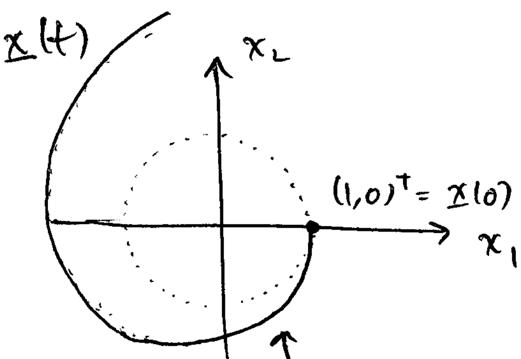
SINCE $\cos(-x) = \cos(x)$, $\sin(-x) = -\sin(x)$.

THEN, THE SOLUTION IS

$$\underline{x}(t) = e^{t}\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix}\underline{x}(0)$$

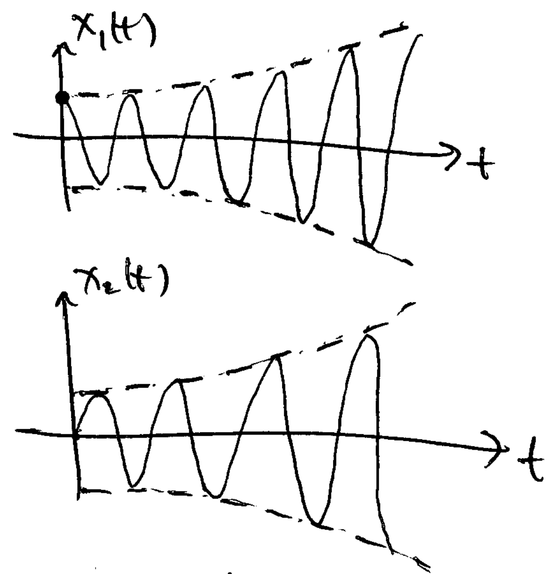
REMARK: REMEMBER, SINCE A IS REAL WE MUST HAVE THAT e^{tA} IS REAL AND $Pe^{tD}P^{-1}$ IS REAL AS WELL! SO THE FINAL ANSWER SHOULD CONSIST ONLY OF REAL TERMS.

WE CAN PLOT THE SOLUTION IN SEVERAL WAYS (GIVEN AN INITIAL CONDITION $\underline{x}(0)$ — FOR EX., $\underline{x}(0) = (1,0)^{T}$):



SPIRAL OUTWARD SINCE HAVE TERM e^{t}.

PHASE PLANE.



$$x_1(t) = e^{t}\cos(2t)$$
$$x_2(t) = -e^{t}\sin(2t).$$

- WE NOTE THAT THE REAL PART OF THE PAIR OF COMPLEX EIGENVALUES $\lambda = 1 \pm 2i$ DETERMINES THE RATE OF GROWTH / DECAY, WHILE THE IMAGINARY PART DETERMINES THE FREQUENCY OF OSCILLATION.

FOR EXAMPLE, IF $\lambda = -1 \pm 2i$ INSTEAD, WE WOULD HAVE HAD A SOLUTION w/ EXPONENTIAL DECAY e^{-t} INSTEAD OF GROWTH (A SPIRAL INWARD IN THE PHASE PLANE) BUT THE SAME FREQUENCY OF OSCILLATION.

REMARK: EVERY n^{th}-ORDER, HOMOGENEOUS, CONST. COEFF. SCALAR ODE

$$(\star) \quad \frac{d^n y}{dt^n} + c_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + c_0 y = 0, \quad y(t) \in \mathbb{R}, \ t \geq 0. \ c_i \in \mathbb{R} \text{ const.}$$

CAN BE WRITTEN AS A SYSTEM OF 1^{st}-ORDER ODE. TO SEE THIS, LET

$$\underline{x}(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ \vdots \\ x_{n-1}(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ \frac{dy}{dt}(t) \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}}(t) \end{pmatrix} \in \mathbb{R}^n \quad \text{FOR ALL } t \geq 0.$$

THEN,

$$\frac{d\underline{x}(t)}{dt} = \begin{pmatrix} \frac{dy}{dt}(t) \\ \frac{d^2y}{dt^2}(t) \\ \vdots \\ \frac{d^n y}{dt^n}(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ -c_{n-1}x_{n-1}(t) - \cdots - c_0 x_0(t) \end{pmatrix}$$

WHERE WE HAVE USED $(\star)$ TO REWRITE $\frac{d^n y}{dt^n}$ IN TERMS OF $y, \frac{dy}{dt}, \ldots, \frac{d^{n-1}y}{dt^{n-1}}$. THEREFORE,

$$\frac{d\underline{x}}{dt} = A\,\underline{x}, \qquad A = \left(\begin{array}{c|ccc} 0 & & I_{n-1} & \\ \hline -c_0 & -c_1 & \cdots & -c_{n-1} \end{array}\right) \Big\updownarrow n$$

$$\overleftrightarrow{\quad n \quad}$$

w/ I_{n-1} AN $(n-1)\times(n-1)$ IDENTITY MATRIX.

NOTE: TO SOLVE THIS LINEAR EVOLUTION SYSTEM, WE NEED TO BE GIVEN

$$\left.\underline{x}(0) = \begin{pmatrix} y(0) \\ \frac{dy}{dt}(0) \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}}(0) \end{pmatrix}\right\} \text{ I.E., } n \text{ INITIAL CONDITIONS.}$$

EX. $\begin{cases} y'' + by' + y = 0 \\ \text{w/ } y(0),\ y'(0) \text{ given.} \end{cases}$ " $' = \frac{d}{dt}$ ".

LET $\underline{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$. THEN, $\underline{x}' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} y \\ y' \end{pmatrix}}_{\underline{x}}$.

$p_A(\lambda) = \lambda^2 + b\lambda + 1$

$\rightsquigarrow$ E-VALUES OF A: $\lambda = \dfrac{-b \pm \sqrt{b^2-4}}{2}$.

SO, IF $|b| < 2$ WE WILL GET OSCILLATIONS ALONG w/ EXPONENTIAL GROWTH/DECAY IN SOLUTION $y(t)$.