10/26/10

1

Vector-valued functions, space curves (14.1)

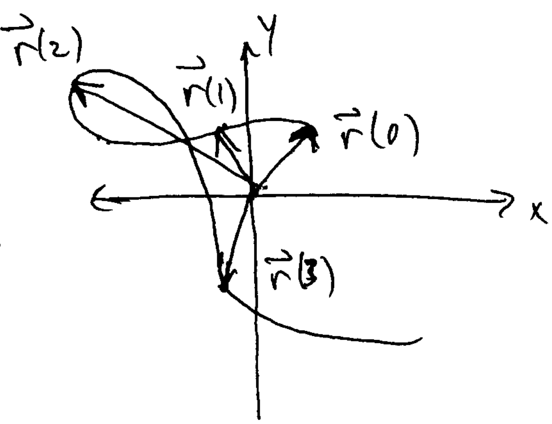
Recall parametric eqn's in 2-D:

$t =$ parameter

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \Rightarrow t \longmapsto (f(t), g(t))$$

In other words, we were considering the vector-valued function

$$\vec{r}(t) = (f(t), g(t)) = f(t)\,\vec{i} + g(t)\,\vec{j}$$



We now do this in 3-D:

$$\vec{r}(t) = (f(t), g(t), h(t)) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

As we vary t, $\vec{r}(t)$ traces out a <u>space curve</u>. Define the <u>limit</u> of $\vec{r}(t)$ as $t \to a$ in terms of its components:

$$\lim_{t\to a} \vec{r}(t) = \left(\lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right)$$

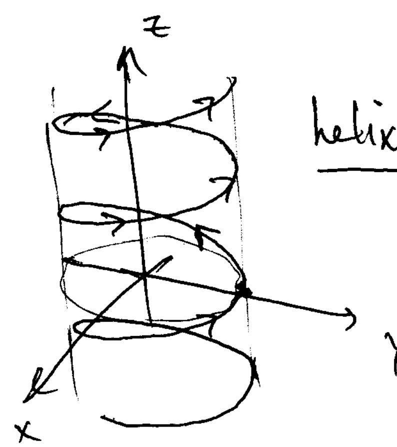
provided these limits exist.

We say $\vec{r}(t)$ is <u>continuous</u> at $t=a$ if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.

<u>Ex.</u> What is the curve associated to

$$\vec{r}(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t\, \vec{k}$$

$$\Rightarrow \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$$



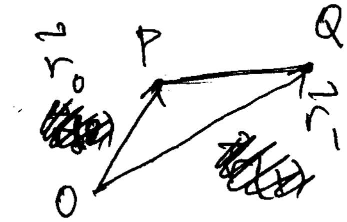
<u>helix</u>.

Note that

$x^2+y^2=1 \Rightarrow$ so, curve lies on surface of cylinder with radius 1.

<u>Ex.</u> What eqn's describe the straight line between the two points $P(1,1,3)$ and $Q(-2,-3,4)$?

$$\vec{r}(t) = \vec{r}_0 + (\vec{r}_1 - \vec{r}_0)\,t$$



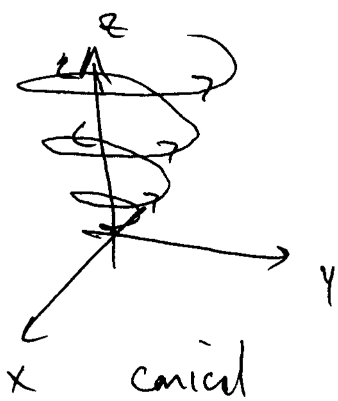
$$0 \le t \le 1$$

$$\boxed{\vec{r}(t) = \vec{r}_0 + (\vec{r}_1 - \vec{r}_0)\,t, \quad 0 \le t \le 1}$$

where $\vec{r}_0 = (1,1,3)$
$\vec{r}_1 = (-2,-3,4)$

vector eqn. for a line.

Ex.
$$\begin{cases} x = at\cos t \\ y = at\sin t \\ z = ct \end{cases}$$



conical spiral

Note that

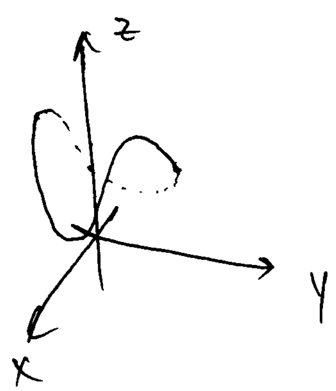
$$x^2 + y^2 = a^2t^2 = \left(\frac{a}{c}\right)^2 z^2$$

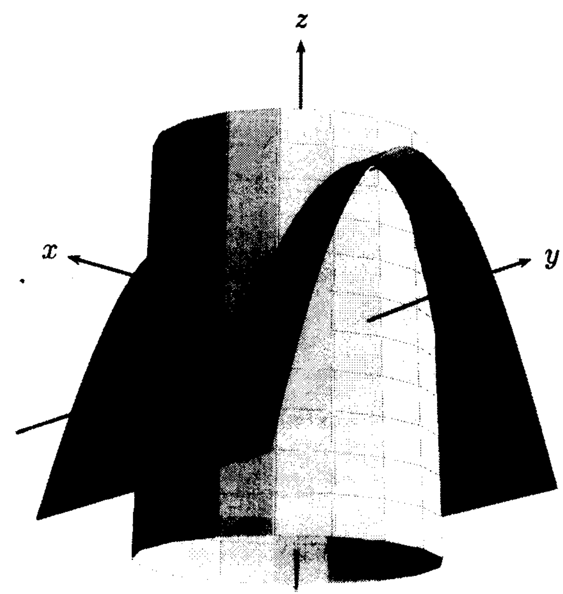
$\Rightarrow$ curve lies on surface of a cone.

Ex.

$$x^2+y^2=1 \quad \text{surface \#1}$$
$$z=1-2x^2 \quad \text{surface \#2}$$

Find parametric eqn's for the intersection of these two surfaces.





Projection of intersection onto x-y plane satisfies $x^2+y^2=1$.

Guess $x(t)=\cos t$, $y(t)=\sin t$, $0\le t\le 2\pi$,

So, by second eqn. $z=1-2x^2$ must have that

$$z(t)=1-2(x(t))^2 = 1-2\cos^2 t$$
$$= -\cos(2t)$$

$$\boxed{\vec{r}(t)=(\cos t,\ \sin t,\ -\cos(2t))}$$

gives one parametrization of the desired curve.

Another parametrization is

$$x(t) = \sin t$$
$$y(t) = \cos t$$
$$z(t) = 1 - 2\sin^2 t = \cos(2t)$$

$$\boxed{\vec{s}(t) = (\sin t,\ \cos t,\ \cos(2t)),\quad t \in [0, 2\pi].}$$

Ex. At which points does the helix $\vec{r}(t) = (\sin t,\ \cos t,\ t)$ intersect the sphere $x^2 + y^2 + z^2 = 10$?

$$\begin{cases} x(t) = \sin t \\ y(t) = \cos t \\ z(t) = t \end{cases}$$

$\Rightarrow$ Find t such that

$$\sin^2 t + \cos^2 t + t^2 = 10$$

$\Rightarrow 1 + t^2 = 10 \Rightarrow t^2 = 9 \Rightarrow t = \pm 3.$

$\Rightarrow$ intersections are at $\vec{r}(3) = (\sin 3,\ \cos 3,\ 3)$

$\vec{r}(-3) = (\sin -3,\ \cos -3,\ -3).$

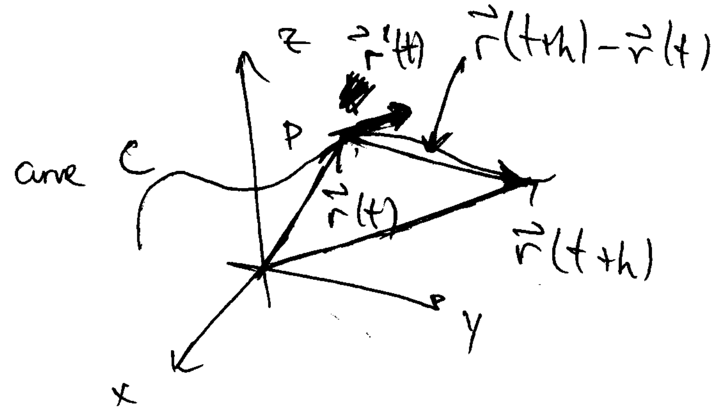
Derivatives and integrals of vector functions (14.2)

Derivatives

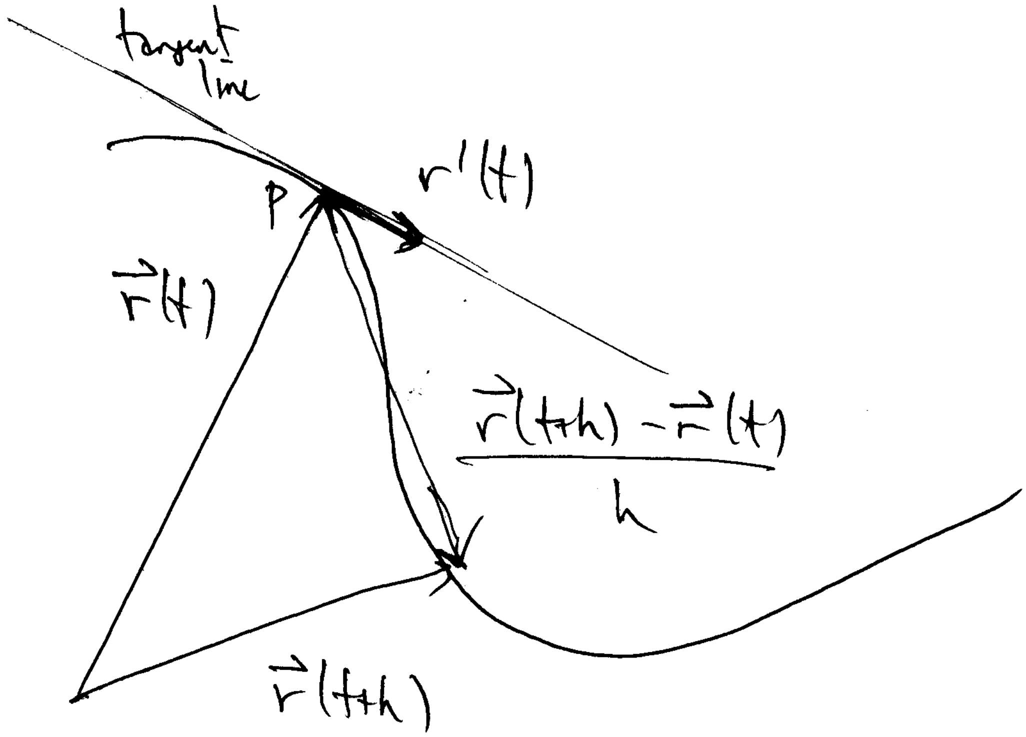
We define the derivative $\vec{r}'(t)$ of a vector-valued function $\vec{r}(t)$ as

$$\vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

if the limit exists.



If $\vec{r}'(t) \neq 0$, $\vec{r}'(t)$ the <u>tangent vector</u> to the curve C at the point P and we let the <u>tangent line</u> to C at point P to be the line parallel to $\vec{r}'(t)$ and going through P.



$\vec{r}\,'(t)$ encodes the <u>direction</u> in which the curve is being traced and the speed at which it is being traced out ~~when the parameter~~ at t.

Define the <u>unit tangent vector</u>

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

which only indicates the direction in which we are moving.

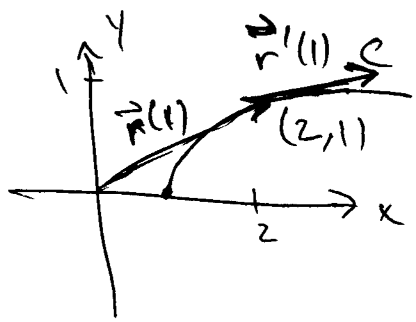
What are the components of $\vec{r}\,'(t)$ if $\vec{r}(t) = (f(t), g(t), h(t))$ where f, g, h are differentiable?

<u>Thm.</u>

$$\boxed{\begin{aligned}\vec{r}\,'(t) &= (f'(t),\ g'(t),\ h'(t)) \\ &= f'(t)\,\vec{i} + g'(t)\,\vec{j} + h'(t)\,\vec{k}.\end{aligned}}$$

<u>Ex.</u> $\vec{r}(t) = (1+t, \sqrt{t})$, $t \geq 0$

What is tangent vector at $t=1$?

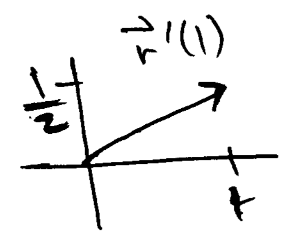


$x(t) = 1+t$
$y(t) = \sqrt{t}$

$\Rightarrow x = 1 + y^2$

$\vec{r}(1) = (2, 1)$

$\vec{r}\,'(t) = \left(1, \frac{1}{2\sqrt{t}}\right)$



$\Rightarrow \vec{r}\,'(1) = \left(1, \frac{1}{2}\right)$

Tangent line at point $P(2,1)$?

$y = mx + b$, $\quad m = \frac{1}{2}$, $b = 0$ $\Rightarrow$ $\boxed{y = \frac{1}{2}x}$.

We can also define second derivatives by

$$\vec{r}''(t) = \frac{d}{dt}(r'(t)) = (\vec{r}\,')'(t).$$

Properties

If $\vec{u}(t)$, $\vec{v}(t)$ are differentiable vector functions, and $f(t)$ is a scalar function:

i) $\frac{d}{dt}(\vec{u}(t) + \vec{v}(t)) = \vec{u}'(t) + \vec{v}\,'(t)$

product rule

ii) $\frac{d}{dt}(f(t)\,\vec{u}(t)) = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$.

(in particular, if $f(t) = c$, then

$$\frac{d}{dt}(c\,\vec{u}(t)) = c\,\frac{d}{dt}\vec{u}(t).)$$

iii) $\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$.

iv) $\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$

v) $\frac{d}{dt}(\vec{u}(f(t))) = f'(t)\,\vec{u}'(f(t))$

chain rule.

Integrals

We define the definite integral

$\int_a^b \vec{r}(t)\,dt$ of a vector-valued function

$\vec{r}(t) = (f(t), g(t), h(t))$. as

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

That is, if $\vec{R}(t)$ is the antiderivative of $\vec{r}(t)$, i.e, $\vec{R}'(t) = \vec{r}(t)$, then

$$\boxed{\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)}$$

Arc length and curvature (14.3)

When we had the parametric eqn's

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad f, g \text{ differentiable}, \quad a \le t \le b.$$

we had arc length formula:

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Now with $\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$, f, g, h differentiable, $a \le t \le b$.

(equiv. to parametric eqn's $\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$ $a \le t \le b$)

we have <u>arc length formula</u>:

$$L = \int_a^b \sqrt{(f')^2 + (g')^2 + (h')^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Concisely, this this

$$\boxed{L = \int_a^b |\vec{r}\,'(t)|\, dt.}$$

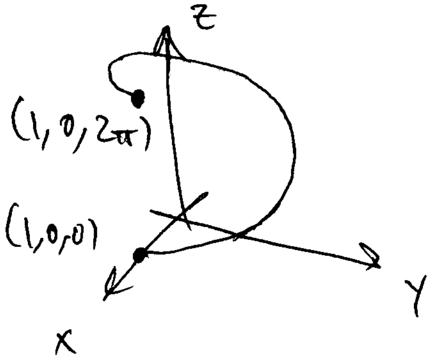
since $|\vec{r}\,'(t)| = |f'(t)\,\vec{i} + g'(t)\,\vec{j} + h'(t)\,\vec{k}|$

$$= \sqrt{(f')^2 + (g')^2 + (h')^2}$$

Remark: Be careful to choose a, b s.t. curve is only traversed once.

Ex. (Arc length of one turn of a helix)

$$\vec{r}(t) = (\cos t)\,\vec{i} + (\sin t)\,\vec{j} + t\,\vec{k}$$



$$0 \le t \le 2\pi$$

$$\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$$

$$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$$

$$= \sqrt{2}.$$

$$\Rightarrow L = \int_0^{2\pi} |\vec{r}\,'(t)|\, dt$$

$$= \int_0^{2\pi} \sqrt{2}\, dt = \boxed{2\sqrt{2}\,\pi.}$$

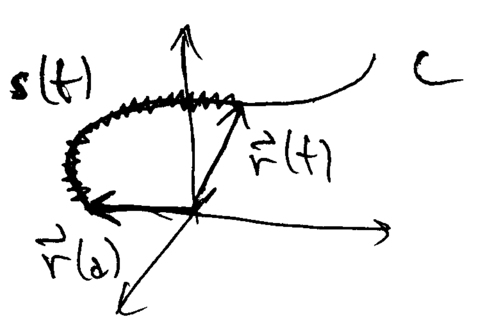
Define the <u>arc length function</u>

$$s(t) = \int_a^t |\vec{r}\,'(u)|\, du$$

$$\Rightarrow \quad \frac{ds}{dt} = |\vec{r}\,'(t)|$$

<u>Note</u>: For any curve C, there are many parametrizations $\vec{r}(t)$. Arc length function gives a useful <u>unique</u> parametrization of C because it only depends on an intrinsic property of the curve (its length).

In other words, choose parametrization that traces out an equal length of the curve in each time interval!



$a \le t \le b$

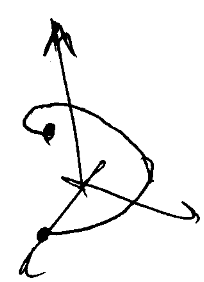
$\vec{r}(t)$

<u>Idea</u>: Invert function $s(t)$ to get $t(s)$, where $t(s)$ is the parameter value s.t. we've traversed length s from $\vec{r}(a)$ (where we started).

$\vec{r}(s) = \vec{r}(t(s))$ is the <u>arc length parametrization</u> of the curve C.

<u>Ex.</u> $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$

$$0 \le t \le 2\pi$$



$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du$$

$$= \int_0^t \sqrt{2}\,du = \sqrt{2}\,t.$$

$$s(t) = \sqrt{2}\,t \longleftrightarrow t(s) = \frac{s}{\sqrt{2}}.$$

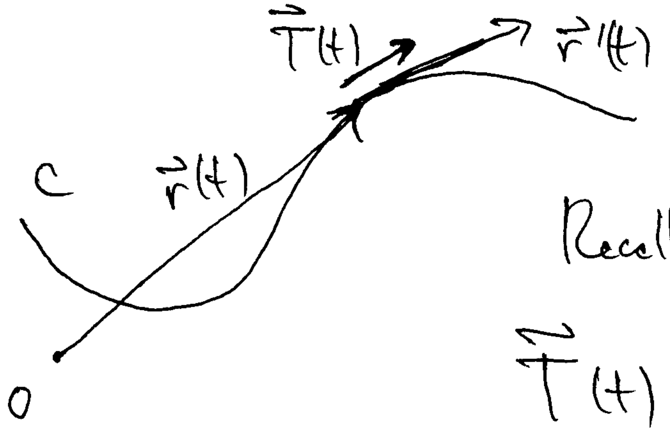
So,
$$\boxed{\vec{r}(s) = \vec{r}(t(s)) = \cos\frac{s}{\sqrt{2}}\,\vec{i} + \sin\frac{s}{\sqrt{2}}\,\vec{j} + \frac{s}{\sqrt{2}}\,\vec{k}}$$

$$0 \le s \le 2\sqrt{2}\pi$$

Curvature

Suppose a curve C is smooth

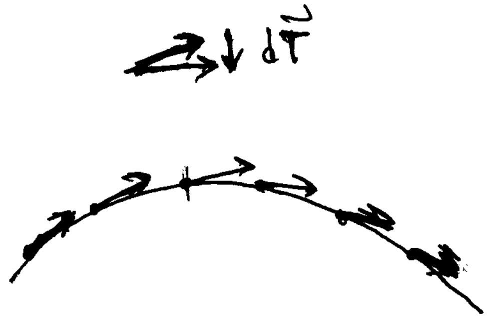
(i.e, has parametrization $\vec{r}(t)$ which is differentiable with $\vec{r}\,'(t) \neq \vec{0}$.)

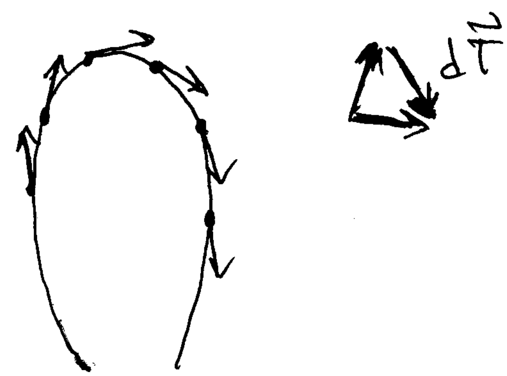


Recall the unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

which gives the direction in which the curve is being traced at at instant t.





Define the curvature

$$K = \left| \frac{d\vec{T}}{ds} \right|$$

where $\vec{T}$ is unit tangent vector and s is the arc length function

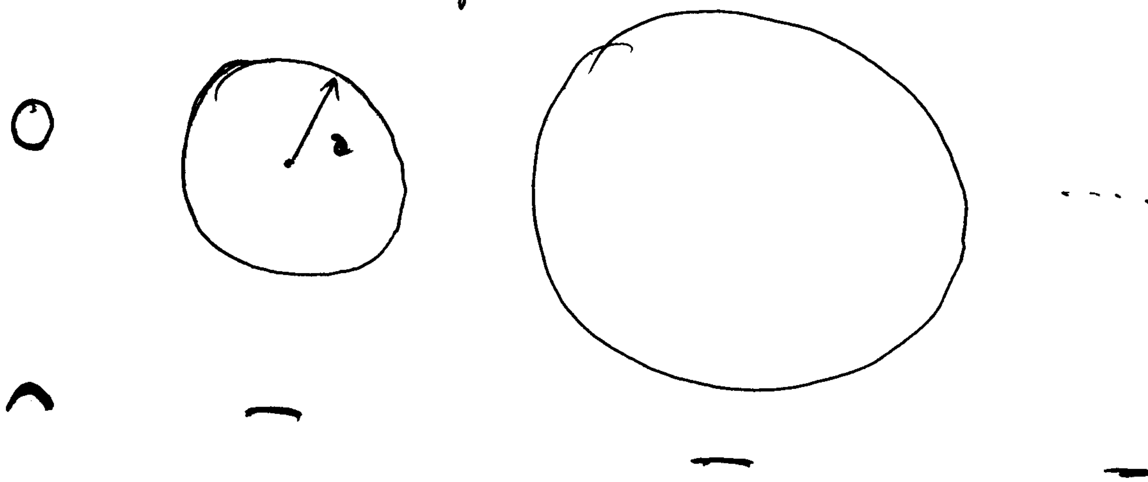
In terms of the parametrization $\vec{r}(t)$, this is

$$\boxed{K(t) = \left|\frac{d\vec{T}}{ds}\right|(t) = \left|\frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}\right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}}$$

Another way to express this is by plugging in $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ and doing a page of calculations (p. 869) is

$$\boxed{K(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}}$$

Ex. Which has greatest curvature ?



$$\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}.$$

$a =$ radius of circle

$$\vec{r}\,'(t) = -a\sin t\,\vec{i} + a\cos t\,\vec{j}$$

$$|\vec{r}\,'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2}$$

$$= a$$

$$\Rightarrow \vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\Rightarrow \vec{T}\,'(t) = -\cos t\,\vec{i} - \sin t\,\vec{j}$$

$$\Rightarrow \boxed{K(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{1}{a}}.$$

Exercise: Check that you get same answer using alternative formula for curvature.

Normal and binormal vector to curve

First, note that $\vec{T}(t)$ and $\vec{T}'(t)$ are orthogonal, i.e.,

$$\vec{T}(t) \cdot \vec{T}'(t) = 0 \quad !$$

Proof:

$$\frac{d}{dt}|T(t)|^2 = \frac{d}{dt}\left(\vec{T}(t) \cdot \vec{T}(t)\right) = 2\,\vec{T}(t) \cdot \vec{T}'(t)$$

$$\frac{d}{dt}|T(t)|^2 = \frac{d}{dt}(1) = 0$$

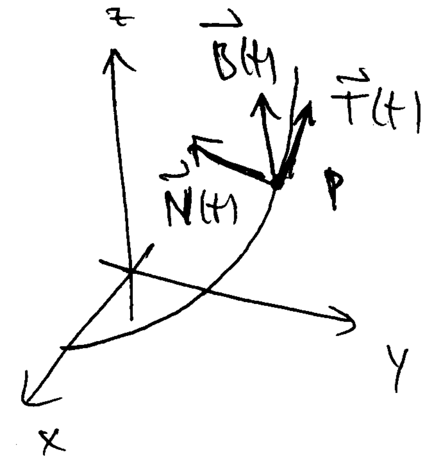
So, we define the principal unit normal vector to the curve C:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

(this vector is orthogonal $\vec{T}(t)$!)

Define the

<u>binormal vector</u> (also a unit vector)



$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

TNB frame

(Frenet frame)

The plane determined by $\vec{N}$ and $\vec{B}$ at the point P on the curve is called the <u>normal plane</u> at P.

<u>Physics : Velocity and acceleration (14.4)</u>

We think of $\vec{r}(t)$ as the location of an object.

$\vec{v}(t) = \vec{r}\,'(t)$ is the <u>velocity</u>

which has corresponding <u>speed</u>

$$|\vec{v}(t)| = |\vec{r}\,'(t)| = \frac{ds}{dt} = \frac{\Delta \text{ distance}}{\Delta \text{ time}}.$$

$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$ is the

<u>acceleration</u>.

<u>Ex.</u> Particle starts at

$\vec{r}(0) = (2, 1, 1)$ with initial velocity.

$\vec{v}(0) = 3\vec{i} - \vec{j} + 2\vec{k}$

It has acceleration

$\vec{a}(t) = 4t\,\vec{i} + 6t\,\vec{j} + \vec{k}$

What is the position $\vec{r}(t)$ and the velocity $\vec{v}(t)$ at time t?

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\,du$$

$$= (3, -1, 2) + \underbrace{\left(\int_0^t 4u\,du, \int_0^t 6u\,du, \int_0^t 1\,du\right)}_{(2t^2,\ 3t^2,\ t)}$$

$$\Rightarrow \boxed{v(t) = (3+2t^2)\vec{i} + (-1+3t^2)\vec{j} + (2+t)\vec{k}.}$$

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\, du$$

$$= \left(2 + 3t + \frac{2}{3}t^3\right)\vec{i}$$

$$+ \left(1 - t + t^3\right)\vec{j}$$

$$+ \left(1 + 2t + \frac{t^2}{2}\right)\vec{k}$$

In general:

$$\boxed{\begin{aligned} \vec{v}(t) &= \vec{v}(t_0) + \int_{t_0}^t \vec{a}(u)\, du \\ \vec{r}(t) &= \vec{r}(t_0) + \int_{t_0}^t \vec{v}(u)\, du. \end{aligned}}$$

Newton's second law:

$$\boxed{\vec{F}(t) = m\,\vec{a}(t) = m\,\vec{r}\,''(t)}$$

Ex. An object with mass m revolves about the origin with constant speed.