Power series centered at a:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

How to represent functions as power series?

→ Use known power series representation and substitution (for special examples):

Ex. Find power series representation of

$$\frac{x^3}{2+x}$$

1) Write $\frac{1}{2}x^3 \cdot \left(\frac{1}{1+\frac{x}{2}}\right) = \frac{1}{2}x^3 \cdot \left(\frac{1}{1-(-\frac{x}{2})}\right)$

2) Know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$

$\Rightarrow \frac{1}{1-(-\frac{x}{2})} = \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$ for $\left|-\frac{x}{2}\right|<1$

i.e, $|x|<2$

3) Put everything together:

$$\frac{x^3}{2+x} = \frac{1}{2}x^3 \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} \text{ for } |x|<2}$$

Differentiation and integration of power series

Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ on some interval centered at a.

$$\frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$\stackrel{?}{=} \sum_{n=0}^{\infty} \frac{d}{dx}\left(c_n (x-a)^n\right) \quad ?$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n \,dx \quad ?$$

<u>Thm.</u> If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$, then

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$
$$= \sum_{n=0}^{\infty} c_n (x-a)^n$$

is continuous and differentiable on $(a-R, a+R)$ with

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$

$$= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

(ii) $$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

Both series in (i) and (ii) have radius of convergence R.

<u>Remark</u>: Radius of convergence stays the same, but interval of convergence may change!

<u>Ex.</u> Find power series representation for $\frac{1}{(1-x)^2}$.

1) Know $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$

and

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x|<1$$

$$= 1 + x + x^2 + x^3 + \cdots$$

2) So, $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$

$$= \sum_{n=0}^{\infty} \frac{d}{dx} x^n$$

$$= \sum_{n=1}^{\infty} n x^{n-1}$$

for $|x| < 1$.

<u>Ex.</u> Find power series representation for $-\ln(1-x)$.

1) Know $-\ln(1-x) = \int \frac{1}{1-x}\, dx$

and

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x|<1.$$

$$= 1 + x + x^2 + x^3 + \cdots$$

$$\Rightarrow \quad -\ln(1-x) = \int\left(\sum_{n=0}^{\infty} x^n\right) dx$$

$$= \sum_{n=0}^{\infty} \left(\int x^n\, dx\right)$$

$\int(1 + x + x^2 + \cdots)$

$= x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$= C + \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for } |x|<1$$

2) Take $x = 0$

$$0 = -\ln(1) = C + 0$$

$$\Rightarrow \quad C = 0$$

$$\Rightarrow \quad \boxed{-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for } |x| < 1}$$

Bonus: What is $-\ln\left(\frac{1}{2}\right)$?

$$\Rightarrow \quad -\ln\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n}$$

$$= \frac{1}{2} + \frac{1}{4 \cdot 2} + \frac{1}{8 \cdot 3} + \frac{1}{16 \cdot 4} + \cdots$$

What to do in general?

Taylor and Maclaurin series (12.10)

Suppose
$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$
$$= \sum_{n=0}^{\infty} c_n (x-a)^n$$
for $|x-a| < R$.

Main question: What are c_n?

$\boxed{n=0:}$
$$f(a) = c_0 + c_1\cancelto{0}{(a-a)} + c_2\cancelto{0}{(a-a)^2} + \cancelto{0}{\dots}$$
$$\Rightarrow \boxed{c_0 = f(a)}$$

$\boxed{n=1:}$
$$f'(x) = 0 + c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$
$$f'(a) = c_1 + 0 + 0 + \dots$$
$$\Rightarrow \boxed{c_1 = f'(a)}$$

$\boxed{n=2:}$ $f''(x) = 2c_2 + 3\cdot 2\, c_3 (x-a)$

$+ 4\cdot 3\, c_4 (x-a)^2$

$+ \cdots$

$f''(a) = 2c_2 + 0 + 0 + \cdots$

$$\Rightarrow \boxed{c_2 = \frac{f''(a)}{2}}$$

In general, nth derivative of f

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

Thm. (Taylor series of f at a)

If f has a power series representation (expansion) at a with radius of convergence R, then it must be

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

for $|x-a| < R$.

When $a=0$, this is called a <u>Maclaurin series</u>.

<u>Ex.</u> Suppose that $f(x) = e^x$ has a power series representation at 0. What is it?

$$f(x) = e^x \qquad\qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f^{(n)}(x) = e^x$$

$$\Rightarrow f^{(n)}(0) = e^0 = 1.$$

$$\Rightarrow \quad f(x) = e^x = \underbrace{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

where does it converge?

<u>Ratio test</u>: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0$$

$$\Rightarrow \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x$$

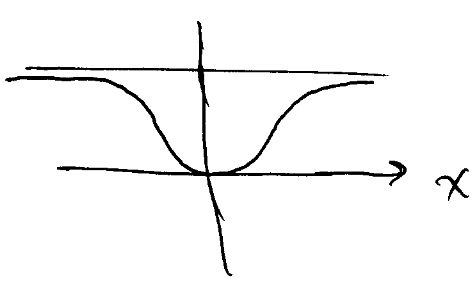
$$(\text{i.e., } R = +\infty)$$

if e^x has a power series representation at a.

<u>Warning</u>: Theorem only says that when f can be represented by a power series at a, it must be equal to the sum of its Taylor series.

Doesn't tell you f has power series representation!

<u>Ex.</u> $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$



$f^{(n)}(0)$?

Question: How to determine if function has power series representation?

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= \lim_{n\to\infty} \underbrace{\sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i}_{T_n(x)}$$

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

nth degree polynomial.

$R_n(x) = f(x) - T_n(x)$ be the remainder. ~~of the Taylor series~~

Thm. If $f(x) = T_n(x) + R_n(x)$

and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a|<R$,

then f has power series representation at a and is equal to the sum of its Taylor series for $|x-a|<R$.

How to show $\lim_{n\to\infty} R_n(x) = 0$?

Usually can try to use Taylor's inequality:

If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$

then $$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for $|x-a| \leq d$

Ex. e^x, $\sin x$, $\cos x$, ...

If f has a power series representation at a, it <u>must</u> <u>be</u> the Taylor series (called Maclaurin series if $a=0$)

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n .$$ with radius of convergence R.

(1) Find nth-degree Taylor polynomial $T_n(x)$, where $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$

(2a) To quantify error between f and T_n we look at the remainder

$$R_n(x) = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(x) - T_n(x)$$

for $|x-a| < R$.

(2b) To <u>prove</u> $f(x) = \lim_{n\to\infty} T_n(x)$ we must show that $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$.

Usually done by using Taylor's inequality:

$$\boxed{\text{Show } |f^{(n+1)}(x)| \leq M \text{ for } |x-a| < d.}$$

$$\Rightarrow \quad |R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| < d$$

$$\longrightarrow 0 \quad \text{as} \quad n \to \infty$$

(since $\sum \frac{b^n}{n!}$ converges for any constant b

$$\Rightarrow \quad \frac{b^n}{n!} \longrightarrow 0 \quad \text{as} \quad n \to \infty \;)$$

Ex. 1) Find Maclaurin series of $f(x) = \sin x$ and its radius of convergence

2) Prove that $\sin x$ equals its Maclaurin series within radius of convergence.

1) $a = 0$.

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

Maclaurin series for $\sin(x)$:

$$0 + \frac{1}{1!}x + 0 + \frac{-1}{3!}x^3$$

$$+ 0 + \frac{1}{5!}x^5 + 0 + \frac{-1}{7!}x^7 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \text{for all } x$$

<u>Ratio test</u>: $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty} |x^2| \cdot \frac{1}{(2n+3)(2n+2)}$$

$$= 0 \text{ for all } x.$$

$$\Rightarrow R = +\infty.$$

2) To <u>prove</u> this equals $\sin(x)$, use Taylor's ineq.

$$f^{(n+1)}(x) = \frac{d^{n+1}}{dx^{n+1}} \sin x = \begin{cases} \pm \sin x \\ \pm \cos x \end{cases}$$

So, $|f^{(n+1)}(x)| \leq \underbrace{1}_{M}$ for all x.

$$\Rightarrow \quad |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \quad \text{for all } x$$

$$\longrightarrow 0 \quad \text{as } n \to \infty.$$

So, $f(x) = \sin(x) = \lim_{n\to\infty} T_n(x) =$

$$\boxed{\begin{aligned} \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x. \end{aligned}}$$

<u>Ex.</u> 1) Find Maclaurin series for $\cos x$ and show that these are equal.

By differentiating both sides of box above,

$$\boxed{\begin{aligned} \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \end{aligned}}$$

2) Find an nth order polynomial that approximates $\cos x$ in the interval $(-1,1)$ with error less than 10^{-3}.

Know $|f^{(n+1)}(x)| \leq 1$ for all x M

Taylor ineq.: $|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1} \leq \frac{1}{(n+1)!}$

Find smallest n s.t. $|R_n(x)| \leq 10^{-3}$ for $-1<x<1$.

$\Rightarrow \frac{1}{(n+1)!} \leq 10^{-3} \Rightarrow n \geq 7$.

$$\boxed{\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cancel{0x^7}}$$

for $-1<x<1$, with error $\leq 10^{-3}$

$R_n(x) = f(x) - T_n(x)$

Ex. $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Find Maclaurin series.

$$f'(x) = \frac{2e^{-1/x^2}}{x^3}$$

$$f''(x) = \frac{2}{x^3}\left(\frac{2e^{-1/x^2}}{x^3}\right) + e^{-1/x^2}\left(\frac{-6}{x^4}\right)$$

$$= e^{-1/x^2}\left(\frac{4}{x^6} - \frac{6}{x^4}\right)$$

nth derivative looks like $\frac{e^{-1/x^2}}{x^k}$ for some k.

$\Rightarrow f^{(n)}(0) = 0$ for all n.

$\Rightarrow$ Maclaurin series is

$$0 + 0 + 0 + \cdots = 0.$$

f is not 0 in any neighborhood of $x=0$

i.e, $|R_n(x)| = e^{-1/x^2} \not\to 0$ as $n \to \infty$.

Multiplication and division of Taylor series

If $f(x)$ and $g(x)$ are equal to their Taylor series in $|x-a|<R$, then

(i) $$f(x)\,g(x) = \left(f(a) + \frac{f'(a)}{1!}x + \cdots\right) \cdot \left(g(a) + \frac{g'(a)}{1!}x + \cdots\right)$$

for $|x-a|<R$.

(ii) Similarly for $\dfrac{f(x)}{g(x)}$ provided $g(a)\neq 0$ and $|x-a|$ small ~~small~~ enough.

Ex. Evaluate $\int_0^1 e^x \cos x \, dx$ using a fourth-order expansion for the integrand.

$$e^x \cos x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

$$\approx 1 + \frac{x}{1!} + \left(\frac{x^2}{2!} - \frac{x^2}{2!}\right) + \left(\frac{x^3}{3!} - \frac{x^3}{2!}\right)$$

$$+\left(\frac{x^4}{4!}+\frac{x^4}{4!}-\frac{x^4}{2!2!}\right)$$

$$= 1 + x - \frac{x^3}{3} + \frac{x^4}{6}$$

(to do this, we multiplied the fourth-order Taylor polynomials of e^x and $\cos x$ and then kept only terms to fourth order).

$$\Rightarrow \int_0^1 e^x \cos x \, dx \approx \int_0^1 \left(1 + x - \frac{x^3}{3} + \frac{x^4}{6}\right) dx$$

$$= x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^5}{30} \Big]_0^1$$

$$= 1 + \frac{1}{2} - \frac{1}{12} - \frac{1}{30}$$

$$= \boxed{1.383}$$

This is a good approx to the true answer, which is 1.378...

List of important Taylor expansions (at a, with radius of convergence R):

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \quad a=0, \ R=1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad a=0, \ R=+\infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \quad a=0, \ R=+\infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \quad a=0, \ R=+\infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \quad a=0, \ R=1.$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots, \quad a=0, \ R=1$$

$\binom{k}{n}$ is called binomial coefficient.

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad a=0, \ R=1.$$