09/30/10

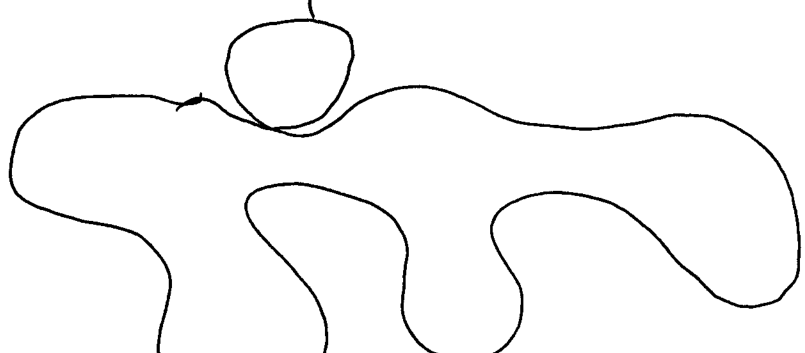
1

Parametric equations (11.1)

How can we write down equations to express curves?

For curves in x-y plane that pass the vertical line test (vert. lines intersect curve <u>exactly</u> once), we can express by a function:

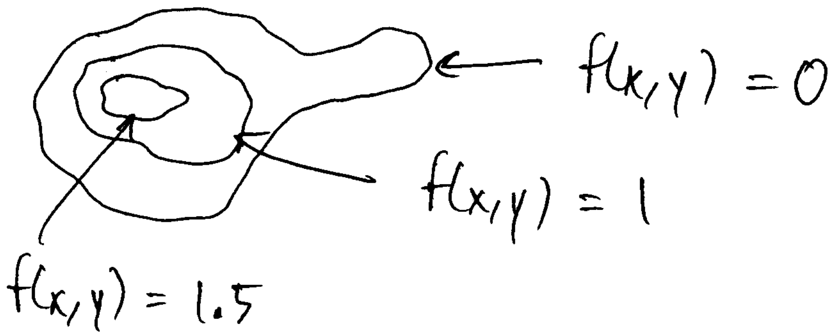
For an arbitrary curve:



<u>One way</u>: Find function $f(x,y)$ s.t.

$$\text{curve} = \{(x,y) : f(x,y) \equiv 0\}.$$

Like a topographical map.



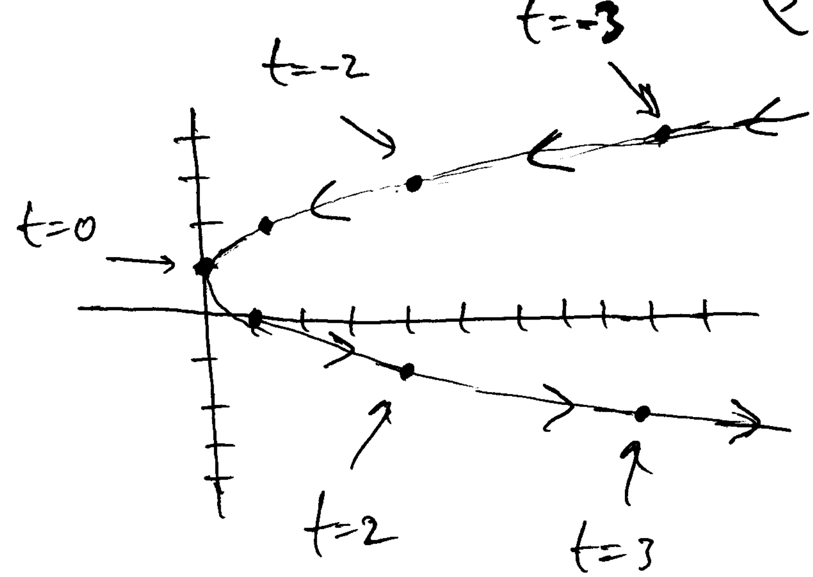
Instead, we consider the following. Suppose x, y are functions of some parameter t (think of as time):

$$x = f(t), \quad y = g(t).$$

These are called parametric eqn's.
For each t, we get a point $(x,y) = (f(t), g(t))$ in the x-y plane.
This traces out a curve (called the parametric curve) as we vary t.

Ex. $\begin{cases} x = t^2 \\ y = 1-t \end{cases} \quad -\infty \leq t \leq \infty.$

t	x	y
-3	9	4
-2	4	3
-1	1	2
0	0	1
1	1	0
2	4	-1
3	9	-2



To show that this is indeed a parabola, eliminate t from the parametric eqn's.

$$\begin{cases} x = t^2 \\ y = 1-t \end{cases}$$

$y = 1-t \Rightarrow t = 1-y$

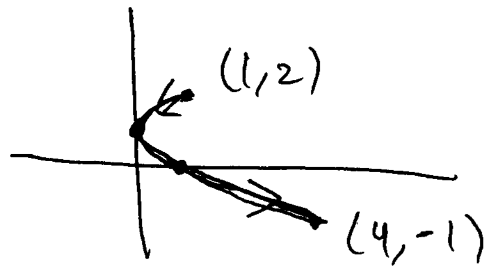
$x = t^2 \Rightarrow \boxed{x = (1-y)^2}$

eqn. for graph in picture.

<u>Note</u>: Eliminating t from parametric eqn's also eliminates our knowledge of "when" we visited a point on the curve, the direction we moved in, our "speed", etc.

If we restricted t to be in a finite interval, we only see part of the curve above:

$-1 \le t \le 2$



If $a \le t \le b$

Initial point: $(f(a), g(a))$

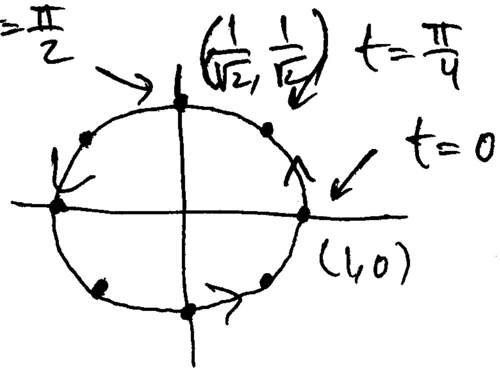
Terminal point: $(f(b), g(b))$.

(i) to sketch parametric curves, plug in increasing values of t and plot points

(ii) eliminate t from parametric eqn's to express curve as eqn. in x, y.

Ex. $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$, $\quad 0 \le t \le 2\pi$

(i)



(ii) $\cos^2 t + \sin^2 t = 1$

$\Rightarrow \boxed{x^2 + y^2 = 1}$

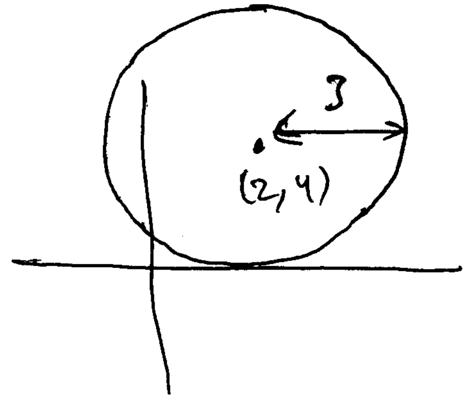
eqn. for circle centered at origin w/ radius 1.

What if $\begin{cases} x = \cos 2t \\ y = \sin 2t \end{cases}$, $0 \le t \le 2\pi$

(ii) $\cos^2(2t) + \sin^2(2t) = 1$

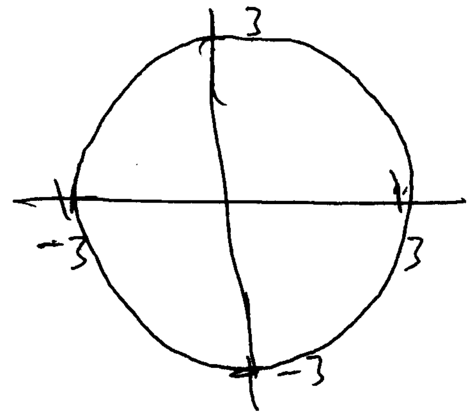
$\Rightarrow \boxed{x^2 + y^2 = 1}$

<u>Ex.</u> Given the circle w/ radius 3 centered at (2, 4), find parametric eqn's which trace out this graph.



$$\begin{cases} x = 2 + 3\cos t \\ y = 4 + 3\sin t \end{cases}$$

$$0 \le t \le 2\pi$$



$$\begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\Rightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\Rightarrow \boxed{x^2 + y^2 = 3^2}$$

$$\cos t = \frac{x-2}{3} \qquad \Rightarrow \left(\frac{x-2}{3}\right)^2 + \left(\frac{y-4}{3}\right)^2 = 1$$

$$\sin t = \frac{y-4}{3} \qquad \Rightarrow \underline{(x-2)^2 + (y-4)^2 = 3^2}$$

Calculus w/ parametric curves (11.2)

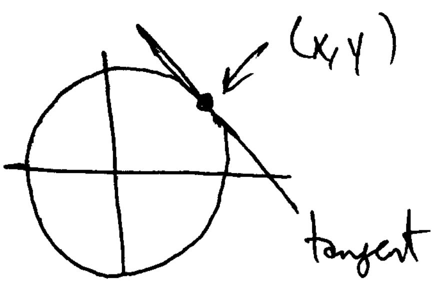
How to find things like area under curve, tangent to curve at a point, arc length, etc.?

Tangent to a curve:

Suppose $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ can be expressed

as $y = F(x)$ by eliminating the parameter t. (only need f to have a continuous derivative which is nonzero).

What is slope of tangent line at a particular point (x,y) on curve?



$$\underline{y = F(x)}$$

$$\Rightarrow g(t) = F(f(t))$$

by plugging in parametric eqns.

$\Rightarrow \quad g'(t) = F'(f(t))\, f'(t)$

by chain rule.

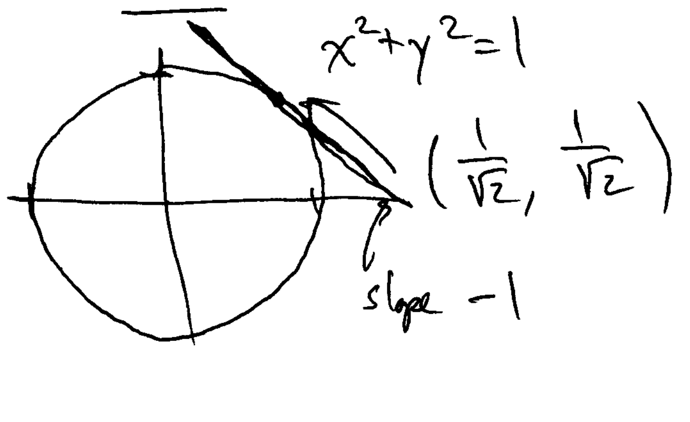
$$\Rightarrow \quad F'(f(t)) = \frac{g'(t)}{f'(t)} \quad \text{if} \quad f'(t) \neq 0.$$

$F'(x)$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0}$$

i.e., can find tangent to curve traced from parametric eqn's from derivatives of the parametric eqn's themselves.

Ex.



Find slope of tangent line to $x^2+y^2=1$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}, \quad 0 \le t \le 2\pi .$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{\cos t}{\sin t}, \quad \sin t \neq 0 .$$

$$\frac{dy}{dt} = \cos t, \quad \frac{dx}{dt} = -\sin t$$

When $t = \frac{\pi}{4}$, $(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

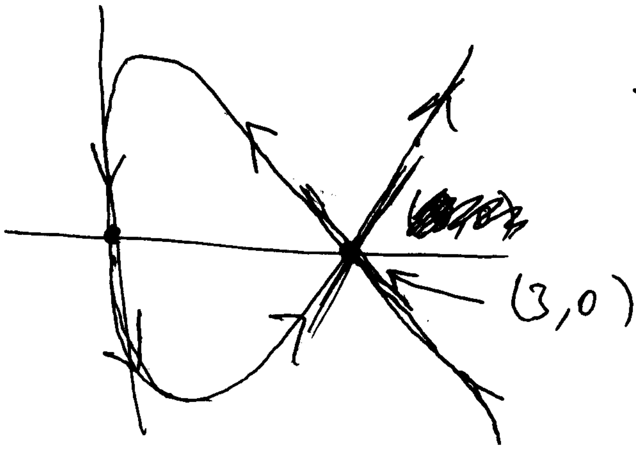
$\Rightarrow$ slope of tangent line at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$= \frac{dy}{dx} = \frac{-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = -1$$

Note: If $\frac{dy}{dt} = 0$, this corresponds to horizontal tangent

If $\frac{dx}{dt} = 0$, vertical tangent

What if curve crosses itself?
(i.e, has multiple tangent lines at a point)



Ex. $\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}$

At $(3,0)$ when $t = \pm\sqrt{3}$.

So, since

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

Plug in $t = \pm\sqrt{3}$:

1) $t = -\sqrt{3} \Rightarrow \frac{dy}{dx} = \frac{3}{2}\left(-\sqrt{3} + \frac{1}{\sqrt{3}}\right)$
$= -\sqrt{3}$.

2) $t = +\sqrt{3} \Rightarrow \frac{dy}{dx} = \frac{3}{2}\left(+\sqrt{3} - \frac{1}{\sqrt{3}}\right)$
$= \sqrt{3}$.

Second derivative ~~[illegible]~~
(to determine convexity of curve at point):

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Plug in $\frac{dy}{dx}$ for y in above eqn:

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}, \quad \text{for } \frac{dx}{dt} \neq 0}$$

<u>Ex.</u> For previous ex., had that

$$\frac{dy}{dx} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}\left(1 + \frac{1}{t^2}\right)}{2t}$$

$\boxed{10/05/10}$

We are discussing parametric eqn's

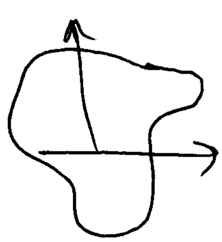
$t \mapsto (x(t), y(t))$ which describes curves in the plane.

"maps to"

<u>Motivation</u>: Gives us a way of describing more general curves than those given by graph of function $y = F(x)$.

Note that $y = F(x)$ can be written parametrically as
$$\begin{cases} x = t \\ y = F(t) \end{cases}$$

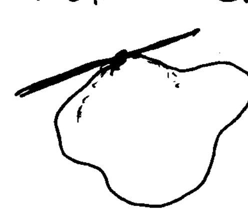
(i) parametric curves C $\longrightarrow$ parametric eqn's $(x(t), y(t))$



C

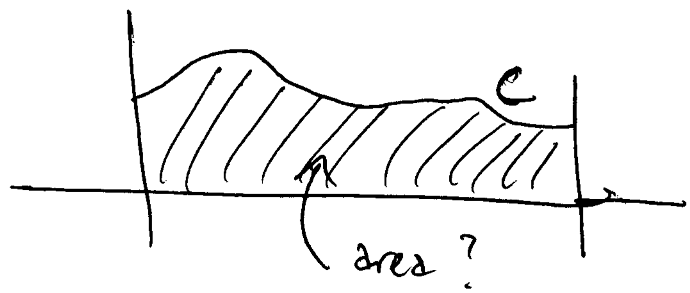
$\longleftarrow$ eliminate t from eqn's to get relationship between x, y.

(ii) tangents, concavity of parametric curves

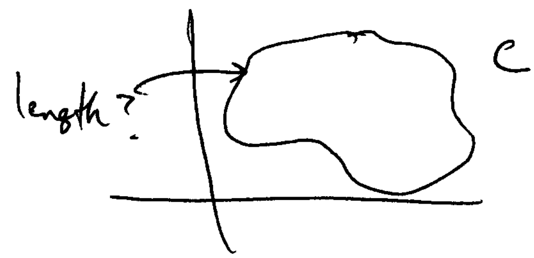


C

(iii) areas under graphs represented by parametric eqn's.

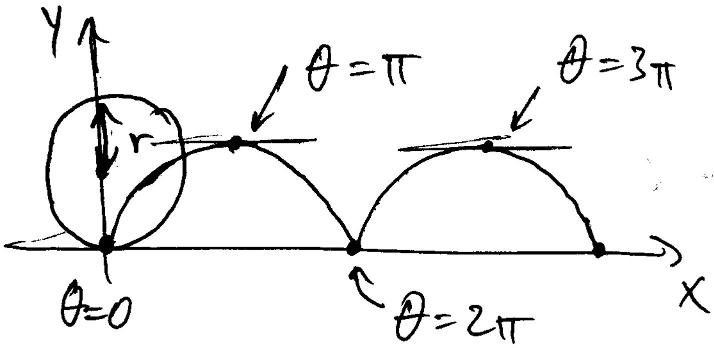


(iv) arc length of parametric curve



similar to what is done for graphs of functions $y = F(x)$.

As a common example for (i) - (iv) we look at curve called cycloid:



$$x = r(\theta - \sin\theta)$$
$$y = r(1 - \cos\theta)$$

Here, r is a constant, and θ is the parameter (plays role of t)

(i) This is an example where it is much easier to discuss parametrically than as the graph of a function.

(ii) Slope of tangent line:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\sin\theta}{r(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

- Horizontal tangents when $\frac{dy}{dx} = 0$,

  i.e, $\sin\theta = 0$ and $1-\cos\theta \neq 0$

  $\Rightarrow \theta = \pi, 3\pi, 5\pi, \ldots$

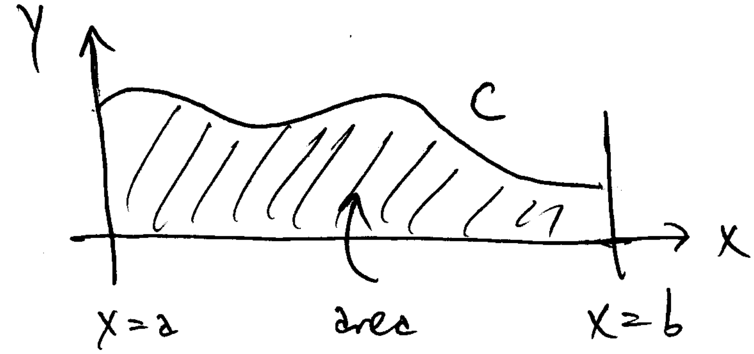
- Vertical tangents when $\frac{dy}{dx} = \pm\infty$,

  i.e, and $1-\cos\theta = 0$

  $\Rightarrow \theta = 0, 2\pi, 4\pi, \ldots$

Concavity: Can verify that $\frac{d^2y}{dx^2} \leq 0$ for all θ.

Areas



For functions $y = F(x)$, area is

$$A = \int_a^b y \, dx .$$

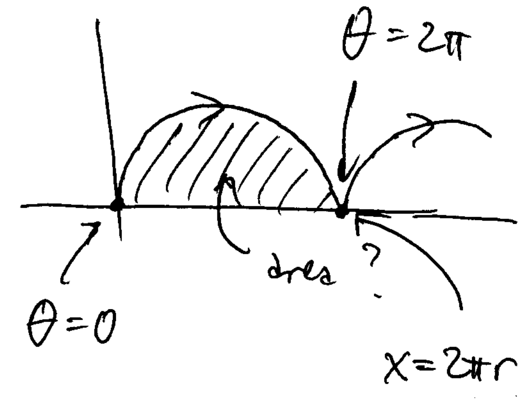
Suppose $y = F(x)$ is represented by parametric eqn's $x = f(t)$, $y = g(t)$.
for $\alpha \le t \le \beta$ s.t. $a = f(\alpha)$
$b = f(\beta)$

By substitution,

$$\boxed{A = \int_a^b y \, \frac{dx}{dt} \, dt = \int_\alpha^\beta g(t) \, f'(t) \, dt}$$

Ex. (Cycloid)

$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$$



(iii) $A = \int_0^{2\pi r} y \, dx$

$$= \int_0^{2\pi} \underbrace{r(1-\cos\theta)}_{g(\theta)} \overbrace{r(1-\cos\theta)}^{f'(\theta)} \, d\theta$$

$$= r^2 \int_0^{2\pi} (1-\cos\theta)^2 \, d\theta$$

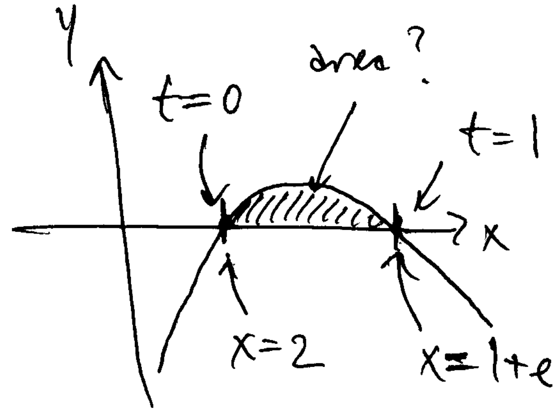
$$= r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= r^2 \int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right) d\theta$$

$$\vdots$$

$$= \boxed{3\pi r^2}.$$



Ex. $x = 1 + e^t$
$y = t - t^2$

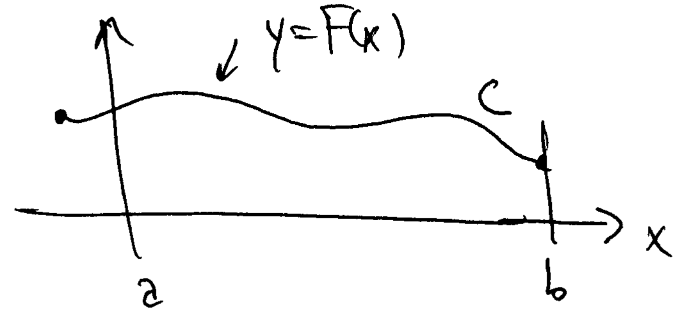
$$A = \int_2^{1+e} y \, dx = \int_0^1 (t - t^2) e^t \, dt$$

$$= 3 - e.$$

← use integration by parts.

Arc length

First consider arc length formula for curve C given by $y = F(x)$:
(with F differentiable)



$$L_C = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Now suppose that C is traced out <u>once</u> by parametric eqns' $x = f(t)$, $y = g(t)$ s.t. $\alpha \le t \le \beta$ and $a = f(\alpha)$, $b = f(\beta)$.

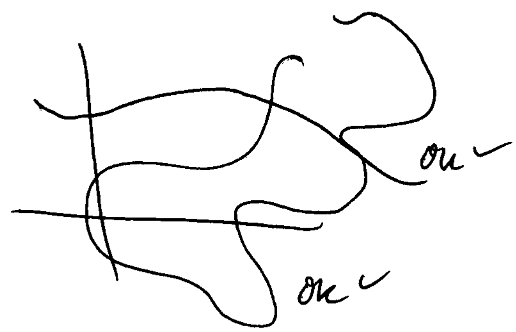
$$L_C = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

$$= \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

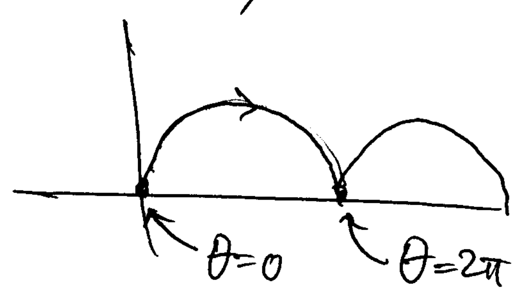
$$\Rightarrow \boxed{L_C = \int_{\alpha}^{\beta} \sqrt{(f'(t))^2 + (g'(t))^2}\, dt}$$

In fact, true for any parametric curve C, not just ones that are graphs over x-axis.



Ex. (Cycloid)

(iv) Find arc length of one arch, i.e, $0 \leq \theta \leq 2\pi$.



$$x = r(\theta - \sin\theta) = f(\theta)$$
$$y = r(1 - \cos\theta) = g(\theta)$$

$$f'(\theta) = r(1 - \cos\theta), \quad g'(\theta) = r\sin\theta$$

$$\Rightarrow \quad L_c = \int_0^{2\pi} \sqrt{(f'(\theta))^2 + (g'(\theta))^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

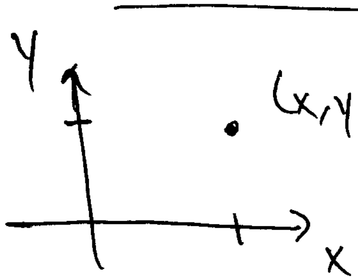
$$= r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

$$\left(\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}\right)$$

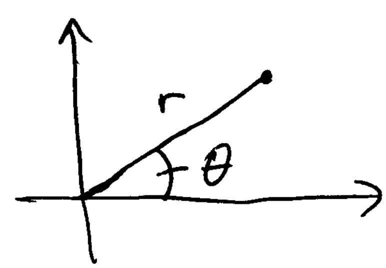
$$= 2r\int_0^{2\pi} \left|\sin\frac{\theta}{2}\right| d\theta$$

$$= 8r.$$

Polar coordinates (11.3)

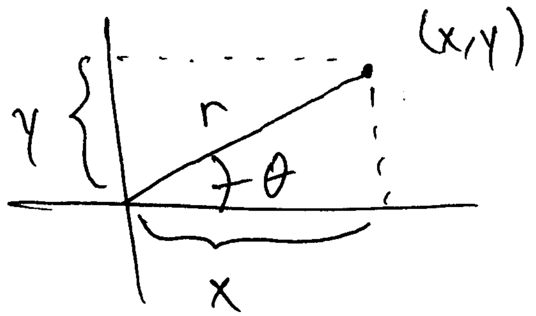


Cartesian coordinates



Polar coordinates

How to go from $(x,y) \Longleftrightarrow (r,\theta)$



$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

- For each (r,θ) there is a unique (x,y).
- For each (x,y) there are many choices for r and θ that give the same coordinate since

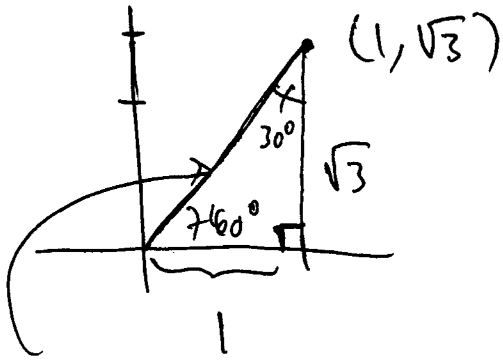
$$(r,\theta) = (r, \theta + 2n\pi) \quad \text{for any integer } n$$

$$(r,\theta) = (-r, \theta + (2n-1)\pi)$$

If we restrict ourselves to $r \geq 0$ and $0 \leq \theta < 2\pi$ then the mapping is unique:

$$\boxed{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}}$$

<u>Ex.</u> $(1, \sqrt{3})$ in Cartesian $\longrightarrow$ polar?



$$\Rightarrow \boxed{\begin{aligned} r &= 2 \\ \theta &= \frac{\pi}{3} \end{aligned}}$$

Polar curves

Consider set of all points in the plane represented by (r, θ) such that the polar equation $r = f(\theta)$ is satisfied.

(more generally, $F(r, \theta) = 0$)

(i) graphing polar curves, changing to equation in Cartesian coords.

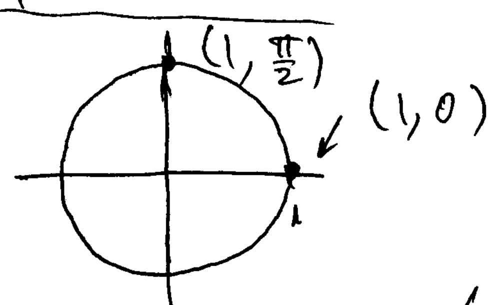
(ii) tangents to polar curves

(iii) areas

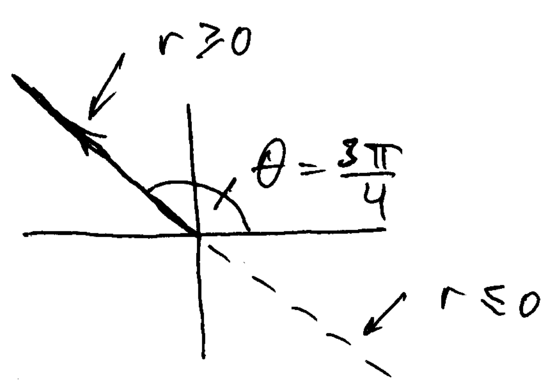
(iv) arc lengths

(i) Graphing polar curves:

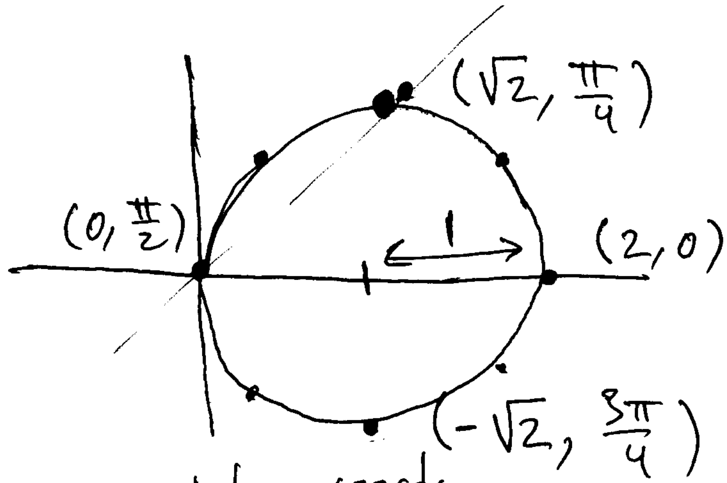
Ex. $r = 1$



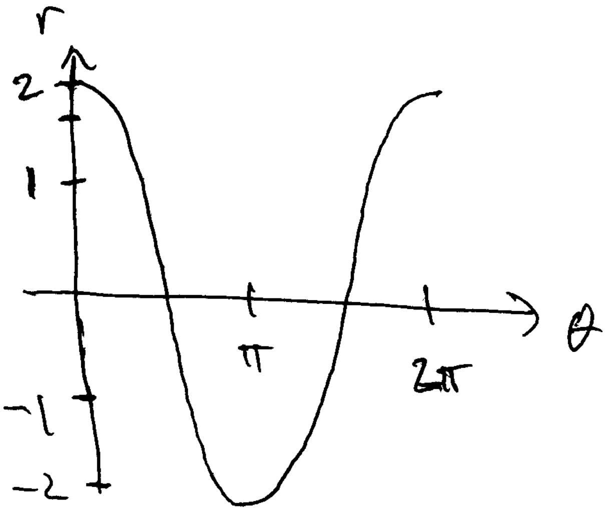
Ex. $\theta = \frac{3\pi}{4}$, $r \geq 0$



Ex. $r = 2\cos\theta$



polar coords.



Cartesian coords.

To verify this is a circle,

$$r = 2\cos\theta$$

$$\Leftrightarrow \quad r^2 = 2r\cos\theta$$

$$= 2x$$

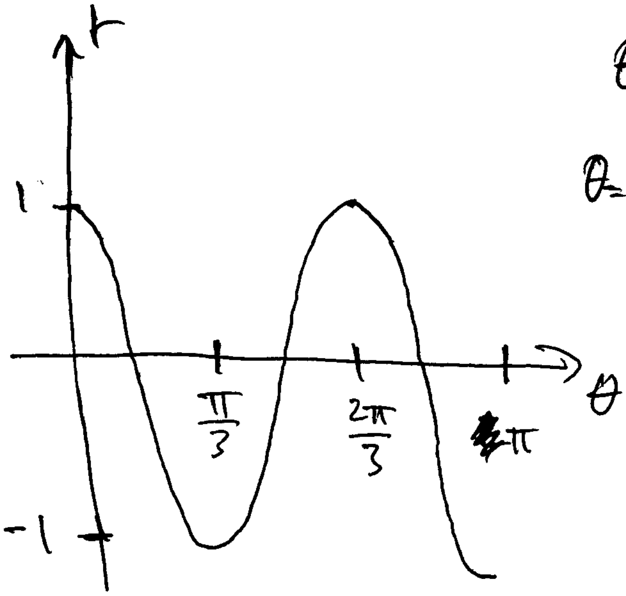
$$\Leftrightarrow \quad x^2 + y^2 = 2x$$

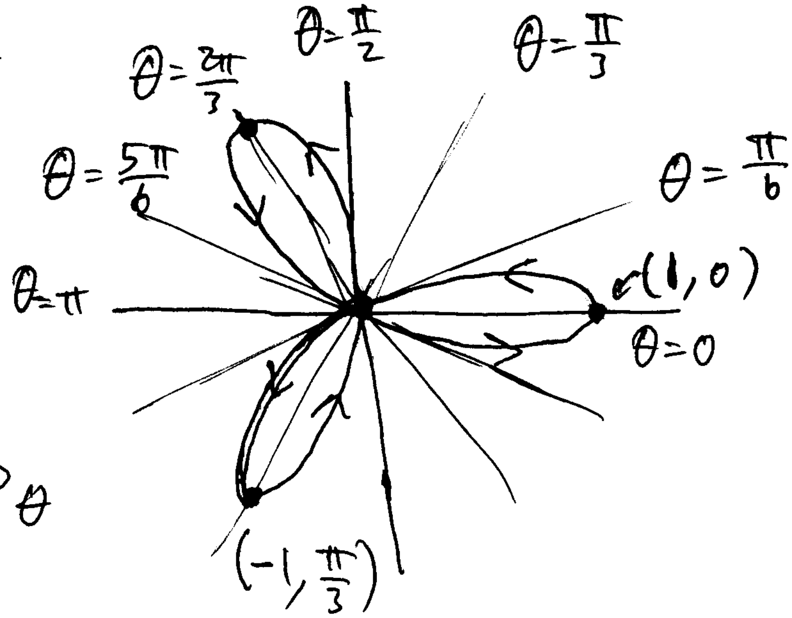
$$\Leftrightarrow \quad x^2 - 2x + 1 + y^2 = 1$$

$$\Leftrightarrow \quad \underbrace{(x-1)^2 + y^2 = 1}_{\text{circle of radius 1}}$$

centered at $(1, 0)$.

Ex. $r = \cos 3\theta$

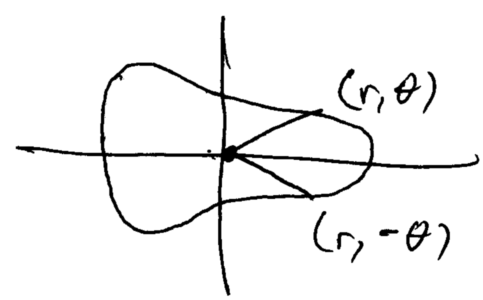




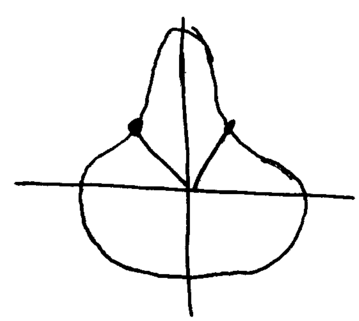
Symmetries

Polar equation $r = f(\theta)$

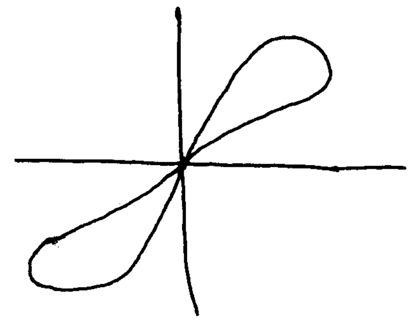
- If equation unchanged under $\theta \leftrightarrow -\theta$, symmetric across $\theta = 0$



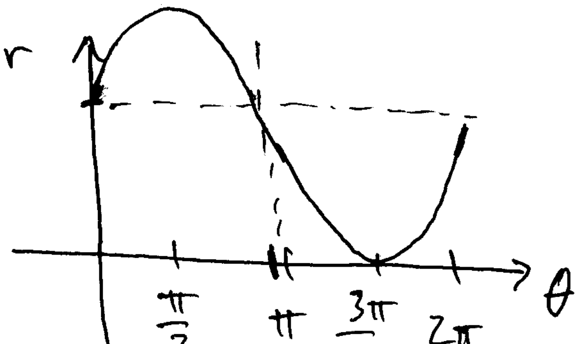
- If unchanged under $\theta \leftrightarrow \pi - \theta$, symmetric across $\theta = \frac{\pi}{2}$

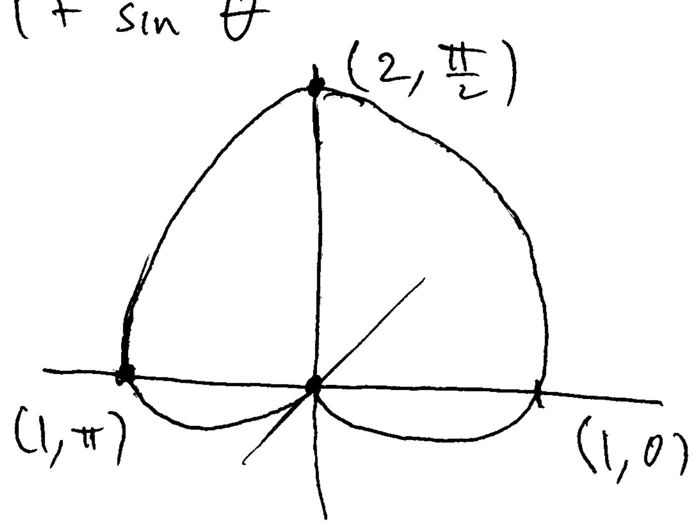


- If $r \leftrightarrow -r$ or $\theta \leftrightarrow \theta + \pi$, unchanged through 180° rotation.



<u>Ex.</u> (Cardioid) $r = 1 + \sin\theta$





(polar curve analogue of cycloid)

Tangents

Consider polar eqn. $r = f(\theta)$. What are slopes to tangent lines?

Write as parametric eqn's for x, y with parameter θ:

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$= \boxed{\frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

$\Rightarrow$ horizontal tangents when $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$.

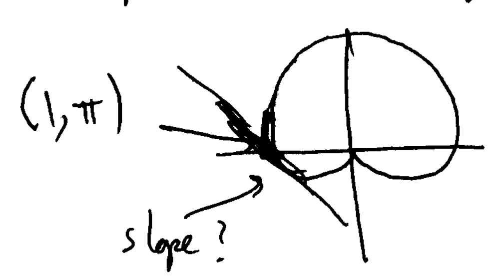
vertical tangents when $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$

Note also that formula simplifies when $r = 0$:

$$\boxed{\frac{dy}{dx} = \tan\theta \quad \text{if} \quad \frac{dr}{d\theta} \neq 0 \quad \text{at } r = 0}$$

Ex. For cardioid, find slope of tangent line at $\theta = \pi$:

$r = 1 + \sin\theta$.



$$\frac{dr}{d\theta} = \cos\theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos\theta \sin\theta + (1+\sin\theta)\cdot\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1+2\sin\theta)}{1-2\sin^2\theta - \sin\theta}$$

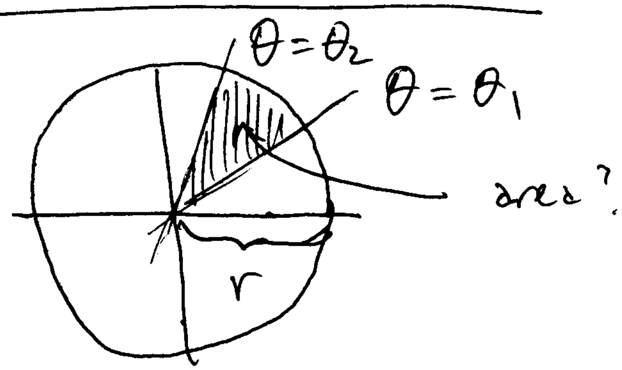
At $\theta = \pi$, this is

$$\frac{dy}{dx} = \frac{-1}{1} = -1.$$

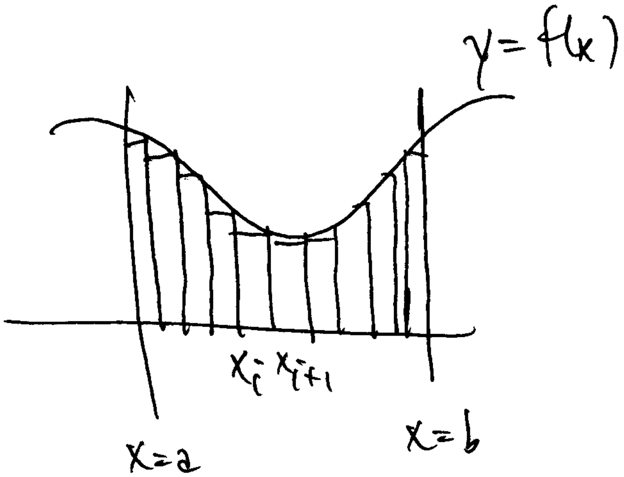
Ex. Find points w/ horizontal tangents ...

Areas and arc length (11.4)

Areas:

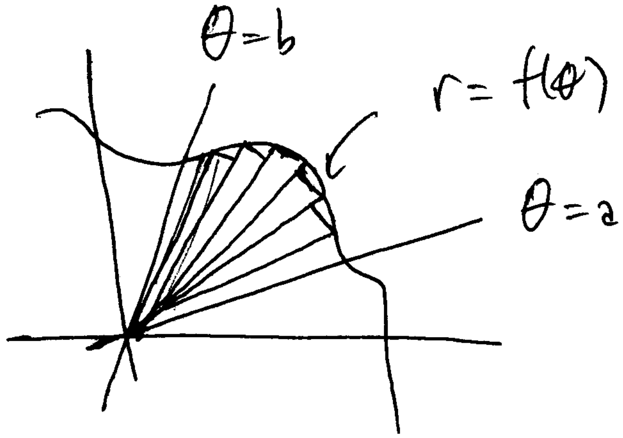


$$A = \pi r^2 \cdot \frac{(\theta_2-\theta_1)}{2\pi} = \frac{1}{2} r^2 (\theta_2-\theta_1)$$



$$A = \sum_i f(x_i)\overbrace{(x_{i+1} - x_i)}^{\Delta x_i}$$

$$\longrightarrow \int_a^b f(x)\,dx$$



$$A = \sum_i \frac{1}{2}\left(f(\theta_i)\right)^2 \overbrace{(\theta_{i+1} - \theta_i)}^{\Delta \theta_i}$$

$$\longrightarrow \int_a^b \frac{1}{2} f^2(\theta)\, d\theta .$$

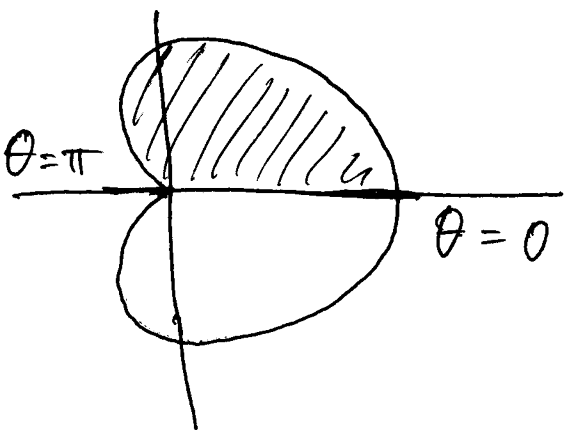
$$\boxed{A = \int_a^b \frac{1}{2} r^2\, d\theta \quad \text{where} \quad r = f(\theta)}$$

<u>Warning</u>: Be careful to choose boundaries of integration so that only sweep out area <u>once</u>!

<u>Note</u>: Will often have to use double angle formulas

$$\boxed{\begin{aligned} \cos^2\theta &= \tfrac{1}{2}(1+\cos(2\theta)) \\ \sin^2\theta &= \tfrac{1}{2}(1-\cos(2\theta)) \end{aligned}}$$

Ex. $r = 1 + \cos\theta$



$$A = \int_0^{\pi} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\pi} \frac{1}{2}(1+\cos\theta)^2 \, d\theta$$

$$= \int_0^{\pi} \frac{1}{2}(1 + 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= \int_0^{\pi} \frac{1}{2}\left(1 + 2\cos\theta + \frac{1}{2}(1+\cos(2\theta))\right) d\theta$$

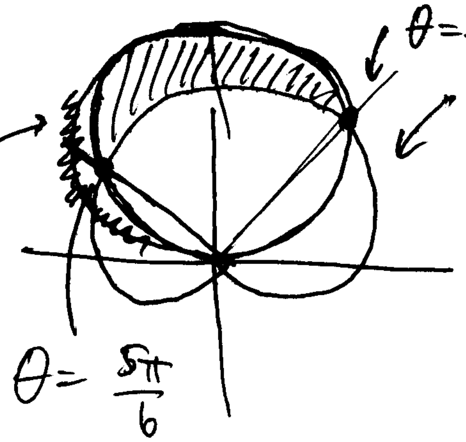
$$= \frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]_0^{\pi}$$

$$= \frac{1}{2}\left(\pi + \frac{\pi}{2}\right) = \boxed{\frac{3\pi}{4}}$$

Ex.

$r = 1 + \sin\theta = f_2(\theta)$

$\theta = \frac{\pi}{6}$



$r = 3\sin\theta$
$= f_1(\theta)$

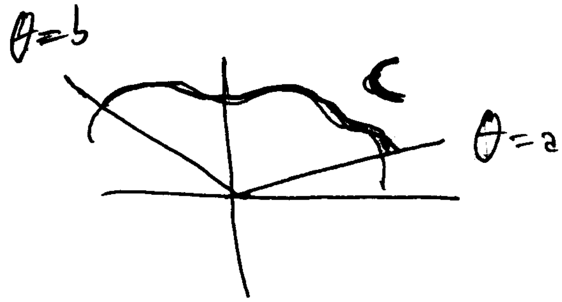
$\theta = \frac{5\pi}{6}$

$$A = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(3\sin\theta)^2\,d\theta - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(1+\sin\theta)^2\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\Big[\underbrace{(3\sin\theta)^2}_{f_1(\theta)^2} - \underbrace{(1+\sin\theta)^2}_{f_2(\theta)^2}\Big]\,d\theta$$

Not $\int \frac{1}{2}(f_1 - f_2)^2\,d\theta$!

Arc length



Again $r = f(\theta)$.

Use arc length formula for parametric eqns x, y w/ parameter θ, $a \le \theta \le b$:

$$L_C = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

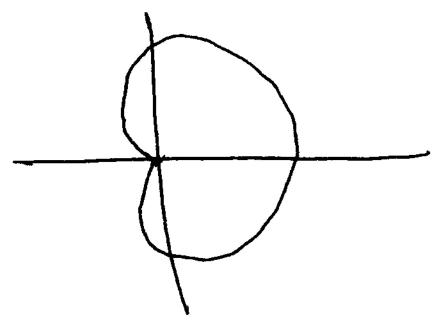
$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta.$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

$$\Rightarrow \boxed{L_c = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}$$

<u>Ex.</u> $r = 1 + \cos\theta$



$$\frac{dr}{d\theta} = -\sin\theta$$

$$L_c = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + \sin^2\theta}$$

$$= \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\; d\theta$$

$$= \boxed{8.}$$