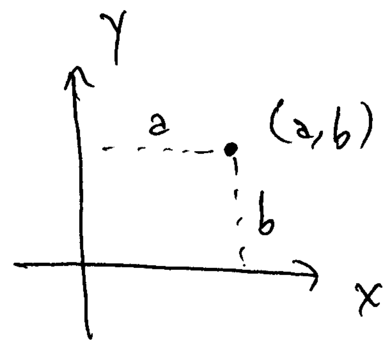
$\boxed{10/12/10}$

3-D coordinate system (13.1)

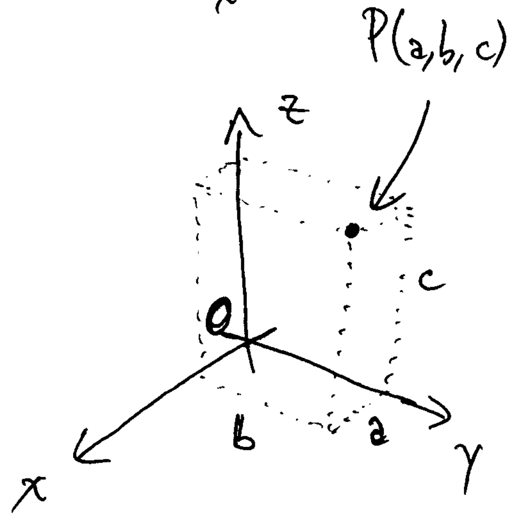
In 2-D : P(a, b)

x-coordinate   y-coordinate



In 3-D : P(a, b, c)

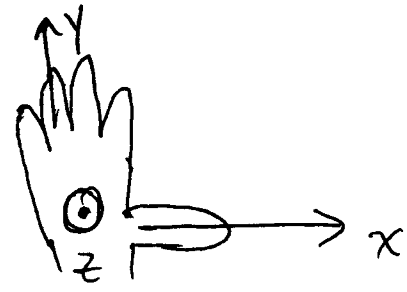
x-   y   z-coordinate



Every point in 3-space is in 1-1 correspondence with a set of coordinates (ordered triple) $(a, b, c) \in \mathbb{R}^3$

"is an element of"

For convention, z-axis is vertical and x- and y-axes parallel to floor. Regardless, if we know the direction of the x- and y-axes, we define the direction of the z-axis by the right-hand rule :



where ⊙ indicates that the z-axis is coming out of your palm.

The projection of $P(a,b,c)$ onto the

x-y axis	has coordinates	$(a, b, 0)$
x-z axis	has coordinates	$(a, 0, c)$
y-z axis	" "	$(0, b, c)$

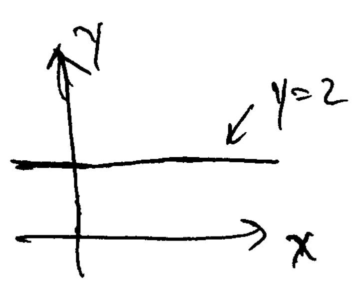
Restricting all 3 coordinates yields a point in $\mathbb{R}^3$.

Restricting only 2 coordinates yields a line in $\mathbb{R}^3$ (or a point in $\mathbb{R}^2$)

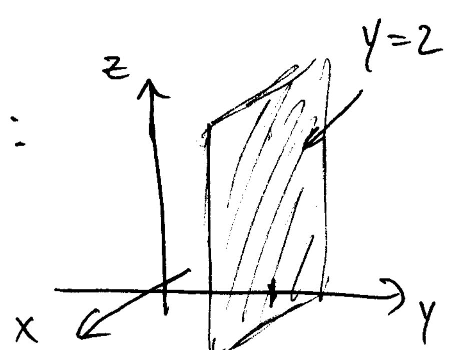
Restricting only 1 coordinate yields a plane in $\mathbb{R}^3$ (or a line in $\mathbb{R}^2$)

Ex. $y = 2$

In 2-D:

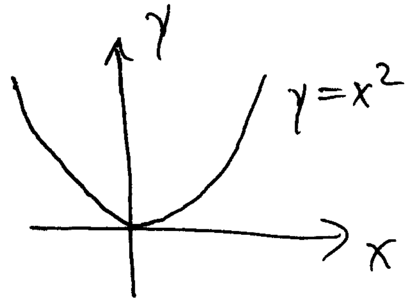


In 3-D:

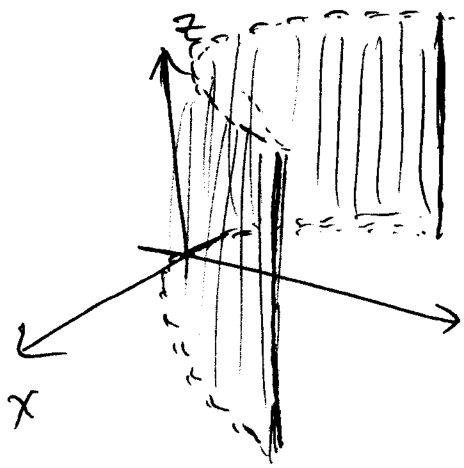


Ex: $y = x^2$

In 2-D:



In 3-D:

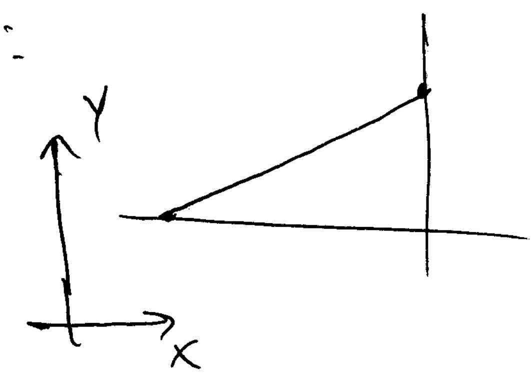


all $(a,b,c) \in \mathbb{R}^3$

s.t.

$a \in \mathbb{R},\ b = a^2,\ c \in \mathbb{R}$

In 2-D:



In 3-D: The <u>distance</u> between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

$$\boxed{|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}}$$

Generalization of Pythagorean theorem.

A <u>sphere</u> ~~the~~ with center $C(h,k,l)$ and radius r is given by all points $P(x,y,z)$ s.t. $|PC| = r$, i.e, $|PC|^2 = r^2$, i.e,

$$\boxed{(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2}$$

<u>Ex.</u> What is intersection of the sphere of radius 5 centered at the point $C(1,2,3)$ with the x-y plane?

(<u>Answer</u>: A circle in x-y plane centered $(1,2)$ w/ radius 4. Why?)

sphere $\Leftrightarrow (x-1)^2 + (y-2)^2 + (z-3)^2 = 5^2$

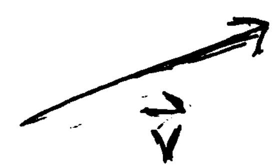
x-y plane $\Leftrightarrow z = 0$

$\Rightarrow (x-1)^2 + (y-2)^2 + (0-3)^2 = 5^2$

$$\Rightarrow \quad (x-1)^2 + (y-2)^2 = 5^2 - 3^2 = 16 = 4^2 .$$

Vectors (13.2)

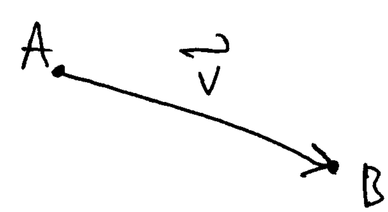
Def. A vector $\vec{v}$ is a quantity that is specified by a magnitude and a direction; usually represented by a directed arrow, e.g.

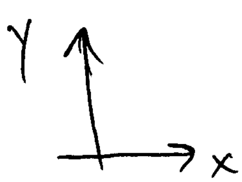


Denote the magnitude of $\vec{v}$ by $|\vec{v}|$.

To contrast, we call a quantity that only has a magnitude a scalar.

Think of $\vec{v}$ as the displacement $\overrightarrow{AB}$ between initial point A and terminal point B.



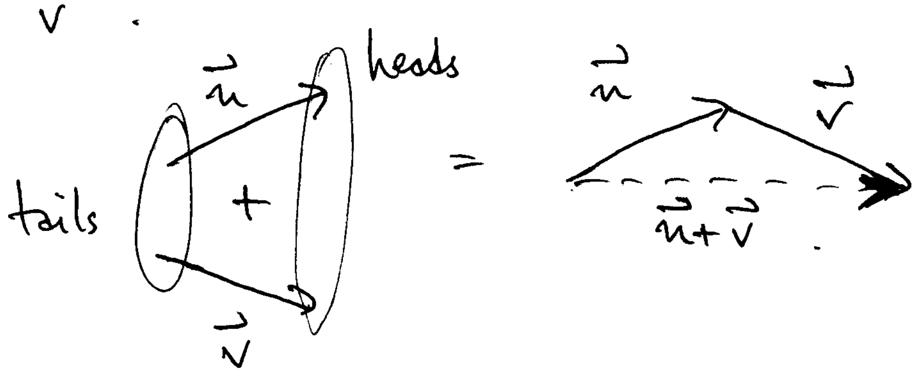


$\vec{v}$ has magnitude $|\overrightarrow{AB}|$ and direction pointing towards B. along the line $\overrightarrow{AB}$.

Note that $\vec{u}$ given by $\overrightarrow{CD}$ is equal to $\vec{v}$ because both have the same magnitude and direction.

Adding vectors

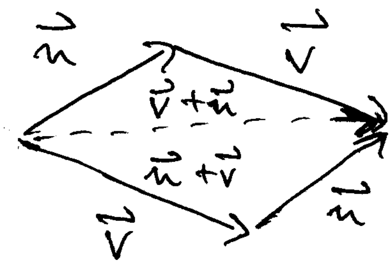
<u>Def.</u> To add vectors $\vec{u}$ and $\vec{v}$; line them up head to tail. The vector $\vec{u}+\vec{v}$ goes from the tail of $\vec{u}$ to the head of $\vec{v}$.



Referred to as <u>triangle law</u>.

Vector addition is commutative, i.e.,

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$, by parallelogram law:



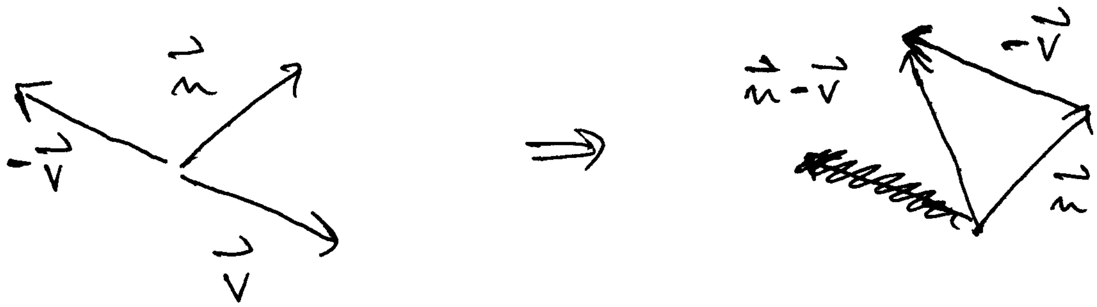
Multiplication of vector with a scalar

Leads to "stretching" and/or reflection of a vector.

Def. If $\vec{v}$ vector, c scalar, then $c\vec{v}$ is the vector with magnitude $|c||\vec{v}|$ in the same direction as $\vec{v}$ if $c>0$ and opposite to $\vec{v}$ if $c<0$. If $c=0$, or $\vec{v}=\vec{0}$ then $c\vec{v} = \vec{0}$.

Subtracting vectors:

Def. $\vec{u} - \vec{v}$ is defined as $\vec{u} + \underbrace{(-1)\vec{v}}_{-\vec{v}}$



Vector components

We will work with vectors *algebraically* by decomposing every vector into simpler parts by introducing a coordinate system.

Every vector $\vec{a}$ is the displacement between the origin (initial point) and an arbitrary terminal point $P(a_1, a_2, a_3)$ in 3-D,

i.e, $\vec{a} = \overrightarrow{OP}$.

We write $\vec{a} = (a_1, a_2, a_3)$ $\left(\begin{array}{l} \text{or } \langle a_1, a_2, a_3\rangle \\ \text{or } [a_1, a_2, a_3] \end{array}\right)$

where a_1, a_2, a_3 are real numbers and are the *components* of the vector $\vec{a}$.

<u>Magnitude</u> (or length) $|\vec{a}|$ of the vector $\vec{a}$ is the length of $|\vec{OP}|$, i.e.,

$$\boxed{|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}}$$

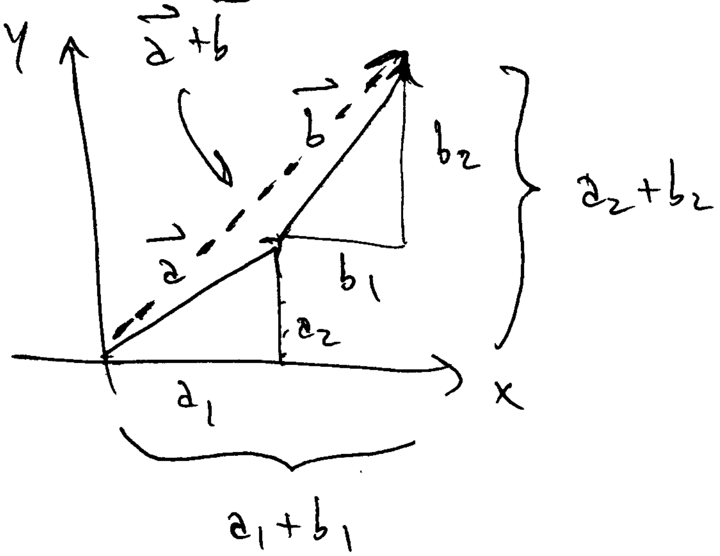
Easy to describe addition, subtraction, scalar multiplication of vectors in component form:

If $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$

i) $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

ii) $\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$

iii) $c\vec{a} = (ca_1, ca_2, ca_3)$ for any scalar c.



Properties: For vectors $\vec{a}, \vec{b}$ and scalars c, d.

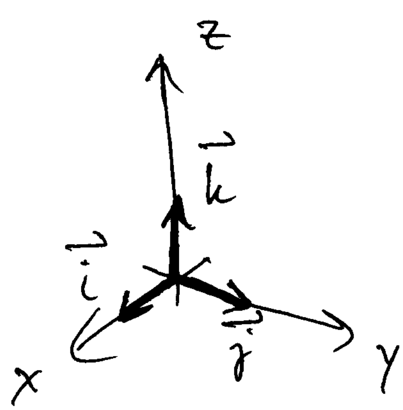
1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2) $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3) $\vec{a} + \vec{0} = \vec{a}$
4) $\vec{a} + (-\vec{a}) = \vec{0}$
5) $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$.
6) $(c+d)\vec{a} = c\vec{a} + d\vec{a}$
7) $(cd)\vec{a} = c(d\vec{a})$
8) $1\vec{a} = \vec{a}$.

Unit vectors

Let $\vec{i} = (1, 0, 0)$

$\vec{j} = (0, 1, 0)$

$\vec{k} = (0, 0, 1)$



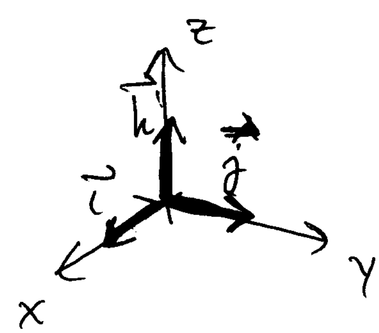
Instead of writing $\vec{a} = (a_1, a_2, a_3)$, we write
$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$.

Vectors $\vec{a}$ has a magnitude <u>and</u> direction. Given coordinate system (x, y, z - axes), we can represent $\vec{a}$ in terms of components:

$$\vec{a} = (a_1, a_2, a_3)$$

In terms of <u>standard basis vectors</u>

$$\vec{i} = (1, 0, 0)$$
$$\vec{j} = (0, 1, 0)$$
$$\vec{k} = (0, 0, 1)$$



$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}.$$

For $\vec{a} \neq \vec{0}$:

Magnitude of $\vec{a}$ is $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

<u>Unit vector</u> in direction of $\vec{a}$: $\vec{n} = \frac{1}{|\vec{a}|} \vec{a}$

vector with length 1

$$\left(|\vec{n}| = \frac{1}{|\vec{a}|} |\vec{a}| = 1 \right)$$

Operations on vectors

Addition : $\vec{a} + \vec{b} = \text{vector}$

Subtraction : $\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = \text{vector}$

Scalar multiplication : $c\,\vec{a} = \text{vector}$

c ↗ scalar

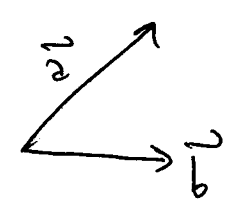
Today we'll discuss :

Dot product : $\vec{a} \cdot \vec{b} = \underline{\underline{\text{scalar}}}$

Cross product : $\vec{a} \times \vec{b} = \underline{\underline{\text{vector}}}$

Dot product (13.3

$\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$



Def. $\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$

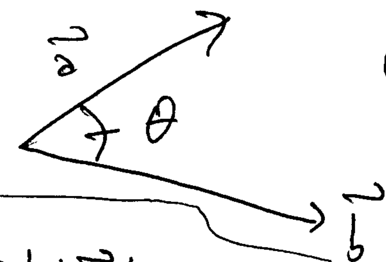
is the dot product (scalar product, inner product)

of $\vec{a}$ and $\vec{b}$.

Think of as "how much" $\vec{a}$ and $\vec{b}$ agree :

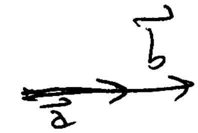
Consider the unique plane which contains $\vec{a}$ and $\vec{b}$:

$0 \leq \theta \leq \pi$



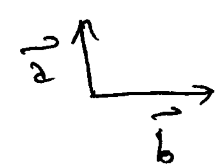
Then, $$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta}$$

Note: If $\theta = 0$, $\cos \theta = 1$



$\Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$

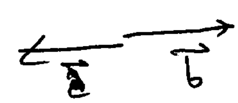
If $\theta = \frac{\pi}{2}$, $\cos \theta = 0$



$\Rightarrow \vec{a} \cdot \vec{b} = 0$

If $\theta = \pi$, $\cos \theta = -1$

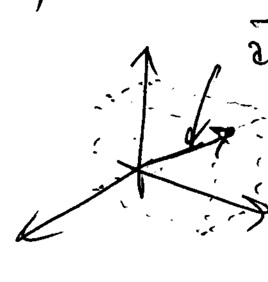
$\Rightarrow \vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$



This tells us how to find the angle between $\vec{a}$ and $\vec{b}$:

$$\boxed{\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}}$$

Ex. Find angle between $\vec{a} = (a_1, a_2, a_3)$ and the x-axis.



i.e., find angle between $\vec{a}$ and $\vec{i}$ $\Rightarrow$

$$\cos\theta = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}$$

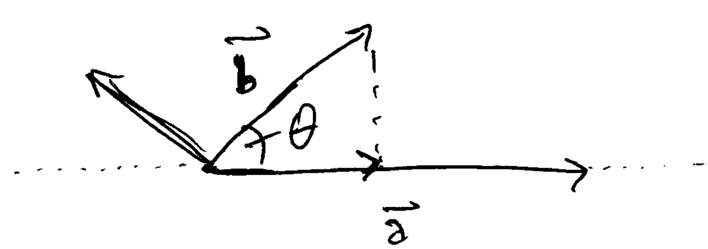
If $\theta = \frac{\pi}{2}$, the vectors are orthogonal to each other, i.e.,

$$\boxed{\vec{a}\cdot\vec{b} = 0 \iff \vec{a}, \vec{b} \text{ are orthogonal}}$$

Vector projection

What is the vector component of $\vec{b}$ that is in the direction of $\vec{a}$?

(Think of as shadow of $\vec{b}$ onto line given by $\vec{a}$)



<u>Scalar projection</u> of $\vec{b}$ onto $\vec{a}$:

(<u>scalar</u> "length" associated to shadow)

$$\text{comp}_{\vec{a}}\,\vec{b} = |\vec{b}|\cos\theta$$
$$= |\vec{b}|\,\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$
$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}.$$

<u>Vector projection</u> of $\vec{b}$ onto $\vec{a}$:

(<u>vector</u> associated to shadow)

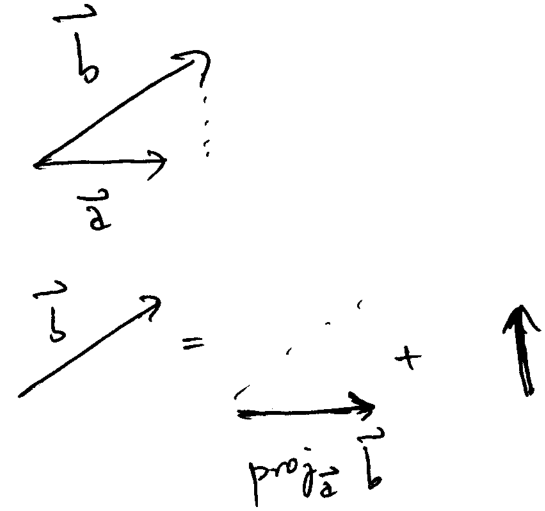
$$\text{proj}_{\vec{a}}\,\vec{b} = \left(\text{comp}_{\vec{a}}\,\vec{b}\right)\underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\substack{\text{unit vector}\\ \text{in direction of } \vec{a}}}$$
$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

<u>Note</u>: $\vec{b} = \underbrace{\text{proj}_{\vec{a}}\,\vec{b}}_{\text{parallel to } \vec{a}} + \underbrace{\left(\vec{b} - \text{proj}_{\vec{a}}\,\vec{b}\right)}_{\text{orthogonal to } \vec{a}}$

$$(\vec{b} - \text{proj}_{\vec{a}}\vec{b}) \cdot \vec{a}$$

$$= \vec{b} \cdot \vec{a} - \left(\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a} \right) \cdot \vec{a}$$

$$= \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} = 0$$



Ex. Find scalar and vector projection of $\vec{b} = (2, 2, 5)$ onto $\vec{a} = (-2, -1, 2)$

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{-4 - 2 + 10}{\sqrt{(-2)^2 + (-1)^2 + 2^2}} = \frac{4}{3}$$

$$\text{proj}_{\vec{a}}\vec{b} = (\text{comp}_{\vec{a}}\vec{b}) \frac{\vec{a}}{|\vec{a}|} = \left(\frac{4}{3}\right) \frac{1}{3} (-2, -1, 2)$$

$$= \left(-\frac{8}{9}, -\frac{4}{9}, \frac{8}{9} \right).$$

Cross product (13.4)

Only defined for 3-dimensional vectors $\vec{a}$ and $\vec{b}$.

To define, let's recall definition of determinant:

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

In 3×3 case:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Let's write $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$

$\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$

We define $\vec{a} \times \vec{b}$ by the symbolic determinant:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$\boxed{\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\vec{i} - (a_1 b_3 - a_3 b_1)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}}$$

<u>Note</u>: $\vec{a} \times \vec{b}$ is a <u>vector</u>!

Properties of cross product

(i) $\vec{a} \times \vec{a} = \vec{0}$

(ii) $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$!

To see this, note that $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$

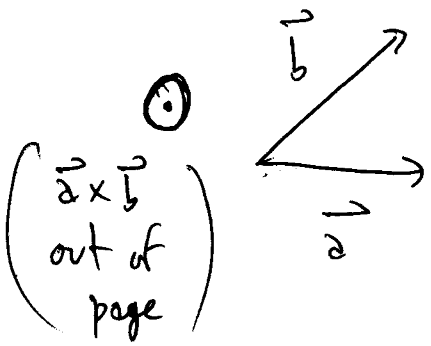
$$
\begin{aligned}
(\vec{a} \times \vec{b}) \cdot \vec{a} &= (a_2 b_3 - a_3 b_2)\, a_1 \\
&\quad - (a_1 b_3 - a_3 b_1)\, a_2 \\
&\quad + (a_1 b_2 - a_2 b_1)\, a_3 \\
&= 0
\end{aligned}
$$

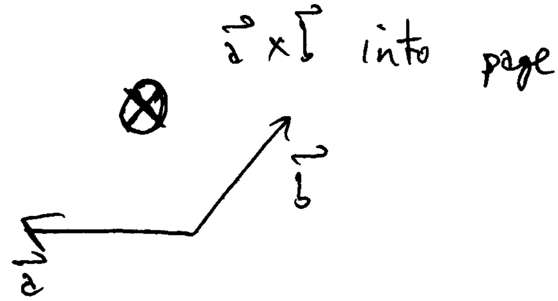
(iii) If θ is the angle $(0 \leq \theta \leq \pi)$ between $\vec{a}$ and $\vec{b}$, then

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$

<u>Note:</u> The closer θ is to $\frac{\pi}{2}$, the larger is $\sin\theta$ and the magnitude of $\vec{a} \times \vec{b}$.

Also note that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.



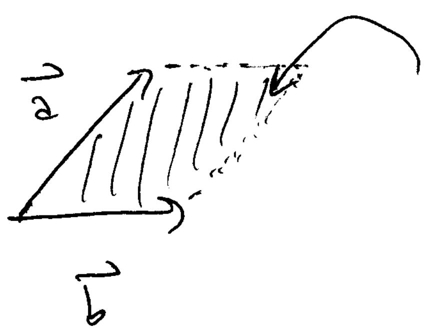


The direction of $\vec{a} \times \vec{b}$ is given by the <u>right-hand rule</u>

(iv) Two vectors (nonzero) $\vec{a}$ and $\vec{b}$ are parallel if and only if

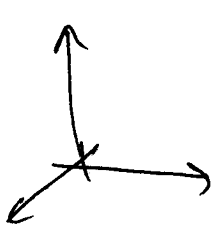
$$\vec{a} \times \vec{b} = \vec{0}.$$

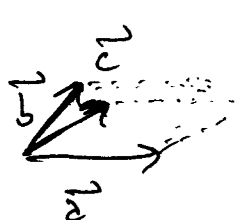
(v)

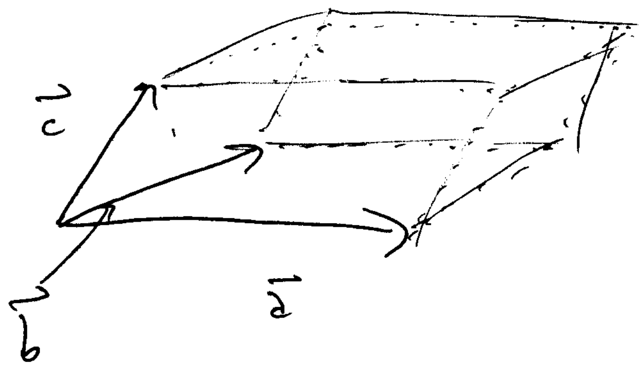


area of parallelogram is $|\vec{a} \times \vec{b}|$

(vi) Given vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, consider the <u>parallelpiped</u>







volume of this parallelpiped?

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

<u>Ex.</u> $\vec{a} = (1, 4, -7)$ are these coplanar?

$\vec{b} = (2, -1, 4)$

$\vec{c} = (0, -9, 18)$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= \underbrace{18\vec{i} - 36\vec{j} - 18\vec{k}}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (1\vec{i} + 4\vec{j} - 7\vec{k}) \cdot (\quad)$$

$$= 18 - 4(36) + 7(18) = 0. \Rightarrow \boxed{\text{yes.}}$$

Properties of dot product (p. 815 in book)

If $\vec{a}, \vec{b}, \vec{c}$ vectors and c is a scalar, then

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (commutativity)
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ (distributivity)
4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$ (associativity with scalar multiple)
5. $\vec{0} \cdot \vec{a} = 0$.

Properties of cross product (p. 826 in book)

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (skew-commutativity)
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$ (associativity with scalar multiple)
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (distributivity from left)
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$ (distributivity from right)
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$