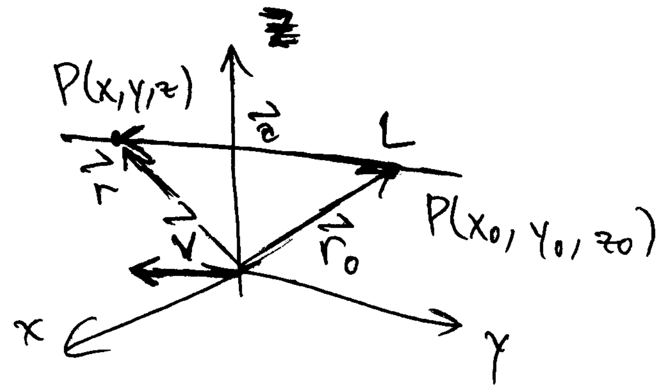
10/19/10

Equations of lines of planes (13.5)

We would like to use vectors to write eqn's describing objects in 3-D space.

Lines:



To specify a line, we need to be given a point $P(x_0, y_0, z_0)$ on the line, and a direction $\vec{v}$ to which this line is parallel.

Let $P(x,y,z)$ also be on the line, and

$\vec{r}_0 = (x_0, y_0, z_0)$

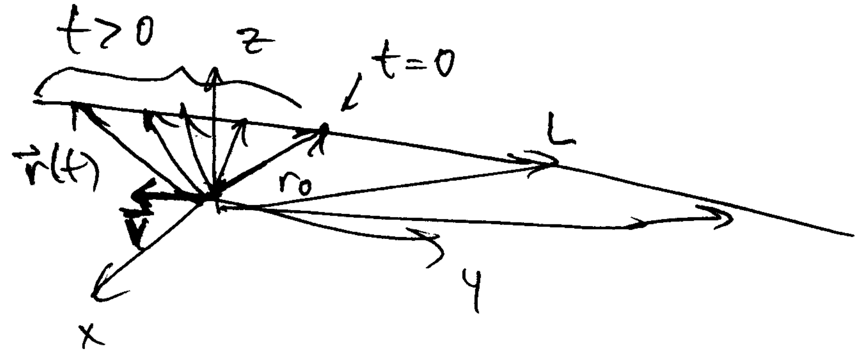
$\vec{r} = (x, y, z)$.

The difference $\vec{a} = \vec{r} - \vec{r}_0$ is parallel to $\vec{v}$, so for some value t, $\vec{a} = t\vec{v}$. Therefore,

$$t\vec{v} = \vec{r} - \vec{r}_0$$

$$\Rightarrow \boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$$ vector eqn. for the line L.

As we change the parameter t, the position vector $\vec{r} = \vec{r}(t)$ traces out L.



Write this in terms of components:

Suppose $\vec{v} = (a, b, c)$, where a, b, c are the <u>direction numbers</u>..

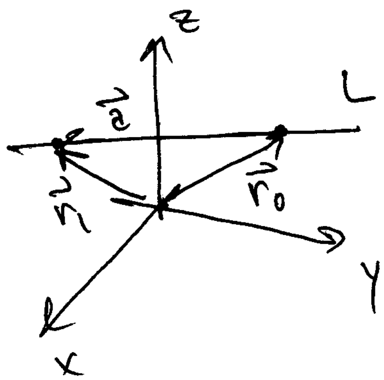
$$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$$

$$\Rightarrow \boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}}$$

parametric eqn's for line L through the point (x_0, y_0, z_0) with direction $\vec{v}$.

Ex. Find eqn's for the line L that goes through points $(1,3,2)$ and $(-4,3,0)$.

Let $\vec{r}_0 = (1,3,2)$ and $\vec{r}_1 = (-4,3,0)$



We find the direction $\vec{v}$ by

$$\vec{r}_1 = \vec{r}_0 + t\vec{v}$$

$$\Rightarrow t\vec{v} = \vec{r}_1 - \vec{r}_0 = (-5,0,-2)$$

(choose $t=1$)

$$\Rightarrow \vec{v} = (-5,0,-2).$$

So, the vector eqn. for L is

$$\boxed{\vec{r} = \vec{r}_0 + t\vec{v} \quad \text{where} \quad \begin{array}{l} \vec{r}_0 = (1,3,2) \\ \vec{v} = (-5,0,-2) \\ t \in \mathbb{R} \end{array}}$$

The parametric eqn's for L are

$$\boxed{\begin{array}{l} x = 1 - 5t \\ y = 3 \\ z = 2 - 2t \end{array} \quad , \quad t \in \mathbb{R}}$$

Remark: Vector and parametric eqn's for a line are non-unique (can be written in many ways!). For ex., we could have started at $\vec{r_1}$ with direction vector $\vec{u} = -2\vec{v}$.

Then,

$$\boxed{\vec{s} = \vec{r_1} + t\vec{u} \quad \text{with} \quad \begin{array}{l} \vec{r_1} = (-4,3,0) \\ \vec{u} = (10,0,4) \\ t \in \mathbb{R} \end{array}}$$

is also a vector eqn. for L.

i.e, in parametric form,

$$\boxed{\begin{array}{l} x = -4 + 10t \\ y = 3 \\ z = 0 + 4t \end{array} , \quad t \in \mathbb{R}}$$

(can go from one parametric representation to the other by transforming $t \longleftrightarrow c(t-t_0)$ for constants c, t_0).

We can eliminate the parameter t from the parametric eqn's here as well. If, for ex., $a, b, c \neq 0$ then

$$\frac{x-x_0}{a} = t, \quad \frac{y-y_0}{b} = t, \quad \frac{z-z_0}{c} = t$$

$$\Rightarrow \boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}, \quad a, b, c \neq 0}$$

<u>symmetric eqn's</u> of L

If one or more of a, b, c are zero (for ex., in previous ex., $b=0$), then can be modified, for ex.:

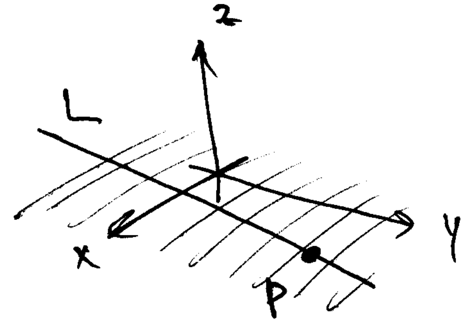
$$\boxed{y = y_0, \quad \frac{x-x_0}{a} = \frac{z-z_0}{c}, \quad a, c \neq 0}$$

<u>Ex.</u> Suppose the line L goes through the point $(2,4,3)$ and has direction vector $\vec{v} = (1,-5,4)$.

Where does this line intersect the x-y plane?

The y-z plane?

The parametric eqn's for L are



$$x = 2 + t$$
$$y = 4 - 5t \quad , \quad t \in \mathbb{R}$$
$$z = 3 + 4t$$

$$\Rightarrow \quad \frac{x-2}{1} = t, \quad \frac{y-4}{-5} = t, \quad \frac{z-3}{4} = t$$

$$x - 2 = \frac{y-4}{-5} = \frac{z-3}{4}.$$

symmetric eqn. for L.

x-y plane : $z = 0$
y-z plane : $x = 0$

$\Rightarrow$ intersection w/ x-y plane given by x, y which satisfy

$$x - 2 = \frac{y-4}{-5} = \frac{0-3}{4}$$

$$\Rightarrow \quad x = \frac{5}{4}, \quad y = 4 + \frac{15}{4} = \frac{31}{4}.$$

$\Rightarrow$ intersection at $\left(\frac{5}{4}, \frac{31}{4}, 0\right)$.

Intersection w/ y-z plane given y, z which satisfy

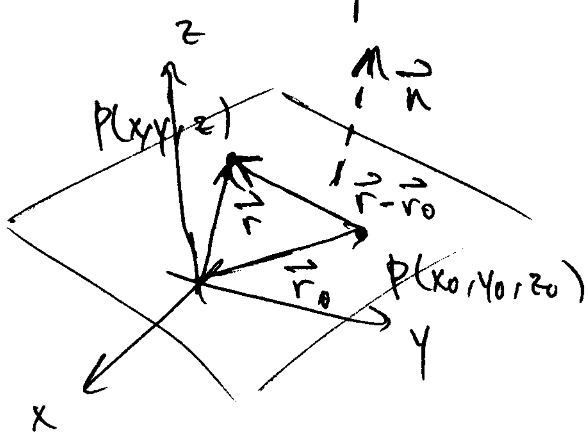
$$0-2 = \frac{y-4}{-5} = \frac{z-3}{4}$$

$\Rightarrow$ $y = 14$, $z = -5$.

$\Rightarrow$ intersection at $(0, 14, -5)$.

Planes.

To uniquely specify a plane, we must specify a point in the plane, say $P(x_0, y_0, z_0)$, and a vector $\vec{n}$ perpendicular to the plane. $\vec{n}$ is called a <u>normal vector</u>.



Any other point $P(x,y,z)$ is given by a vector $\vec{r}$ s.t. $\vec{r}-\vec{r_0}$ is orthogonal to $\vec{n}$ since $\vec{r}-\vec{r_0}$ is parallel to the plane:

$$\Rightarrow \boxed{\vec{n}\cdot(\vec{r}-\vec{r_0}) = 0}$$

<u>vector eqn.</u> of the plane.

In components, if $\vec{n} = (a, b, c)$, $\vec{r} = (x, y, z)$

$\vec{r}_0 = (x_0, y_0, z_0)$

then $(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$

$$\Rightarrow \boxed{\begin{array}{l} ax + by + cz + d = 0 \\ \quad \text{where } d = -(ax_0 + by_0 + cz_0) \end{array}}$$

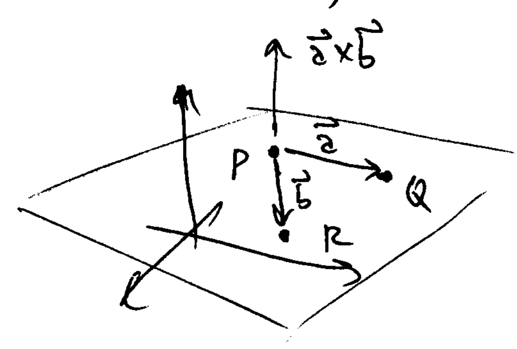
<u>scalar eqn.</u> of the plane through $P_0(x_0, y_0, z_0)$ w/ normal vector $\vec{n}$.

<u>Ex.</u> Find plane that goes through points $P(1, 3, 2)$, $Q(3, -1, 6)$, $R(5, 2, 0)$

(check that these points are not on same line)

Consider $\vec{a} = \overrightarrow{PQ}$, $\vec{b} = \overrightarrow{PR}$

$= (2, -4, 4)$ $\quad = (4, -1, -2)$



Since $\vec{a}, \vec{b}$ are parallel to plane,

$\vec{n} = \vec{a} \times \vec{b}$ specifies a normal vector.

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\vec{i} + 20\vec{j} + 14\vec{k}$$

$\Rightarrow$ plane given $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$

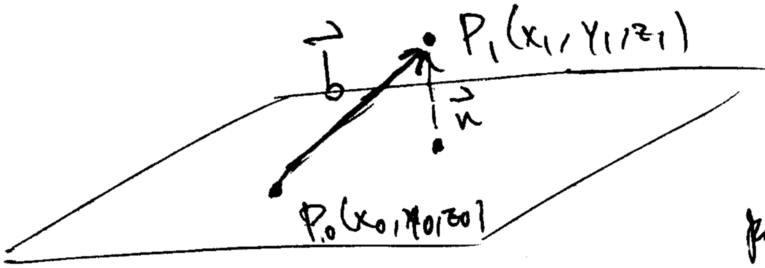
with $\vec{r_0} = (1,3,2)$

$\vec{r} = (x, y, z)$

$$\Rightarrow \quad 12(x-1) + 20(y-3) + 14(z-2) = 0$$

$$\Rightarrow \quad \boxed{6x + 10y + 7z = 50}$$

What is the distance between a point $P_1(x_1, y_1, z_1)$ to a given plane?



Plane given by point $P_0(x_0, y_0, z_0)$ and $\vec{n} = (a, b, c)$

$$D = \left| \mathrm{comp}_{\vec{n}} \vec{b} \right| = \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|}$$

$$= \frac{\left| (a,b,c) \cdot (x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0) \right|}{|(a,b,c)|}$$

$$\Rightarrow \boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}} \quad \text{where } d = -(ax_0 + by_0 + cz_0).}$$

Ex. Are the planes

$$5x + y - z = 1 \quad , \quad 10x + 2y - 2z = 5$$

parallel? If so, what is distance between them?

Normal vector to first plane is

$$\vec{n}_1 = (5, 1, -1)$$

and to second plane is

$$\vec{n}_2 = (10, 2, -2)$$

$\vec{n}_2 = 2\vec{n}_1 \Rightarrow$ parallel.

Distance between them:

(1) See where plane 1 intersects x-axis.

$\Rightarrow$ $y, z = 0$ in $5x + y - z = 1$

$\Rightarrow$ $x = \frac{1}{5}$. $\Rightarrow$ plane 1 intersects x-axis at $(\frac{1}{5}, 0, 0)$

Now we find the distance D from the point $(\frac{1}{5}, 0, 0)$ (which lies on plane 1) to plane 2:

$$D = \frac{\left| 10 \cdot \frac{1}{5} + 2 \cdot 0 - 2 \cdot 0 - 5 \right|}{\sqrt{10^2 + 2^2 + (-2)^2}}$$

$$= \frac{3}{\sqrt{108}} = \frac{3}{\sqrt{36 \cdot 3}} = \frac{3}{6\sqrt{3}} = \boxed{\frac{\sqrt{3}}{6}}.$$

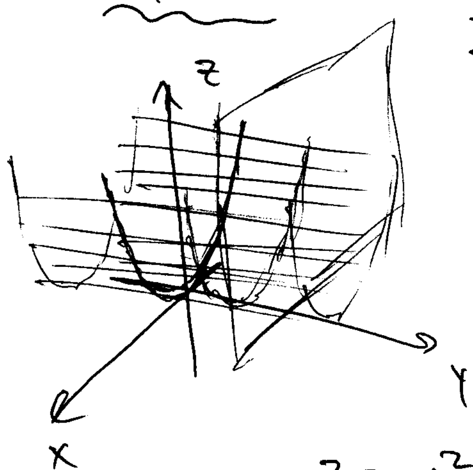
$\boxed{10/21/10}$

We've considered

<u>planes</u>: $ax + by + cz + d = 0$

<u>sphere</u>: $x^2 + y^2 + z^2 = r^2$,

Cylinders



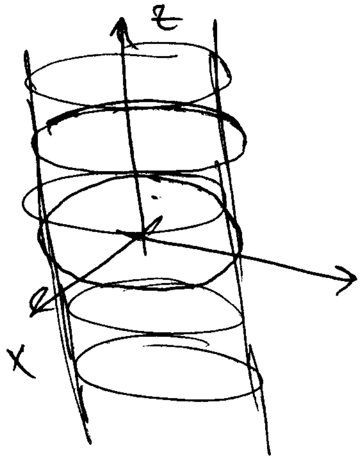
<u>Idea</u>: Take a graph in 2-D, for ex., in x-z ~~axis~~ plane, then drag along the y-axis to form surface.

$z = x^2$ (parabolic cylinder)

Slicing the surface along planes parallel to coordinate axis yields <u>traces</u> (cross-sections) of the surface. We use traces to sketch surfaces.

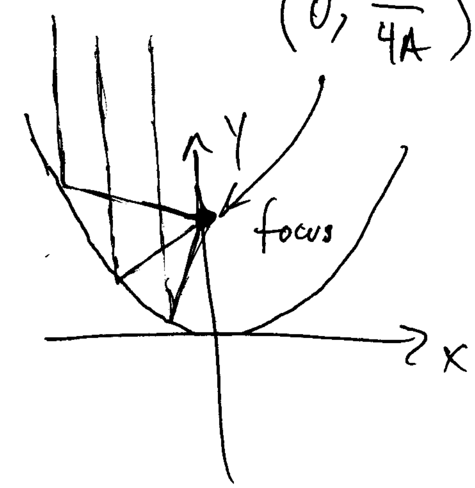
For a cylinder, all the traces taken from planes parallel to the axis of the cylinder are identical.

Ex. In 3-D, $x^2+y^2=4$ gives a cylinder along the z-axis, horizontal traces (slicing w/ planes $z=k$, $k \in \mathbb{R}$) are circles of radius 2.



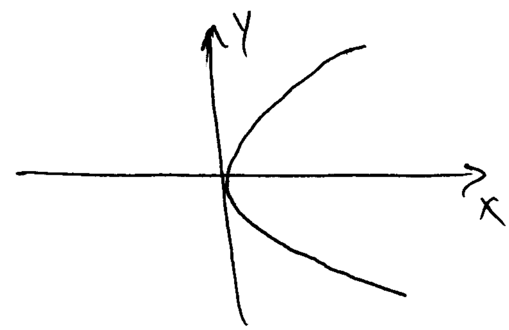
$$x^2+y^2=2^2$$

Conic sections (2 min. review)

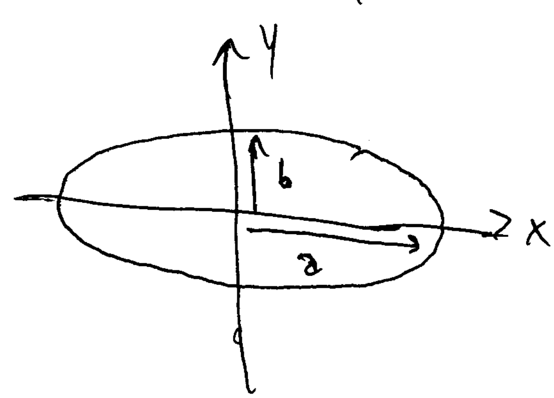


Parabola: $y = Ax^2$

$$x = By^2$$

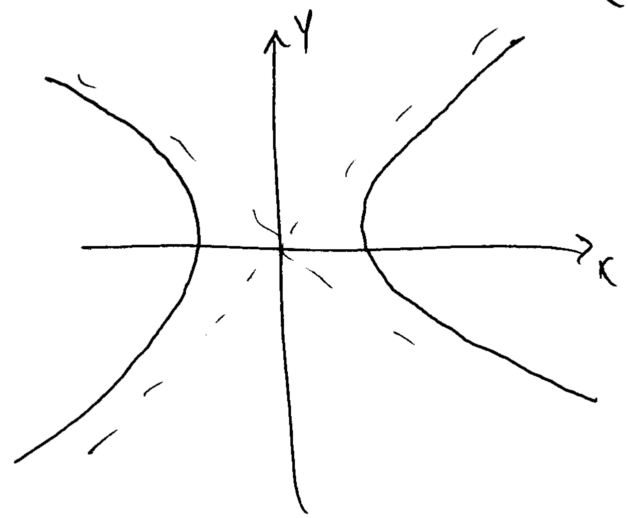


Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

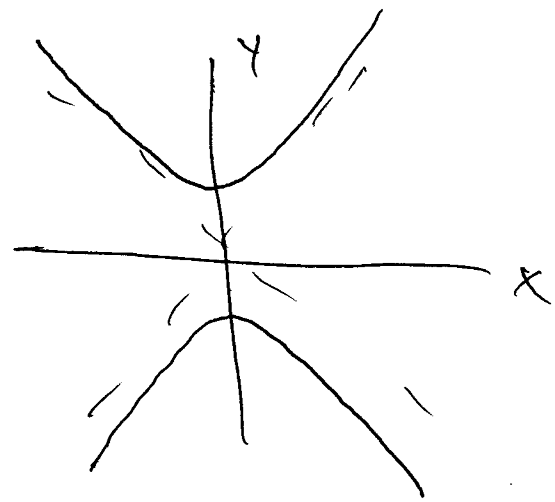


Hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$



Quadric surfaces (13.6)

Graph of all (x,y,z) such that the second-order eqn.

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

(A through J constants)

Can reduce by translations and rotations to either

I. quadratic in z — II. linear in z

$$\boxed{Ax^2 + By^2 + Cz^2 + J = 0} \quad \text{-or-} \quad \boxed{Ax^2 + By^2 + Iz = 0}$$

Ellipsoid (in class I.).

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$$

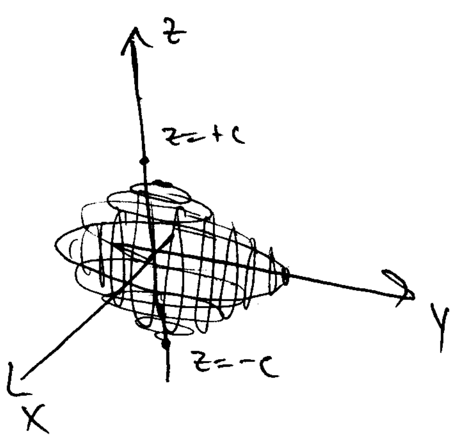
- horizontal traces are ellipses
  vertical traces are ellipses

Check this:

Horizontal traces are given by considering

$\boxed{z = k, \; k \in \mathbb{R}:}$ $$\underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}}_{\text{ellipse in } x\text{-}y \text{ plane}}$$

if $|k| < c$



Vertical traces given by considering

$\boxed{y = k,\ k \in \mathbb{R}:}$

$$\underbrace{\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 - \frac{k^2}{b^2}}$$

ellipse in x-z plane
if $|k| < b$

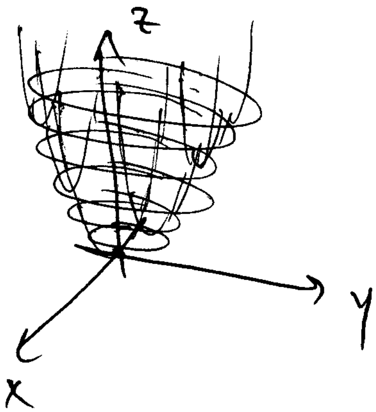
Elliptic paraboloid (in class II.)

$$\boxed{z = Ax^2 + By^2, \quad A \text{ and } B \text{ of same sign}}$$

- horizontal traces are ellipses
  vertical traces are parabolas

Check this: say $A = B = 1$.

$\boxed{z = k,\ k \in \mathbb{R}:}$ $\quad k = x^2 + y^2$



$$\Rightarrow \underbrace{\frac{x^2}{k} + \frac{y^2}{k} = 1}$$

ellipse (circle) if $k > 0$
nothing if $k < 0$

$\boxed{y = k,\ k \in \mathbb{R}:}$ $\underbrace{z = x^2 + k^2}_{\text{parabola in x-z plane}}$

<u>Hyperbolic paraboloid</u> (in class II.)

$$\boxed{z = Ax^2 + By^2, \quad A, B \text{ of different sign}}$$

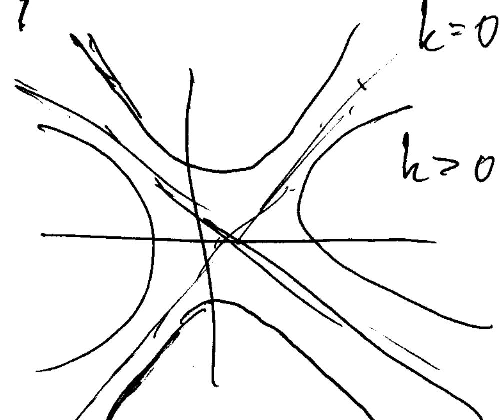
- horizontal traces are hyperbolas
- vertical traces are parabolas

<u>Check this:</u> say ~~[illegible]~~ $A = 1,\ B = -1$

$\boxed{z = k,\ k \in \mathbb{R}:}$ $k = Ax^2 + By^2$

$$= x^2 - y^2$$

$$\Rightarrow \underbrace{\frac{x^2}{k} - \frac{y^2}{k} = 1}_{\text{hyperbola!}}$$

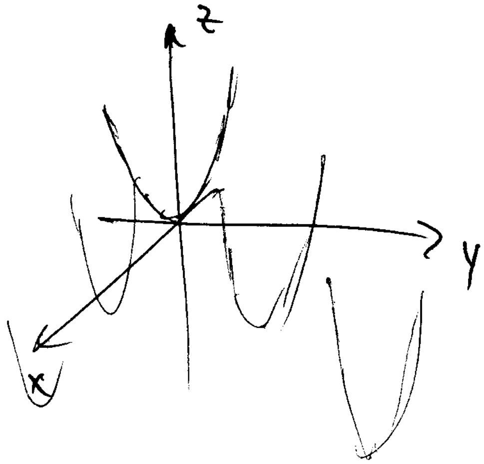


$\boxed{y = k,\ k \in \mathbb{R}:}$ $\quad \underbrace{z = x^2 - k^2}$

parabola in x-z plane

$\boxed{x = k,\ k \in \mathbb{R}:}$ $\quad z = \underbrace{k^2 - y^2}$

parabola in y-z plane

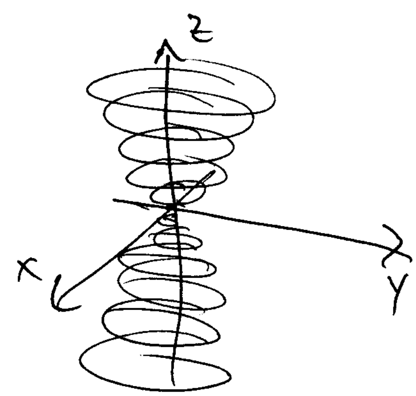


<u>Double cone</u> (in class I)

$$\boxed{z^2 = Ax^2 + By^2 \quad \Big| \quad A, B > 0}$$

- horizontal traces are ellipses, vertical traces are hyperbolas

<u>Check this:</u> say $A = B = 1$.



$\boxed{z = k,\ k \in \mathbb{R}:}$ $\quad \underbrace{k^2 = x^2 + y^2}_{\text{ellipse ! (circle)}}$

$\boxed{y=k,\ k\in\mathbb{R}:}$ $z^2 = x^2 + k^2$

$$\Rightarrow \underbrace{z^2 - x^2 = k^2}$$

hyperbola in x-z plane.

Cone is important because <u>every</u> conic section is the trace along some slice!

Hyperboloid of one sheet (in class I.)

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1}$$

- horizontal traces are ellipses, vertical traces are hyperbolas.

<u>Hyperboloid of two sheets</u> (in class I.)

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1}$$

- horizontal traces are ellipses if $|z| > c$, else nothing
  vertical traces are hyperbolas

$\rightarrow$ website: http://www.math.umn.edu/~rogness/quadrics/

<u>Ex.</u> $\underline{x^2} - \underline{y^2} + \underline{z^2} - \underline{4x} - \underline{2y} - \underline{2z} + 4 = 0$

<u>Idea</u>: Translate the surface by completing the square for x, y, z to reduce to form I. or II.

$$x^2 - 4x + 4 - 4$$
$$-y^2 - 2y - 1 + 1$$
$$+z^2 - 2z + 1 - 1$$
$$+4$$
$$= 0$$

$$\Leftrightarrow \boxed{(x-2)^2 - (y+1)^2 + (z-1)^2 = 0}$$

Let $\tilde{x} = x - 2$
$\tilde{y} = y + 1$
$\tilde{z} = z - 1$

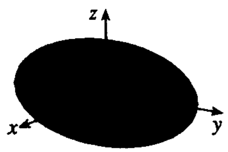
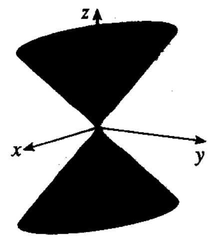
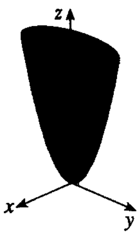
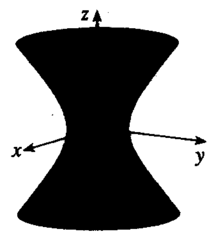
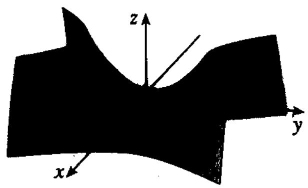
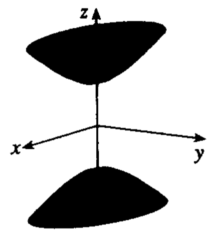
$$\Rightarrow \tilde{x}^2 - \tilde{y}^2 + \tilde{z}^2 = 0$$
$$\Rightarrow \tilde{y}^2 = \tilde{x}^2 + \tilde{z}^2$$

$\Rightarrow$ cone with axis $x=2,\ z=1$.

Vector-valued functions, space curves (14.1)

Next time ...

TABLE 1 Graphs of quadric surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.