M408D (54690/95/00), Midterm #2 Solutions

Multiple choice questions (20 points)

See last two pages.

Question #1 (25 points)

Define the vector-valued function

$$\boldsymbol{r}(t) = \langle e^t, 2, 3e^t \rangle.$$

a) At what point $P(x_0, y_0, z_0)$ does the curve $\boldsymbol{r}(t)$ intersect the surface $y = x^2 + 1$?

Solution: We need to find a t such that $2 = e^{2t} + 1$, i.e., $t = 0$. This corresponds to the point $\boldsymbol{r}(0) = \langle 1, 2, 3 \rangle$.

b) Find $\boldsymbol{r}'(t)$ and $|\boldsymbol{r}'(t)|$ to determine the unit tangent vector $\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$ at the point $P(1, 2, 3)$.

Solution: $\boldsymbol{r}'(t) = \langle e^t, 0, 3e^t \rangle$ and $|\boldsymbol{r}'(t)| = \sqrt{10}e^t$ so $\boldsymbol{T}(t) = \frac{1}{\sqrt{10}}\langle 1, 0, 3 \rangle$ and the unit tangent vector at $P(1, 2, 3)$ is $\frac{1}{\sqrt{10}}\langle 1, 0, 3 \rangle$.

c) What is the arc length $L = \int_a^b |\boldsymbol{r}'(u)| du$ of the curve between the points $P(1, 2, 3)$ and $Q(e, 2, 3e)$?

Solution: $L = \int_0^1 \sqrt{10}e^u du = \sqrt{10}(e - 1)$.

d) Write a vector equation of the form $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r_0}) = 0$ for the plane normal to the curve at the point $P(1, 2, 3)$.

Solution: A normal vector for the plane normal to the curve at $P(1, 2, 3)$ is simply the tangent vector $\boldsymbol{r}'(0) = \langle 1, 0, 3 \rangle$. So the equation is $\langle 1, 0, 3 \rangle \cdot (\langle x, y, z \rangle - \langle 1, 2, 3 \rangle) = 0$.

Question #2 (25 points)

Define the vectors

$$\boldsymbol{a} = \langle -3, 1, 1 \rangle$$
$$\boldsymbol{b} = \langle 4, 0, 3 \rangle$$
$$\boldsymbol{c} = \langle 2, 3, 4 \rangle.$$

a) What is $\boldsymbol{b} \times \boldsymbol{c}$?

Solution: $\boldsymbol{b} \times \boldsymbol{c} = \langle -9, -10, 12 \rangle$.

b) Determine the volume of the parallelepiped determined by $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ using the scalar triple product $V = |\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})|$.

Solution: $V = 29$.

c) Are $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ coplanar (i.e., do the lines which pass through the origin with directions $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ lie in the same plane), and **why** or **why not**? [Hint: Use part (b).]

Solution: No, the vectors are not coplanar because if they were than the triple scalar product would be zero (i.e., we would have had $V = 0$ in part (b)).

d) Find the vector projection $\mathbf{proj}_{\boldsymbol{b}}\boldsymbol{a}$ of $\boldsymbol{a}$ onto $\boldsymbol{b}$. [Hint: If you do not remember the definition, recall that the vector projection of $\boldsymbol{u}$ onto a *unit* vector $\boldsymbol{e}$ is $(\boldsymbol{u} \cdot \boldsymbol{e})\boldsymbol{e}$ (i.e., the component of $\boldsymbol{u}$ in the direction $\boldsymbol{e}$). Then $\mathbf{proj}_{\boldsymbol{b}}\boldsymbol{a}$ is the vector projection of $\boldsymbol{a}$ onto the unit vector $\frac{\boldsymbol{b}}{|\boldsymbol{b}|}$.]

Solution: $\mathbf{proj}_{\boldsymbol{b}}\boldsymbol{a} = \langle -\frac{36}{25}, 0, -\frac{27}{25} \rangle$.

Question #3 (15 points)

Consider the surface consisting of all points $P(x, y, z)$ equidistant from the point $P(0, 0, 1)$ and the plane $z = -1$.

a) Using the formula for the distance between two points, write an equation for this surface.

Solution: The distance between $P(x, y, z)$ and $P(0, 0, 1)$ is $d_1 = \sqrt{x^2 + y^2 + (z - 1)^2}$. Since the point on the plane $z = -1$ closest to $P(x, y, z)$ is $P(x, y, -1)$, the distance to the plane is $d_2 = |z + 1|$. Therefore, since $d_1^2 = d_2^2$ we have that the equation for the surface is $x^2 + y^2 + (z - 1)^2 = (z + 1)^2$, i.e.,

$$x^2 + y^2 = 4z.$$

b) Is this quadric surface a cone or a paraboloid? [Hint: Remember that for surfaces symmetric about the z-axis, vertical traces of cones are hyperbolas while vertical traces of paraboloids are parabolas.]

Solution: Setting $y = k$ where k is a constant, we see that the vertical traces of the surface are of the form $z = \frac{1}{4}x^2 + \frac{1}{4}k^2$, which is the equation for a parabola in the xz-plane. Therefore, the quadric surface is a paraboloid.

Question #4 (15 points)

Let a curve C in the xy-plane be given by the parametric equations

$$x(t) = t^2, \qquad y(t) = t^3 - 3t.$$

a) Find $dy/dx = \frac{dy/dt}{dx/dt}$ and compute the slope of the tangent line to the curve at the point $P(4,2)$.

Solution: We have that $dy/dx = \frac{3t^2 - 3}{2t}$. Since the curve is at the point $P(4,2)$ at $t = 2$ we have that the slope of the tangent line is $dy/dx = 9/4$.

b) At what two points $P\,(x, y)$ and $Q(x, y)$ does the curve have a horizontal tangent?

Solution: We have that $y'(t) = 3t^2 - 3 = 0$ when $t = -1$ and $t = 1$. Since $x'(\pm 1) = 1 \neq 0$ we have that $dy/dx = 0$ (horizonal tangent) at the points $(x(-1),\, y(-1))$ and $(x(1),\, y(1))$, i.e., at $P(1,2)$ and $Q(1,-2)$.

This print-out should have 4 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

CalC13c02a
001 10.0 points

Determine the dot product of the vectors

$$\mathbf{a} = \langle 1, 2, -3 \rangle, \quad \mathbf{b} = \langle -1, 2, 1 \rangle.$$

1. $\mathbf{a} \cdot \mathbf{b} = -2$

2. $\mathbf{a} \cdot \mathbf{b} = -8$

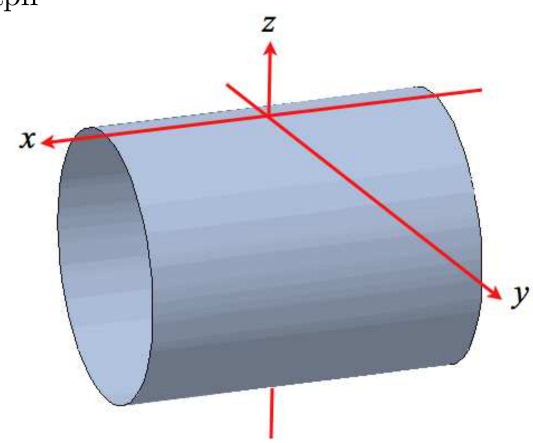
3. $\mathbf{a} \cdot \mathbf{b} = -4$

4. $\mathbf{a} \cdot \mathbf{b} = -6$

5. $\mathbf{a} \cdot \mathbf{b} = 0$ **correct**

CalC13f03b
002 10.0 points

Which one of the following equations has graph



when the circular cylinder has radius 2.

1. $y^2 + z^2 + 2z = 0$

2. $z^2 + x^2 + 4x = 0$

3. $x^2 + z^2 - 4z = 0$

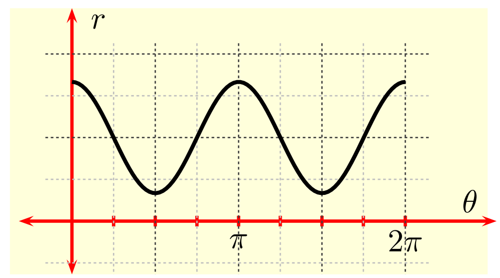
4. $x^2 + z^2 - 2z = 0$

5. $y^2 + z^2 + 4z = 0$ **correct**

6. $z^2 + x^2 + 2x = 0$

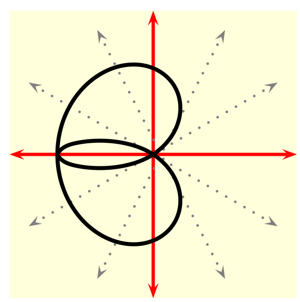
CalC11c26a
003 10.0 points

Use the graph in Cartesian coordinates



of r as a function of θ to determine which one of the following is the graph of the corresponding polar function?

1.



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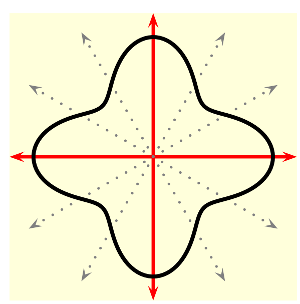

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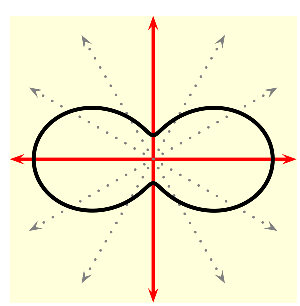
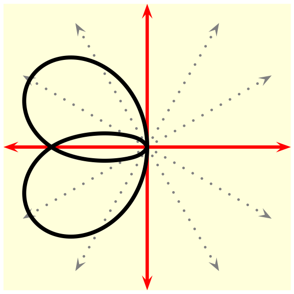
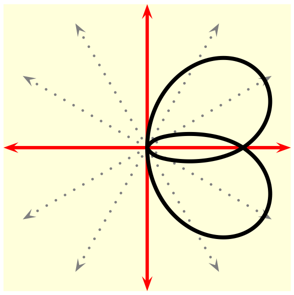
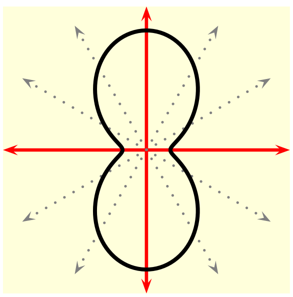


correct

4. 

5. 

6. 

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004 10.0 points

Find a polar representation for the curve whose Cartesian equation is

$$x^2 + (y+2)^2 = 4 .$$

1. $r = 2\cos\theta$

2. $r = 4\sin\theta$

3. $r + 4\cos\theta = 0$

4. $r = 2\sin\theta$

5. $r + 2\cos\theta = 0$

6. $r + 4\sin\theta = 0$ **correct**

7. $r + 2\sin\theta = 0$

8. $r = 4\cos\theta$