M408D (54690/95/00), Quiz #5 Solutions

Question #1

Define the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

a) Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist, and if so, what is it? Is $f(x,y)$ continuous at $(0,0)$?

Solution: This problem is similar to one done in lecture. No, the limit does not exist and therefore f is **not** continuous at the origin. To see this we note that along the path $y=0$ we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} 0 = 0$$

while along the path $y = x^2$ we find

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

b) Now define

$$h(x,y) = \ln(x+4y).$$

Find h_x, h_y, and h_{xy}.

Solution: Taking partial derivatives we find that

$$h_x = \frac{1}{x+4y}$$
$$h_y = \frac{4}{x+4y}$$
$$h_{xy} = -\frac{4}{(x+4y)^2}.$$

c) Suppose that x and y are functions of s and t given by

$$x(s,t) = s e^{2t}$$
$$y(s,t) = \sin(s^2 t).$$

Find $\partial h/\partial s$ as a function of s and t. [Hint: Use part (b)!]

Solution: Find we see that

$$\frac{\partial x}{\partial s} = e^{2t}, \qquad \frac{\partial y}{\partial s} = 2st\cos(s^2 t).$$

Therefore, by the chain rule and part (b) we find

$$\begin{aligned} \frac{\partial h}{\partial s} &= \frac{\partial h}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial s} \\ &= \frac{e^{2t}}{x+4y} + \frac{8st\cos(s^2 t)}{x+4y} \\ &= \frac{e^{2t}}{se^{2t} + 4\sin(s^2 t)} + \frac{8st\cos(s^2 t)}{se^{2t}+4\sin(s^2 t)}. \end{aligned}$$

Question #2

Let

$$f(x,y,z) = ze^{xy}.$$

a) Determine the gradient $\nabla f(x,y,z)$.

Solution: The gradient is the vector

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \langle yze^{xy}, x\,ze^{xy}, e^{xy} \rangle.$$

b) What is the directional derivative of f at the point $(0,0,0)$ in the direction of the vector $\boldsymbol{v} = \langle 4,4,2 \rangle$? That is, find $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u}$ at $(0,0,0)$ with $\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$.

Solution: Since $|\boldsymbol{v}| = \sqrt{4^2+4^2+2^2} = 6$ we have that $\boldsymbol{u} = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$. So,

$$\nabla f \cdot \boldsymbol{u} = \frac{2}{3} y z e^{xy} + \frac{2}{3} x\, z e^{xy} + \frac{1}{3} e^{xy}$$

and

$$D_{\boldsymbol{u}} f\Big|_{(0,0,0)} = \nabla f \cdot \boldsymbol{u}\Big|_{(0,0,0)} = \frac{1}{3}.$$