M408D (54690/95/00), Quiz #6 Solutions

Question #1

Let

$$f(x, y) = 4 + x^3 + y^3 - 3xy.$$

a) Find the two critical points of f.

Solution: Since $\nabla f = (3x^2 - 3y, 3y^2 - 3x)$, we have critical points for (x, y) such that

$$x^2 - y = 0, \qquad y^2 - x = 0.$$

Substituting the first equation $y = x^2$ into the second equation, we have that

$$x^4 - x = 0.$$

So $x = 1$ or $x = 0$ and critical points occur at $(1, 1)$ and $(0, 0)$.

b) Classify the critical points (i.e., determine if they correspond to local maxima, minima, or saddle points).

[Hint: If you've forgotten, remember to use the determinant $D(x,\ y)$ of the matrix of second partial derivatives. If $D > 0$ at the point $(x_0,\ y_0)$, the function behaves the same way in both the x- and y-directions. If $D < 0$, the opposite behavior occurs.]

Solution: The determinant of the Hessian of f is

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9.$$

So $D(1, 1) = 27 > 0$ and $f_{xx}(1, 1) = 6 > 0$ implies that $(1, 1)$ is a local minimum of f, while $D(0, 0) = -9 < 0$ implies that $(0, 0)$ is a saddle point.

c) (Bonus: +2 point) Using the method of Lagrange multipliers, find extrema of f subject to the constraint

$$3xy = 12.$$

Begin by writing out the equations in x, y, and the multiplier λ that need to be solved.

Solution: With constraint function $g(x, y) = 3xy$, we must solve

$$\nabla f = \lambda \nabla g$$
$$g = 12.$$

Explicitly, this gives the set of equations

$$3x^2 - 3y = 3\lambda y$$
$$3y^2 - 3x = 3\lambda x$$
$$3xy = 12.$$

Simplifying, this is

$$(1+\lambda)y = x^2$$
$$(1+\lambda)x = y^2$$
$$3xy = 12.$$

Note that $x \neq 0$ and $y \neq 0$ by the third equation. This implies $\lambda \neq -1$ (otherwise this gives $x = y = 0$). We can then divide the first equation by the second to get $x^2/y^2 = y/x$, so $x^3 = y^3$ and therefore $x = y$. Plugging this into the third equation we get $x = y = \pm 2$. Therefore, the only solutions are $(x, y, \lambda) = (2, 2, 1)$ and $(x, y, \lambda) = (-2, -2, -3)$.

Question #2

Evaluate

$$\iint_D 2\sin(y^2)\, dA$$

where the domain D in the xy-plane is a triangle with vertices $(0, 0)$, $(0, \sqrt{\pi})$, and $(\sqrt{\pi}, \sqrt{\pi})$. At the very least, write a proper expression (with limits of integration) for the double integral.

[Hint: Set this up as an iterated integral. Note that it is only possible to evaluate this explicitly by choosing the correct order of integration!]

Solution: The integral can either be written as

$$\int_0^{\sqrt{\pi}} \int_0^{y} 2\sin(y^2) dx\, dy$$

or as

$$\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} 2\sin(y^2) dy\, dx.$$

It is much easier to evaluate the first expression rather than the second. Doing this, we see that

$$\begin{aligned}
\int_0^{\sqrt{\pi}} \int_0^{y} 2\sin(y^2) dx\, dy &= \int_0^{\sqrt{\pi}} [2x]_{x=0}^{x=y} \sin(y^2)\, dy \\
&= \int_0^{\sqrt{\pi}} 2y \sin(y^2)\, dy \\
&= \int_0^{\pi} \sin(u)\, du \\
&= [-\cos(u)]_{u=0}^{u=\pi} \\
&= 2,
\end{aligned}$$

where we have used the substitution $u = y^2$ in the third equality.