M408D (54690/95/00), Sample Final Exam

Part 1:

Multiple choice questions (5 points each)

See last four pages, questions 1-4.

Question 1 (25 points)

Define

$$h(x,y) = 2x^3 + xy^2 + 5x^2 + y^2.$$

a) Find all critical points of h.

b) Classify all critical points (i.e., determine if they are local maxima, minima, or saddle points) by using the second derivative test. To begin, compute the Hessian of $h(x,y)$.

c) What are the absolute maximum and minimum values of h on the domain $D_1 = \{(x,y): 0 \leq x \leq 4, 0 \leq y \leq 5\}$?

d) Using the method of Lagrange multipliers ($\nabla f = \lambda \nabla g$, $g = 0$ with constraint function g), determine the extreme values of the function

$$f(x,y) = xy^2$$

on the ellipse $x^2 + \frac{1}{4}y^2 = 1$.

Question #2 (20 points)

Evaluate the following integrals. Remember that iterated integrals are sometimes easier to evaluate after switching the order of integration, or after changing to different coordinates.

a) $\int_1^4 \int_1^2 \left(\frac{x}{y} - \frac{y}{x}\right) dy\, dx$

b) $\int_0^1 \int_{3y}^3 e^{x^2} dx\, dy$

c) $\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\, dx\, dy$

Question #3 (20 points)

Consider the linear transformation $T: (u,v) \to (x,y)$ given by

$$x = \frac{1}{2}(u - v)$$
$$y = \frac{1}{2}(u + v) \quad .$$

a) Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transformation.

b) Determine the region in the uv-plane which maps to the trapezoidal region R in the xy-plane with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, $(0, -1)$.

c) Use parts (a) and (b) to evaluate the integral

$$\iint_R e^{(x+y)/(x-y)} dA$$

where R is the trapezoidal region of the xy-plane defined in part (b).

Question #4 (15 points)

Define the function

$$f(x, y, z) = x \sin(yz).$$

a) Determine $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.

b) Find the directional derivative $D_{\boldsymbol{u}} f$ of f at the point $P(2,\ 0,\ 1)$ in the direction of the vector $\boldsymbol{v} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$.

c) Suppose

$$x(s, t) = \cos(s^2 + t)$$
$$y(s, t) = e^{-2st}$$
$$z(s, t) = s^3 - 2st^2 + 4.$$

Use the chain rule to determine $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

Part 2:

Multiple choice questions (5 points each)

See last four pages, questions 5-8.

Question 1 (25 points)

Determine whether the following series are convergent or divergent. Justify your use of any test.

a) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

b) $\sum_{n=6}^{\infty} (-1)^n \frac{3 + 2n + 4n^2}{n^2 + n + \sin(n)}$

c) $\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$

d) $\sum_{n=1}^{\infty} \frac{n}{(\ln n)^n}$

Question #2 (20 points)

Find the Taylor series of the following functions about the given point a (using any method) and find the corresponding interval of convergence of the series.

a) $f(x) = 2x\cos(x^2)$; $a = 0$

b) $f(x) = x^{-2}$; $a = 1$

Question #3 (20 points)

Define the vector-valued function

$$\boldsymbol{r}(t) = \langle 2e^t, 3t^2, te^{4t} \rangle.$$

a) What is $\boldsymbol{r}''(t)$?

b) Write an integral in t for the arc length of the curve between the points $P(2,\ 0,\ 0)$ and $Q(2e, 3, e^4)$. Do not attempt to evaluate the integral.

c) At what point $P(x_0, y_0, z_0)$ does the curve $\boldsymbol{r}(t)$ intersect the surface $16z = x^4$?

d) Find the unit tangent vector $\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$ at the point $P(2, 0, 0)$.

Question #4 (15 points)

Define the vectors

$$\boldsymbol{a} = \langle 4, 2, 0 \rangle$$
$$\boldsymbol{b} = \langle 1, -3, 5 \rangle$$
$$\boldsymbol{c} = \langle -2, 2, 1 \rangle.$$

a) Find $\boldsymbol{c} \cdot (\boldsymbol{b} \times \boldsymbol{a})$.

b) What is the cosine of the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{c}$?

c) Determine the equation for the plane parallel to the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ that passes through the point $P(1, 1, 1)$ (write in the form $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r_0}) = 0$).

This print-out should have 8 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

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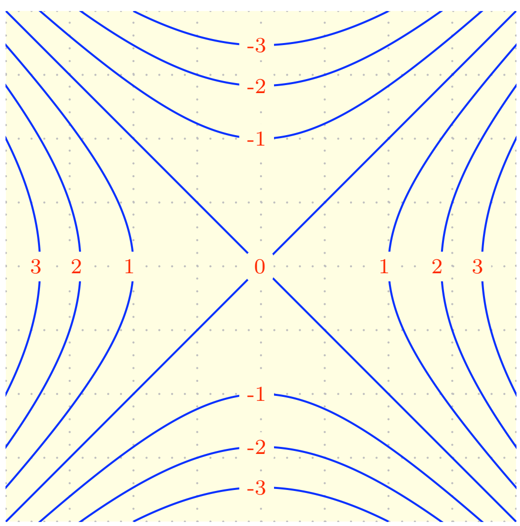
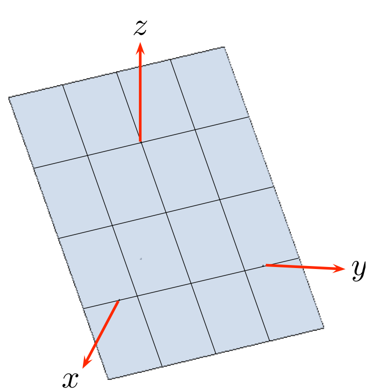
001

Which of the following surfaces could have contour map





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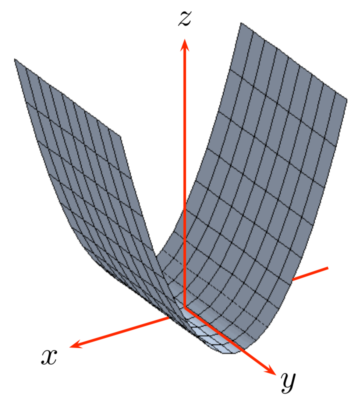

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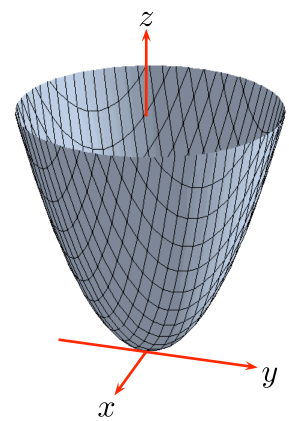

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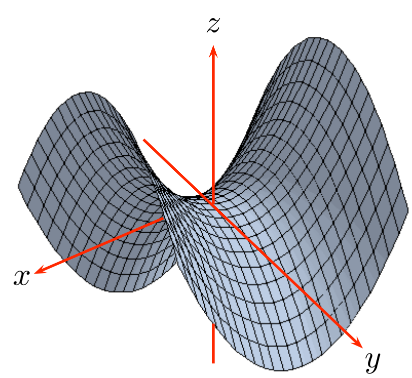

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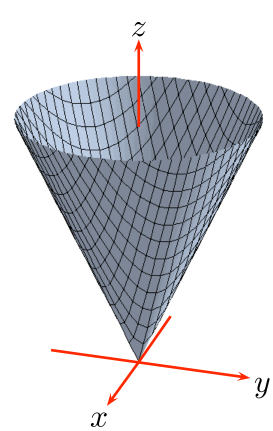


5.

CalC15b11s
002

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{6xy}{\sqrt{x^2+y^2}}$, if it exists.

1. 12

2. 3

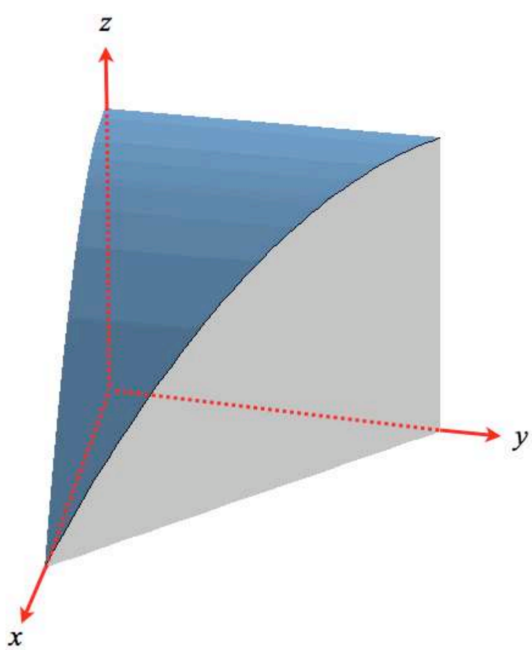
3. The limit does not exist.

4. 6

5. 0

CalC16c22s
003

The solid shown in



is bounded by the cylinder

$$z = 4 - x^2,$$

the xy-plane and the planes

$$x = 0, \quad y = 0, \quad x + y = 2.$$

Find the volume of this solid.

1. volume $= \dfrac{43}{6}$ cu. units

2. volume $= \dfrac{41}{6}$ cu. units

3. volume $= \dfrac{22}{3}$ cu. units

4. volume $= 7$ cu. units

5. volume $= \dfrac{20}{3}$ cu. units

CalC16d24s
004

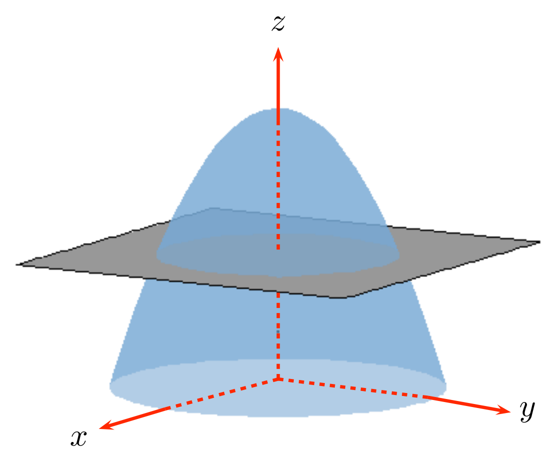
The plane

$$z = 2$$

and the paraboloid

$$z = 8 - 6x^2 - 6y^2$$

enclose a solid as shown in



Use polar coordinates to determine the volume of this solid.

1. volume $= \dfrac{5}{2}\pi$

2. volume $= \dfrac{7}{2}\pi$

3. volume $= 2\pi$

4. volume $= \dfrac{3}{2}\pi$

5. volume $= 3\pi$

CalC12b04exam2
005

If the n^{th} partial sum of $\sum_{n=1}^{\infty} a_n$ is given by

$$S_n = \frac{4n+1}{n+3},$$

what is a_n when $n \geq 2$?

1. $a_n = \dfrac{13}{n(n+3)}$

2. $a_n = \dfrac{11}{(n+3)(n+4)}$

3. $a_n = \dfrac{13}{(n+3)(n+2)}$

4. $a_n = \dfrac{11}{(n+3)(n+2)}$

5. $a_n = \dfrac{13}{(n+3)(n+4)}$

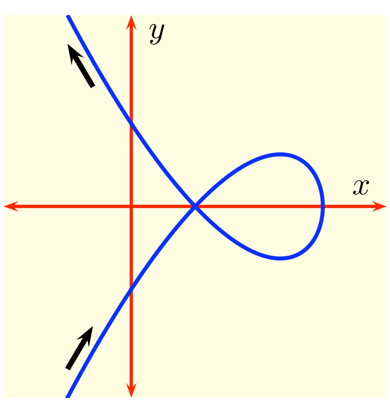
6. $a_n = \dfrac{11}{n(n+3)}$

CalC11a25a
006

Which one of the following could be the graph of the curve given parametrically by

$$x(t) = t^3 - 2t, \qquad y(t) = 3 - t^2,$$

where the arrows indicate the direction of increasing t?

1.


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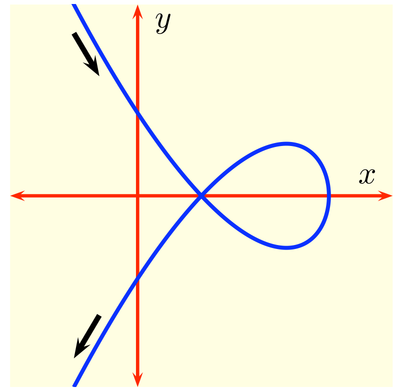

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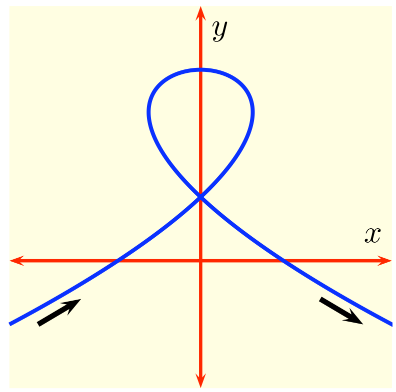

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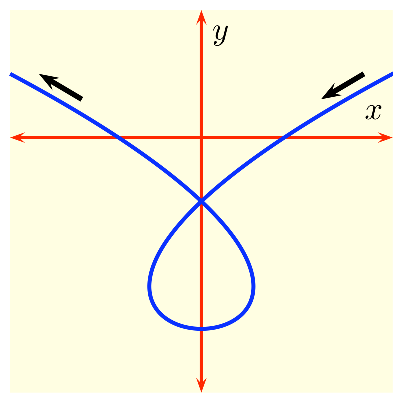

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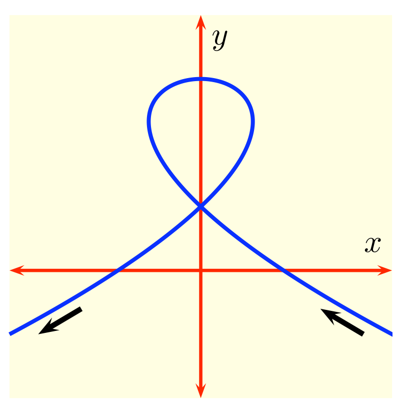
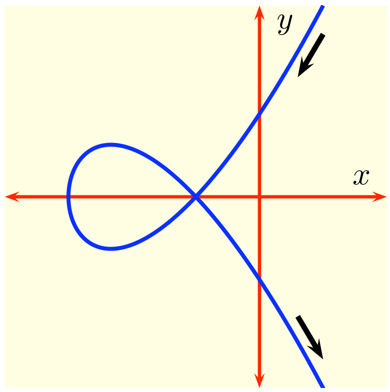


6. 

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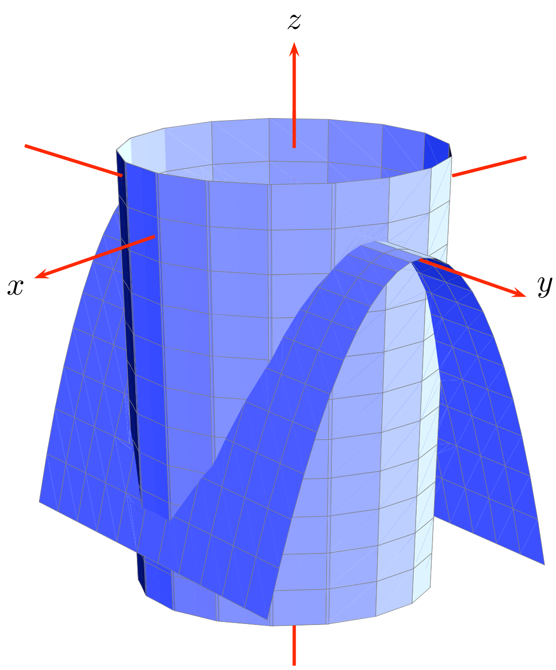
007

The curve of intersection of the surfaces shown in



is the graph of which of the following vector functions?

1. $\mathbf{r}(t) = \langle \sin t, \cos t, \cos 2t - 1 \rangle$

2. $\mathbf{r}(t) = \langle \cos t, \sin t, 1 - \cos 2t \rangle$

3. $\mathbf{r}(t) = \langle \cos t, \sin t, \cos 2t - 1 \rangle$

4. $\mathbf{r}(t) = \langle \sin t, \cos t, \cos 2t \rangle$

5. $\mathbf{r}(t) = \langle \cos t, \sin t, \cos 2t \rangle$

6. $\mathbf{r}(t) = \langle \sin t, \cos t, 1 - \cos 2t \rangle$

CalC11b17a
008

Determine all values of t for which the curve given parametrically by

$$x = t^3 - 2t^2 + 1, \quad y = 2t^3 + t^2 - 4t$$

has a vertical tangent?

1. $t = \frac{4}{3}$

2. $t = 0, \quad \frac{4}{3}$

3. $t = -\frac{1}{3}$

4. $t = \frac{1}{3}$

5. $t = 0, \quad \frac{1}{3}$

6. $t = 0, \quad -\frac{4}{3}$

7. $t = -\frac{4}{3}$

8. $t = 0, \quad -\frac{1}{3}$