This print-out should have 10 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Determine if

$$\lim_{x \to 0} \Big(\frac{5}{x} - \frac{10}{e^{2x}-1}\Big)$$

exists, and if it does, find its value.

1. limit $= \dfrac{10}{3}$

2. limit $= 5$ **correct**

3. limit $= 10$

4. limit does not exist

5. limit $= \dfrac{5}{2}$

6. limit $= 0$

Explanation:

Now

$$\frac{5}{x} - \frac{10}{e^{2x}-1} = 5\Big(\frac{e^{2x}-1-2x}{x(e^{2x}-1)}\Big) = \frac{f(x)}{g(x)}$$

where f, g are everywhere differentiable functions such that

$$\lim_{x \to 0} f(x) = 0, \quad \lim_{x \to 0} g(x) = 0.$$

Thus L'Hospital's rule can be applied. But

$$f'(x) = 10e^{2x} - 10,$$

while

$$g'(x) = (e^{2x}-1) + 2xe^{2x},$$

so

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x \to 0} \Big(\frac{10(e^{2x}-1)}{e^{2x}+2xe^{2x}-1}\Big).$$

As f' and g' are differentiable functions such that

$$\lim_{x \to 0} f'(x) = 0, \quad \lim_{x \to 0} g'(x) = 0$$

we have to apply L'Hospital's rule again. But

$$f''(x) = 20e^{2x}, \quad g''(x) = 4e^{2x} + 4xe^{2x},$$

from which it follows that

$$\lim_{x \to 0} f''(x) = 20, \quad \lim_{x \to 0} g''(x) = 4.$$

Consequently, the limit exists and

$$\boxed{\text{limit} = 5}.$$

002 10.0 points

Find the n^{th} term, a_n, of an infinite series $\sum_{n=1}^{\infty} a_n$ when the n^{th} partial sum, S_n, of the series is given by

$$S_n = \frac{2n}{n+1}.$$

1. $a_n = \dfrac{2}{n(n+1)}$ **correct**

2. $a_n = \dfrac{1}{n^2}$

3. $a_n = \dfrac{1}{n(n+1)}$

4. $a_n = \dfrac{1}{n}$

5. $a_n = \dfrac{1}{2n^2}$

6. $a_n = \dfrac{1}{2n}$

Explanation:

Since $S_n = a_1 + a_2 + \cdots + a_n$, we see that

$$a_1 = S_1, \quad a_n = S_n - S_{n-1} \ (n > 1).$$

But

$$S_n = \frac{2n}{n+1} = 2 - \frac{2}{n+1}.$$

Thus $a_1 = 1$, while

$$a_n = \frac{2}{n} - \frac{2}{n+1}, \quad (n > 1).$$

Consequently,

$$\boxed{a_n = \frac{2}{n} - \frac{2}{n+1} = \frac{2}{n(n+1)}}$$

for all n.

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003 10.0 points

Determine whether the series

$$\sum_{n=0}^{\infty} 3\left(\frac{2}{5}\right)^n$$

is convergent or divergent, and if convergent, find its sum.

1. convergent, sum $= 5$ **correct**

2. convergent, sum $= \dfrac{15}{7}$

3. convergent, sum $= \dfrac{16}{3}$

4. convergent, sum $= -\dfrac{16}{3}$

5. divergent

Explanation:

The given series is an infinite geometric series

$$\sum_{n=0}^{\infty} a\,r^n$$

with $a = 3$ and $r = \frac{2}{5}$. But the sum of such a series is

(i) convergent with sum $\dfrac{a}{1-r}$ when $|r| < 1$,

(ii) divergent when $|r| \geq 1$.

Consequently, the given series is

$$\boxed{\text{convergent, sum} = 5}.$$

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004 10.0 points

Let h be a continuous, positive, decreasing function on $[2, \infty)$. Compare the values of the series

$$A = \sum_{n=3}^{10} h(n)$$

and the integral

$$B = \int_2^{10} h(z)\,dz.$$

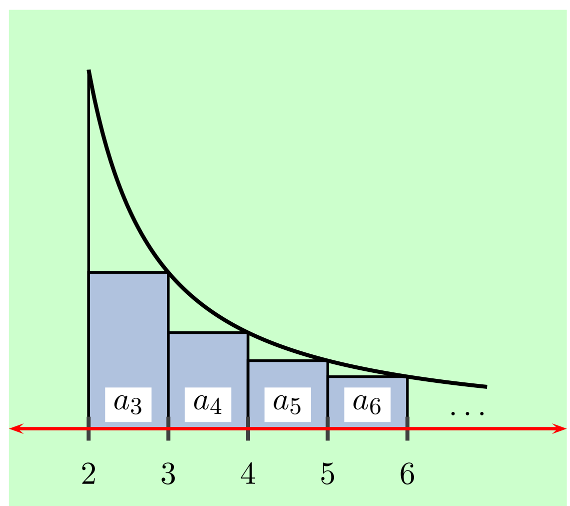
1. $A > B$

2. $A = B$

3. $A < B$ **correct**

Explanation:

In the figure



the bold line is the graph of h on $[2, \infty)$ and the areas of the rectangles the terms in the series

$$\sum_{n=3}^{\infty} a_n, \qquad a_n = h(n).$$

Clearly from this figure we see that

$$a_3 = h(3) < \int_2^3 h(z)\,dz,$$

$$a_4 = h(4) < \int_3^4 h(z)\,dz,$$

while

$$a_5 = h(5) < \int_4^5 h(z)\,dz,$$

$$a_6 = h(6) < \int_5^6 h(z)\,dz\,,$$

and so on. Consequently,

$$\boxed{A < B}\,.$$

keywords: Szyszko

005 10.0 points

Which one of the following properties does the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{5n^2+1}{3^n}$$

have?

1. divergent

2. conditionally convergent

3. absolutely convergent **correct**

Explanation:

The given series has the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n\,, \quad b_n = \frac{5n^2+1}{3^n}$$

of an alternating series. But the denominator is increasing very fast, so first let's check if the series is absolutely convergent rather than simply conditionally convergent. We use the Ratio test, for then

$$\left|\frac{(-1)^n b_{n+1}}{(-1)^{n-1} b_n}\right| = \frac{b_{n+1}}{b_n} = \frac{1}{3}\frac{5(n+1)^2+1}{5n^2+1}\,.$$

But

$$\frac{5(n+1)^2+1}{5n^2+1} = \frac{5n^2+10n+6}{5n^2+1} \longrightarrow 1$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty} \left|\frac{(-1)^n b_{n+1}}{(-1)^{n-1} b_n}\right| = \frac{1}{3} < 1\,.$$

Consequently, by the Ratio test, the given series is

$$\boxed{\text{absolutely convergent}}\,.$$

006 10.0 points

Determine whether the series

$$\sum_{n=0}^{\infty} \frac{2}{\sqrt{n+5}} \cos n\pi$$

is conditionally convergent, absolutely convergent or divergent.

1. absolutely convergent

2. divergent

3. conditionally convergent **correct**

Explanation:

Since $\cos n\pi = (-1)^n$, the given series can be rewritten as the alternating series

$$\sum_{n=0}^{\infty} (-1)^n \frac{2}{\sqrt{n+5}} = \sum_{n=0}^{\infty} (-1)^n f(n)$$

with

$$f(x) = \frac{2}{\sqrt{x+5}}\,.$$

Now

$$f(n) = \frac{2}{\sqrt{n+5}} > \frac{2}{\sqrt{n+6}} = f(n+1)$$

for all n, while

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{2}{\sqrt{n+5}} = 0\,.$$

Consequently, by the Alternating Series Test, the given series converges. On the other hand, by the Limit Comparison Test and the p-series test with $p = 1/2$, we see that the series

$$\sum_{n=0}^{\infty} f(n)$$

is divergent. Consequently, the given series is

$$\boxed{\text{conditionally convergent}}\,.$$

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007 10.0 points

Determine which, if any, of the following series diverge.

$$(A) \quad \sum_{n=1}^{\infty} \frac{(5n)^n}{n!}$$

$$(B) \quad \sum_{n=1}^{\infty} \frac{5^n}{(n+2)^n}$$

1. both of them

2. B only

3. neither of them

4. A only **correct**

Explanation:

To check for divergence we shall use either the Ratio test or the Root test which means computing one or other of

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|, \qquad \lim_{n\to\infty} |a_n|^{1/n}$$

for each of the given series.

(A) The ratio test is the better one to use because

$$\left|\frac{a_{n+1}}{a_n}\right| = 5\left(\frac{n!}{(n+1)!}\frac{(n+1)^{n+1}}{n^n}\right).$$

Now

$$\frac{n!}{(n+1)!} = \frac{1}{n+1},$$

while

$$\frac{(n+1)^{n+1}}{n^n} = (n+1)\left(\frac{n+1}{n}\right)^n.$$

Thus

$$\left|\frac{a_{n+1}}{a_n}\right| = 5\left(\frac{n+1}{n}\right)^n \longrightarrow 5e > 1$$

as $n \to \infty$, so series (A) diverges.

(B) The root test is the better one to apply because

$$|a_n|^{1/n} = \frac{5}{n+2} \longrightarrow 0$$

as $n \to \infty$, so series (B) converges.

Consequently, of the given infinite series,

$$\boxed{\text{only } A \text{ diverges}}\,.$$

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008 10.0 points

Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{1}{n}\frac{x^n}{5^n}.$$

1. interval of cgce $= [-5,\, 5)$ **correct**

2. interval of cgce $= (-\infty,\, \infty)$

3. interval of cgce $= \left(-\frac{1}{5},\, \frac{1}{5}\right)$

4. interval of cgce $= [-5,\, 5]$

5. interval of cgce $= (0,\, 5)$

6. interval of cgce $= \left(-\frac{1}{5},\, 0\right]$

7. converges only at origin

8. interval of cgce $= (-5,\, 5)$

Explanation:

When

$$a_n = \frac{1}{n}\frac{x^n}{5^n},$$

then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n5^n}{(n+1)5^{n+1}}\frac{x^{n+1}}{x^n}\right| = \left|\frac{nx}{5(n+1)}\right|.$$

In this case

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{5}.$$

So by the Ratio Test, the given series

(i) *converges when* $|x| < 5$ *and*

(ii) *diverges when* $|x| > 5$.

It remains to check for convergence at $x = \pm 5$. Now when $x = -5$, the series reduces to

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

which is the alternating harmonic series, hence convergent by the Alternating Series Test. On the other hand when $x = 5$, the series reduces to

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

which is the harmonic series, hence divergent. Consequently

$$\boxed{\text{interval of convergence} = [-5,\, 5)}.$$

009 10.0 points

Find a power series representation for the function

$$f(x) = \tan^{-1}(3x).$$

1. $f(x) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{2n+1}$

2. $f(x) = \sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n}$

3. $f(x) = \sum_{n=0}^{\infty}(-1)^n\frac{3^{2n+1}}{2n+1}x^{2n+1}$ **correct**

4. $f(x) = \sum_{n=0}^{\infty}\frac{3^{2n+1}}{2n+1}x^{2n+1}$

5. $f(x) = \sum_{n=0}^{\infty}(-1)^n\frac{3^{2n}}{2n+1}x^{2n}$

6. $f(x) = \sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1}$

Explanation:

We know that

$$\tan^{-1}x = \int_0^x \frac{1}{1+t^2}\,dt.$$

On the other hand,

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots$$

on the interval $(-1,\, 1)$. Replacing x by $-x^2$ we thus see that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

on $(-1,\, 1)$. But then

$$f(x) = \int_0^x\left\{\sum_{n=0}^{\infty}(-1)^n t^{2n}\right\}dt$$

$$= \sum_{n=0}^{\infty}\left\{(-1)^n\int_0^x t^{2n}\,dt\right\}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$$

on $(-1,\, 1)$. Consequently,

$$\boxed{\tan^{-1}(3x) = \sum_{n=0}^{\infty}(-1)^n\frac{3^{2n+1}}{2n+1}x^{2n+1}}$$

is a power series representation for $\tan^{-1}(3x)$ on $(-\frac{1}{3},\, \frac{1}{3})$.

010 10.0 points

Find the degree 2 Taylor polynomial of f centered at $x = 2$ when

$$f(x) \ = \ 5x \ln x\,.$$

1. $10 + 2\ln 5(x-2) + \dfrac{5}{4}(x-2)^2$

2. $10 + 5\ln 2(x-2) + \dfrac{5}{2}(x-2)^2$

3. $10\ln 2 + 5(\ln 2 + 1)(x-2) + \dfrac{5}{2}(x-2)^2$

4. $10\ln 2 + 5(\ln 2 + 1)(x-2) + \dfrac{5}{4}(x-2)^2$
correct

5. $10 + 5(\ln 2 + 1)(x-2) + \dfrac{5}{4}(x-2)^2$

6. $10\ln 2 + 5\ln 2(x-2) + \dfrac{5}{4}(x-2)^2$

Explanation:

The degree 2 Taylor polynomial of f centered at $x = 2$ is given by

$$T_2(x) \ = \ f(2) + f'(2)(x-2) + \frac{1}{2!}f''(2)(x-2)^2\,.$$

When $f(x) = 5x\ln x$, therefore,

$$f'(x) \ = \ 5\ln x + 5, \qquad f''(x) \ = \ \frac{5}{x}\,.$$

But when $f(2) \ = \ 10\ln 2$,

$$f'(2) \ = \ 5(\ln 2 + 1)\,, \qquad f''(2) \ = \ \frac{5}{2}\,.$$

Consequently, the degree 2 Taylor polynomial centered at $x = 2$ of f is

$$\boxed{10\ln 2 + 5(\ln 2 + 1)(x-2) + \frac{5}{4}(x-2)^2}\,.$$