M427L (55200), Quiz #10 Solutions

Question #1 (5 pts.)

Let $\boldsymbol{c}(t) = (1, t, t^2)$, $0 \leq t \leq 1$ be a path in $\mathbb{R}^3$.

a) With $f(x, y, z) = \sqrt{z(x + 4y^2)}$, compute $\int_{\boldsymbol{c}} f(x, y, z)\, ds$.

Solution: [*Question based on p. 427, #2, 3.*] We have that $\boldsymbol{c}'(t) = (0,\ 1,\ 2t)$, so $ds = \sqrt{1+4t^2}dt$. Therefore, since $x(t) = 1$, $y(t) = t$, and $z(t) = t^2$,

$$\begin{aligned}\int_{\boldsymbol{c}} f(x,y,z)ds &= \int_0^1 \sqrt{t^2(1+4t^2)}\sqrt{1+4t^2}dt \\ &= \int_0^1 t(1+4t^2)dt = \frac{3}{2}.\end{aligned}$$

b) Evaluate

$$\int_{\boldsymbol{c}} y \sin z dx + xyz dy + dz.$$

Solution: [*Question based on p. 447, #2.*] Since $x'(t) = 0$, $y'(t) = 1$, and $z'(t) = 2t$, we have that

$$\begin{aligned}\int_{\boldsymbol{c}} y \sin z dx + xyz dy + dz &= \int_0^1 (x(t)y(t)z(t)y'(t) + z'(t))dt \\ &= \int_0^1 (t^3 + 2t)dt = \frac{5}{4}.\end{aligned}$$

c) How do your answers to parts (a) and (b) change if the path of integration is taken to be $\boldsymbol{e}(t) = (1, 1-t, (1-t)^2)$, $0 \leq t \leq 1$? [Hint: There is a quick way to find the answer.]

Solution: Since $\boldsymbol{e}(t)$, $0 \leq t \leq 1$, traces out the same curve as $\boldsymbol{c}(t)$, $0 \leq t \leq 1$, but in the opposite orientation, we have that $\int_{\boldsymbol{e}} f ds = 3/2$ as before and $\int_{\boldsymbol{e}} y \sin z dx + xyz dy + dz = -5/4$.

Question #2 (3 pts.)

Let C be the **top half** of the ellipse $x^2 + 9y^2 = 4$ in the xy-plane, oriented in a counterclockwise direction. Find $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$, where

$$\boldsymbol{F}(x, y, z) = e^{yz}\boldsymbol{i} + xze^{yz}\boldsymbol{j} + xye^{yz}\boldsymbol{k}.$$

Solution: [*Question based on p. 448, #12 and p. 449, #15.*] Note that $\boldsymbol{F} = \nabla f$ for $f(x, y, z) = xe^{yz}$, so it is a gradient vector field. Therefore, the integral only depends on the endpoints $(2, 0, 0)$ and $(-2, 0, 0)$ of C and is

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_C \nabla f \cdot d\boldsymbol{s} = f(-2,0,0) - f(2,0,0) = -4.$$

Alternatively, one can compute the line integral directly but this is more time-consuming. To do this, consider the parametrization

$$\boldsymbol{c}(t) = \left(2\cos t, \frac{2}{3}\sin t, 0\right), \qquad 0 \le t \le \pi.$$

This traces out the curve C in a counterclockwise direction. Since $x(t) = 2\cos t$, $y(t) = \frac{2}{3}\sin t$, and $z(t) = 0$,

$$\begin{aligned} \int_C \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_0^\pi \left(e^{y(t)z(t)} x'(t) + x(t)z(t)e^{y(t)z(t)} y'(t) + x(t)y(t)e^{y(t)z(t)} z'(t)\right) dt \\ &= \int_0^\pi -2\sin t\, dt = 2[\cos t]_0^\pi = -4. \end{aligned}$$