M427L (55200), Quiz #12 Solutions

Question #1 (4 pts.)

Let S be the sphere of radius 2 (i.e., the surface given by $x^2+y^2+z^2=4$). Suppose there is an electric field $\boldsymbol{E}(x,y,z)=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$. Compute the electric flux

$$\int\int_S \boldsymbol{E}\cdot d\boldsymbol{S}$$

through the surface, where S is oriented so that the unit normal points outward. [Hint: Remember that the vector area element can be written as $d\boldsymbol{S}=\boldsymbol{n}\,dS$, with $\boldsymbol{n}$ the unit normal to the surface and dS the surface area element. Also recall that the surface area of a sphere of radius R is $4\pi R^2$.]

Solution: [*Question based on p. 497, #2, 3 and p. 498, #9.*] Since the unit normal to the sphere S is simply the radial vector

$$\boldsymbol{n}=\frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{\sqrt{x^2+y^2+z^2}}$$

we have that the electric flux is

$$\int\int_S \boldsymbol{E}\cdot d\boldsymbol{S}=\int\int_S (\boldsymbol{E}\cdot\boldsymbol{n})dS=\int\int_S \sqrt{x^2+y^2+z^2}\,dS=2\int\int_S dS=32\pi,$$

where we have used that $\sqrt{x^2+y^2+z^2}=2$ on S.

Question #2 (4 pts.)

Let C be the boundary of the region bounded by $y=x$ and $y=x^2$, oriented in a counterclockwise direction. Consider the line integral

$$\int_C (xy+y^2)dx+x^2\,dy.$$

a) Evaluate the above quantity **using Green's theorem**. [Hint: Remember that the line integral is $\int_C \boldsymbol{F}\cdot d\boldsymbol{s}$, where $\boldsymbol{F}(x,y)=P(x,y)\boldsymbol{i}+Q(x,y)\boldsymbol{j}=(xy+y^2)\boldsymbol{i}+x^2\boldsymbol{j}$.

Solution: [*Question based on p. 528, #1, 2.*] Green's theorem states that for a vector field $\boldsymbol{F}$ in the plane

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{s}=\int\int_D (\mathrm{curl}\,\boldsymbol{F})\cdot\boldsymbol{k}\,dA$$

where D is the region bounded by the curve C. Alternatively, this can be written as

$$\int_C P dx+Q dy=\int\int_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA.$$

Since $\partial Q/\partial x=2x$ and $\partial P/\partial y=x+2y$, by Green's theorem we have that

$$\int_C (xy+y^2)dx+x^2\,dy=\int_0^1\int_{x^2}^x (x-2y)dy\,dx=-\int_0^1 (x^3-x^4)dx=-\frac{1}{20}.$$

b) Compute the line integral directly to verify that your answer to (a) is correct.

Solution: We evaluating the line integral directly by splitting the integral along C into two pieces C_1 and C_2, one along $y = x^2$ and the other along $y = x$. Using the parametrization $\boldsymbol{c}(t) = \left(t, t^2\right)$, $0 \le t \le 1$, the first integral is

$$\int_{C_1} \left(xy + y^2\right)dx + x^2\,dy = \int_0^1 \left(t^3 + t^4\right)dt + \int_0^1 t^2(2t)dt = \frac{19}{20}.$$

Using the parametrization $\boldsymbol{c}(t) = (1-t, 1-t)$, $0 \le t \le 1$, the integral over C_2 is

$$\int_{C_2} \left(xy + y^2\right)dx + x^2\,dy = -\int_0^1 2(1-t)^2 dt - \int_0^1 (1-t)^2 dt = -1.$$

Therefore,

$$\int_C \left(xy + y^2\right)dx + x^2\,dy = \frac{19}{20} - 1 = -\frac{1}{20}$$

as in part (a).