M427L (55200), Quiz #13 Solutions

Question #1 (4 pts.)

Let $S = \{(x,\, y,\, z)\colon x^2 + y^2 + z^2 = 1,\, z \geq 0\}$ be the upper half of the unit sphere (without a bottom) oriented outward. Evaulate

$$\int\int_S (\nabla \times \boldsymbol{V}) \cdot d\boldsymbol{S}$$

where $\boldsymbol{V} = -y\boldsymbol{i} + x\boldsymbol{j} + \big(e^{x-y}\sin z\big)\boldsymbol{k}$. [Hint: Don't evaluate this directly.]

Solution: [*Question based on p. 547, #9.*] By Stokes theorem, we have that

$$\int\int_S (\nabla \times \boldsymbol{V}) \cdot d\boldsymbol{S} = \int_{\partial S} \boldsymbol{V} \cdot d\boldsymbol{s}$$

where the oriented boundary ∂S of S is simply the unit circle in the xy-plane, to be traversed counterclockwise. Parametrizing ∂S by $\boldsymbol{c}(t) = (\cos t,\, \sin t,\, 0)$, $0 \leq t \leq 2\pi$, we find the desired answer:

$$\int_0^{2\pi} \boldsymbol{V}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t)dt = \int_0^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0)dt = 2\pi.$$

Note that this would be significantly more complicated to evaluate directly since one would need to compute $\nabla \times \boldsymbol{V}$, find the unit normal $\boldsymbol{n}$ to S, and then compute a double integral. To see this, note that

$$\nabla \times \boldsymbol{V} = \big(-e^{x-y}\sin z\big)\boldsymbol{i} + \big(-e^{x-y}\sin z\big)\boldsymbol{j} + 2\boldsymbol{k}.$$

Since $\boldsymbol{n} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is the outward unit normal we have that

$$\int\int_S (\nabla \times \boldsymbol{V}) \cdot \boldsymbol{n}\, dS = \int\int_S \big[-(x+y)e^{x-y}\sin z + 2z\big]dS.$$

This first part of the integral yields 0 due to the symmetry of the term $(x+y)e^{x-y}$ and the domain of integration (this can be checked directly but it involves a computation). Using spherical coordinates with $z = \cos\phi$ and $dS = \sin\phi\, d\theta\, d\phi$, the second part yields

$$\int\int_S 2z\, dS = \int_0^{\pi/2}\int_0^{2\pi} 2\cos\phi \sin\phi\, d\theta\, d\phi = 2\pi\big[\sin^2\phi\big]_0^{\pi/2} = 2\pi$$

as before.

Question #2 (4 pts.)

Let $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and define S to be outward oriented surface of the unit cube $W = [0,1] \times [0,1] \times [0,1]$.

a) Directly evaluate the surface integral

$$\int\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

by finding the unit normal $\boldsymbol{n}$ to the surface and using that $d\boldsymbol{S} = \boldsymbol{n}\, dS$. *Show your work!* [Hint: Compute the integral over each face of the cube separately. Each of these integrals is easy to evaluate by noting the particular form of the integrand.]

Solution: [*Question based on p. 574, #4.*] First we compute the three integrals along the xy-, xz-, and yz-planes. Note that for the face $S_{z=0}$ in the xy-plane we have

$$\int\int_{S_{z=0}} \boldsymbol{F} \cdot d\boldsymbol{S} = \int_0^1 \int_0^1 (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot (-\boldsymbol{k}) dx\, dy = -\int_0^1 \int_0^1 z\, dx\, dy = 0$$

since $z = 0$ on this surface. The other two integrals are also 0 by similar reasoning. Finally, the other three integrals are evaluated as follows. For the face $S_{z=1}$ parallel to the xy-plane we have that

$$\int\int_{S_{z=1}} \boldsymbol{F} \cdot d\boldsymbol{S} = \int_0^1 \int_0^1 (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{k}\, dx\, dy = \int_0^1 \int_0^1 z\, dx\, dy = 1$$

since $z = 1$ on this surface. The remaining integrals also yield 1. Putting these integrals together we have that

$$\int\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = 3.$$

b) Re-evaluate the surface integral in part (a) using the divergence theorem.

Solution: Since $\operatorname{div} \boldsymbol{F} = 3$ everywhere, we use the divergence theorem to obtain

$$\int\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int\int_V (\operatorname{div} \boldsymbol{F}) dV = 3 \times \text{volume}(V) = 3.$$