M427L (55200), Quiz #3 Solutions

Question #1 (4 pts.)

At which point on the surface $z = f(x, y) = y^2 - x^2 + 3x$ is the tangent plane *orthogonal* to the line $\boldsymbol{r}(t) = 10t\boldsymbol{i} + 4t\boldsymbol{j} - 2t\boldsymbol{k}$, $t \in \mathbb{R}$?

Solution: [*Question based on Differentiation (2.3), extra problem #1.*] The tangent plane at any point (x_0, y_0, z_0) on the surface is

$$\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

and therefore has normal vector $\boldsymbol{n} = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0), -1\right) = (-2x_0 + 1, 2y_0, -1)$. Since we are looking for a plane which is orthogonal to $\boldsymbol{r}(t)$ for any $t \in \mathbb{R}$, we must have that $\boldsymbol{n} = c\boldsymbol{r}(1)$ for some constant c. Therefore, this yields the system of equations

$$\begin{aligned} -2x_0 + 3 &= 10c \\ 2y_0 &= 4c \\ -1 &= -2c. \end{aligned}$$

This implies $c = 1/2$, so the point on the surface we are looking for is $(x_0, y_0) = (-1, 1)$.

Question #2 (4 pts.)

Compute the following limits, if they exist (as a finite number). If the limit does not exist, state this and **justify your answer**.

a) (*2 pts.*)

$$\lim_{(x,y)\to(0,0)} \frac{-2x^3y + xy^2}{x^6 + y^2}$$

Solution: [*Question based on problem done in lecture.*] This limit does not exist. To see this, consider two paths along which we obtain different values for the limit. For example, if $y = 0$ (i.e., travel along the x-axis) then the limit as $x \to 0$ is 0. On the other hand, if we balance the terms in the denominator by letting $y = y(x) = x^3$ then as $x \to 0$ we get the limit

$$\lim_{x\to 0} \frac{-2x^6 + x^7}{2x^6} = -1.$$

b) (*2 pts.*) (Hint: The limit exists.)

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x - y)}{e^{-y} - e^{-x}}$$

Solution: [*Question based on p. 125, #8(b).*] First, let us rewrite

$$\frac{\sin(x - y)}{e^{-y} - e^{-x}} = e^x \cdot \frac{\sin(x - y)}{e^{x-y} - 1} = e^x \cdot \frac{\sin(z)}{e^z - 1}$$

with $z = x - y$. In addition, since $z \to 0$ as $(x, y) \to (0, 0)$, L'Hopital's rule gives

$$\lim_{z \to 0} \frac{\sin(z)}{e^z - 1} = \lim_{z \to 0} \frac{\cos(z)}{e^z} = 1.$$

Therefore

$$\lim_{(x,y) \to (0,0)} \frac{\sin(x - y)}{e^{-y} - e^{-x}} = \left(\lim_{x \to 0} e^x\right)\left(\lim_{z \to 0} \frac{\sin(z)}{e^z - 1}\right) = 1$$

since we are considering the product of two functions of one variable, each of which are continuous at 0.