M427L (55200), Quiz #4 Solutions

Question #1 (4 pts.)

The *eikonal equation*

$$\|\nabla u\| = 1$$

is a nonlinear partial differential equation (PDE) that arises in the study of electromagnetic wave propogation (it has implications in geometric optics, fluid flow, and heat transfer as well). In 2-D, it is often useful to make a change from Euclidean coordinates to polar coordinates, in which case the eikonal equation reads as

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = 1.$$

Show that these two equations are equivalent by transforming one to the other using the chain rule. [Hint: Choose the direction which is easier, since the other requires implicit differentiation.]

Solution: [*Question based on p. 160, #8.*] It is easier to transform the second equation to the first. With $x = r\cos\theta$ and $y = r\sin\theta$, we use the chain rule to obtain

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\\ \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}.\end{aligned}$$

Another way to write this is as a transformation of differential operators:

$$\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial \theta}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix} = \begin{pmatrix}\frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}}\\ -y & x\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}.$$

Therefore,

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

and we have that $\|\nabla u\| = 1$.

Alternatively, one could go from the first equation to the second. This requires the use of implicit differentiation on the expressions $r^2 = x^2 + y^2$ and $\tan\theta = y/x$:

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{x}{r} = \cos\theta & \frac{\partial r}{\partial y} &= \frac{y}{r} = \sin\theta\\ \frac{\partial \theta}{\partial x} &= -\frac{1}{\sec^2\theta}\cdot\frac{y}{x^2} = -\frac{\sin\theta}{r} & \frac{\partial \theta}{\partial y} &= \frac{1}{\sec^2\theta}\cdot\frac{1}{x} = \frac{\cos\theta}{r}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}.\end{aligned}$$

(In terms of differential operators this is

$$\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\dfrac{\sin\theta}{r} \\ \sin\theta & \dfrac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial \theta} \end{pmatrix},$$

which is exactly what one would obtain upon inverting the matrix in the previous calculation.) It is easy to check that this confirms the equivalence between the two expressions for the PDE.

Question #2 (4 pts.)

Suppose you are hiking up the face of an active volcano in order to take measurements. The elevation profile (in km) of the terrain is described by

$$z = 3 + (x^2 + 2y^2)e^{-x^2-2y^2},$$

where x and y are the east-west and north-south coordinates (in km).

a) (*3 pts.*) You are at location $(x, y) = (-1, -1)$. The maximum grade you can hike up safely is 20% (that is, a maximum slope of 0.2 along any tangent line to the surface). If your goal is to reach the rim as directly as possible, along which two directions could you move at this point? Use the approximation $e^{-3} \approx 1/20$ when needed.

Solution: [*Question based on p. 173, #21.*] The gradient of the elevation profile is

$$\nabla z(x, y) = (1 - x^2 - 2y^2)e^{-x^2-2y^2}(2x\boldsymbol{i} + 4y\boldsymbol{j}).$$

Therefore, the gradient vector at $(1, 1)$ is $\nabla z|_{(-1,-1)} = 4e^{-3}\boldsymbol{i} + 8e^{-3}\boldsymbol{j} \approx \frac{1}{5}(\boldsymbol{i} + 2\boldsymbol{j})$. We seek *unit* direction vectors $\boldsymbol{n} = a\boldsymbol{i} + b\boldsymbol{j}$ (so $a^2 + b^2 = 1$) such that the directional derivative of $z(x, y)$ in the direction of $\boldsymbol{n}$ is 0.2. Therefore,

$$\frac{1}{5} = \nabla z|_{(-1,-1)} \cdot \boldsymbol{n} = \frac{1}{5}(a + 2b)$$

and we must solve the system of equations $a + 2b = 1$ and $a^2 + b^2 = 1$. Substituting the first into the second yields $1 = (1 - 2b)^2 + b^2$, so $b(5b - 4) = 0$. So, either $b = 0$ and $a = 1$ or $b = 4/5$ and $a = -3/5$ and we should move along the directions

$$\boldsymbol{n} = \boldsymbol{i} \qquad \text{or} \qquad \boldsymbol{n} = -\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}.$$

b) (*1 pt.*) Tremors force you to evacuate. If you are still standing at $(-1, -1)$ in which direction should you move initially to *descend* as fast as possible?

Solution: [*Question based on p. 172, #7.*] The direction of steepest ascent is the gradient vector, so the direction of steepest descent at $(-1, -1)$ is

$$\boldsymbol{u} = -\nabla z|_{(-1,-1)} = -\frac{1}{5}(\boldsymbol{i} + 2\boldsymbol{j}).$$

(Technically, if we want a unit direction vector we should normalize this as $\boldsymbol{u}/\|\boldsymbol{u}\|$.)