M427L (55200), Quiz #8 Solutions

Question #1 (4 pts.)

Consider a 3-D vector field of the form

$$\boldsymbol{V} = \nabla\phi + \operatorname{curl}\boldsymbol{F}$$

where

$$\phi(x,y,z) = \frac{1}{12}x^4y^4z^4, \qquad \operatorname{curl}\boldsymbol{F}(x,y,z) = \frac{1}{2}x^2y^2z^2(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}).$$

a) Compute $\nabla^2\phi$ (i.e., $\operatorname{div}\nabla\phi = \phi_{xx}+\phi_{yy}+\phi_{zz}$).

Solution:

$$\nabla^2\phi = x^2y^4z^4 + x^4y^2z^4 + x^4y^4z^2 = x^4y^4z^4\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right).$$

b) What is $\nabla\times\operatorname{curl}\boldsymbol{F}$?

Solution:

$$\begin{aligned}
\nabla\times\operatorname{curl}\boldsymbol{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{1}{2}x^2y^2z^2 & \dfrac{1}{2}x^2y^2z^2 & \dfrac{1}{2}x^2y^2z^2 \end{vmatrix} \\
&= \left(x^2yz^2 - x^2y^2z\right)\boldsymbol{i} + \left(x^2y^2z - xy^2z^2\right)\boldsymbol{j} + \left(xy^2z^2 - x^2yz^2\right)\boldsymbol{k} \\
&= x^2y^2z^2\left[\left(\frac{1}{y}-\frac{1}{z}\right)\boldsymbol{i} + \left(\frac{1}{z}-\frac{1}{x}\right)\boldsymbol{j} + \left(\frac{1}{x}-\frac{1}{y}\right)\boldsymbol{k}\right].
\end{aligned}$$

c) Find $\operatorname{div}\boldsymbol{V}$ and $\operatorname{curl}\boldsymbol{V}$. [Hint: There is a quick way to find these answers!]

Solution: [*Question based on p. 312, #23, 28, 32 and problem done in lecture.*] The divergence and curl of $\boldsymbol{V}$ can be found easily using that curls are divergence-free and gradients are curl-free. That is, since $\operatorname{div}\operatorname{curl}\boldsymbol{F} = 0$,

$$\operatorname{div}\boldsymbol{V} = \operatorname{div}\nabla\phi + \operatorname{div}\operatorname{curl}\boldsymbol{F} = \nabla^2\phi = x^4y^4z^4\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)$$

by part (a). Similarly, since $\nabla\times\nabla\phi = 0$,

$$\operatorname{curl}\boldsymbol{V} = \nabla\times\nabla\phi + \nabla\times\operatorname{curl}\boldsymbol{F} = x^2y^2z^2\left[\left(\frac{1}{y}-\frac{1}{z}\right)\boldsymbol{i} + \left(\frac{1}{z}-\frac{1}{x}\right)\boldsymbol{j} + \left(\frac{1}{x}-\frac{1}{y}\right)\boldsymbol{k}\right]$$

by part (b). Note that it would be significantly more difficult to derive these expressions without using the above properties for the divergence and curl.

Question #2 (4 pts.)

a) Evaluate

$$\int_0^4 \int_0^{\sqrt{x}} 3e^{12y-y^3}\,dy\,dx.$$

Solution: [*Question based on p. 353-354, #2c, 11.*] The integral can be evaluated after reversing the order of integration. The region of integration is

$$D = \{(x, y) : 0 \le y \le \sqrt{x}, \quad 0 \le x \le 4\} = \{(x, y) : y^2 \le x \le 4, \quad 0 \le y \le 2\}.$$

Therefore,

$$\begin{aligned} \int_0^4 \int_0^{\sqrt{x}} 3e^{12y-y^3} dy dx &= \int_0^2 \int_{y^2}^4 3e^{12y-y^3} dx dy \\ &= \int_0^2 3(4-y^2)\, e^{12y-y^3} dy \\ &= \int_0^{16} e^u \, du \\ &= e^{16} - 1. \end{aligned}$$

b) Write a triple integral which gives the volume of the *top half* of a sphere of radius 1 centered at the origin. *Do not* evaluate this integral!

Solution: [*Question based on p. 365, #26 and problem done in lecture.*]

$$V = \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-z^2-y^2}}^{\sqrt{1-z^2-y^2}} dx dy dz.$$