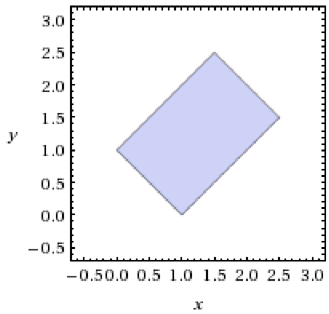
M427L (55200), Quiz #9 Solutions

Question #1 (5 pts.)

Let R be the region bounded by $y=1-x$, $y=4-x$, $y=1+x$, and $y=-1+x$.

a) Sketch the region R in the x-y plane.

Solution:



b) Evaluate

$$\int\int_R (x+y)^2\, e^{x-y}\, dx\, dy$$

by using a change of coordinates $(x,\, y) \mapsto (u,\, v)$ for some choice of u and v. Note that evaluating this without a change of coordinates would require integrating by parts several times.

Solution: [*Question based on p. 390-392, #3, 28.*] Let $u=x+y$ and $v=x-y$. The map from $(x,\, y)$ to (u, v) takes the region R to the region $D=\{(u, v)\colon 1\le u\le 4, -1\le v\le 1\}$. Furthermore, since $x=(u+v)/2$ and $y=(u-v)/2$ we find the Jacobian of the transformation to be

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2.$$

Therefore,

$$\begin{aligned} \int\int_R (x+y)^2\, e^{x-y}\, dx\, dy &= \int\int_D u^2\, e^v \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \\ &= \frac{1}{2}\int_{-1}^{1}\int_{1}^{4} u^2\, e^v\, du\, dv \\ &= \frac{21}{2}\big(e-e^{-1}\big). \end{aligned}$$

Question #2 (3 pts.)

In the last quiz you were asked to give an expression for the volume

$$V = \int\int\int_H dx\, dy\, dz$$

of a *hemisphere* H of radius 1. You are now asked to compute this using spherical coordinates. To do this, recall that the mapping $(\rho,\theta,\phi)\mapsto(x,y,z)$ is given by

$$x=\rho\cos\theta\sin\phi,\qquad y=\rho\sin\theta\sin\phi,\qquad z=\rho\cos\phi$$

with Jacobian

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=-\rho^2\sin\phi.$$

a) Write V as an integral in terms of ρ, θ, and ϕ with appropriate boundaries of integration.

Solution: [*Question based on p. 392, #23 and problem done in lecture.*] The integral is

$$\int\!\!\int\!\!\int_H\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right|d\rho d\theta d\phi=\int_0^{\pi/2}\int_0^{2\pi}\int_0^1\rho^2\sin\phi\, d\rho d\theta d\phi.$$

b) Evaluate this integral explicitly.

Solution:

$$\int_0^{\pi/2}\int_0^{2\pi}\int_0^1\rho^2\sin\phi\, d\rho d\theta d\phi=\left[\frac{\rho^3}{3}\right]_0^1\cdot 2\pi\cdot[-\cos\phi]_0^{\pi/2}=\frac{2}{3}\pi.$$

This agrees with the basic notion that the hemisphere has half the volume of the sphere of radius 1.