M427L (55200), Sample Midterm #1 Solutions

1. Define the vectors

$$\boldsymbol{a} = (4, 2, 0)$$
$$\boldsymbol{b} = (1, -3, 5)$$
$$\boldsymbol{c} = (-2, 2, 1).$$

a) What is the volume of the parallelpiped whose edges are given by the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$?

Solution: $\boldsymbol{c} \cdot (\boldsymbol{b} \times \boldsymbol{a}) = 74$.

b) What is the cosine of the angle between the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$?

Solution: $\cos\theta = \frac{\boldsymbol{b} \cdot \boldsymbol{c}}{\|\boldsymbol{b}\|\|\boldsymbol{c}\|} = \frac{-1}{\sqrt{35}}$.

c) Determine the equation for the plane parallel to the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ that passes through the point $(1, 1, 1)$ (write in the form $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r_0}) = 0$).

Solution: $\boldsymbol{n} = \boldsymbol{a} \times \boldsymbol{b} = (10, -20, -14)$ and $\boldsymbol{r_0} = (1, 1, 1)$.

2. Define

$$h(x, y) = 2x^3 + xy^2 + 5x^2 + y^2.$$

a) Find all critical points of h.

Solution: $(0, 0)$, $(-5/3, 0)$, $(-1, \pm 2)$.

b) Classify all critical points (i.e., determine if they are local maxima, minima, or saddle points) by using the second derivative test. To begin, compute the Hessian of $h(x, y)$.

Solution: $(0, 0)$ local minimum, $(-5/3, 0)$ local maximum, $(-1, \pm 2)$ saddle points.

c) What are the absolute maximum and minimum values of h on the domain $D_1 = \{(x, y)\colon 0 \le x \le 4, 0 \le y \le 5\}$?

Solution: $h(4, 5) = 333$ absolute maximum, $h(0, 0) = 0$ absolute minimum.

3.

a) Find the equation of the tangent plane to the surface $x^2 - e^{xy} + z^2 = 1$ at $(1, 0, 1)$.

Solution: With $f(x, y, z) = x^2 - e^{xy} + z^2$, $\nabla f|_{(1,0,1)} = (2, -1, 2)$ is tangent to surface at $(1, 0, 1)$. Therefore, vector equation for plane is $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r_0}) = 0$ with $\boldsymbol{n} = (2, -1, 2)$ and $\boldsymbol{r_0} = (1, 0, 1)$.

b) Find the equation of the line perpendicular to the surface in part (a) at the point $(1, 0, 1)$.

Solution: $\boldsymbol{r}(t) = \boldsymbol{r}_0 + t\boldsymbol{n}$ with $\boldsymbol{n}, \boldsymbol{r_0}$ as above.

c) What is the distance between the plane found in part (a) and the origin?

Solution: Let $\boldsymbol{a} = (1, 0, 1) - (0, 0, 0) = (1, 0, 1)$, and $\boldsymbol{u} = \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} = \frac{1}{3}(2, -1, 2)$. The desired distance is $d = |\boldsymbol{a} \cdot \boldsymbol{u}| = \frac{4}{3}$.

4.

a) Determine the maximum and minimum values of the function

$$f(x, y) = xy^2$$

on the ellipse $x^2 + \frac{1}{4}y^2 = 1$.

Solution: Max. value $= \frac{8}{3\sqrt{3}}$, min. value $= \frac{-8}{3\sqrt{3}}$.

b) Moving clockwise along the ellipse, is the function increasing or decreasing at the point $(0, -2)$?

Solution: Decreasing, since the directional derivative of f at $(0,\ -2)$ in the direction $(-1,\ 0)$ (unit tangent vector to ellipse at $(0,\ -2)$ when traversing clockwise) is $\nabla f_{(0,-2)} \cdot (-1, 0) = -4 < 0$.

5. Define the function

$$f(x, y, z) = x \sin(yz).$$

a) Determine $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.

Solution: $\frac{\partial f}{\partial x} = \sin(yz)$, $\frac{\partial f}{\partial y} = xz\cos(yz)$, $\frac{\partial f}{\partial z} = xy\cos(yz)$.

b) Does f satisfy Laplace's equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$?

Solution: No.

c) Suppose

$$x(s, t) = \cos(s^2 + t)$$
$$y(s, t) = e^{-2st}$$
$$z(s, t) = s^3 - 2st^2 + 4.$$

Determine $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

Solution:

$$\frac{\partial f}{\partial s} = -2s\sin(s^2 + t)\sin(yz) - 2te^{-2st}xz\cos(yz) + (3s^2 - 2t^2)xy\cos(yz)$$

$$\frac{\partial f}{\partial t} = -\sin(s^2 + t)\sin(yz) - 2se^{-2st}xz\cos(yz) - 4stxy\cos(yz).$$

d) Find the directional derivative of f at the point $(2, 0, 1)$ in the direction of the vector $\boldsymbol{v} = -\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$. How does this value compare to the maximal rate of increase at $(2, 0, 1)$?

Solution: The gradient at $(2,\ 0,\ 1)$ is $\nabla f|_{(2,0,1)} = (0,\ 2,\ 0)$, and directional derivative in direction $(-1, 1, 4)$ is $\nabla f|_{(2,0,1)} \cdot \frac{1}{\sqrt{18}}(-1, 1, 4) = \frac{\sqrt{2}}{3}$. Maximal rate of increase is $\|\nabla f(2, 0, 1)\| = 2$.

6. Consider the vector-valued function

$$\boldsymbol{r}(t) = \left(2e^t, 3t^2, te^{4t}\right).$$

a) What is $\boldsymbol{r}''(t)$?

   Solution: $\left(2e^t, 6, 8e^{4t} + 16te^{4t}\right)$.

b) At what point does the curve $\boldsymbol{r}(t)$ intersect the surface $16z = x^4$?

   Solution: When $t = 1$, i.e., the point $(2e, 3, e^4)$.

c) Find the tangent vector to the curve at the point $(2, 0, 0)$.

   Solution: $(2, 0, 1)$.

7. Does the limit

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y - z}{x^4 + y^2 + z}$$

exist? Show why or why not.

Solution: No. Take limit along paths $x = z = 0$ and $y = x^2$, $z = x^4$ to obtain different values.