M427L (55200), Sample Midterm #2 Solutions

1.

a) Draw the vector field $\boldsymbol{F}(x,y) = (y, -x)$.

Solution: None given.

b) Find an expression for a flow line $\boldsymbol{c}(t)$ for $\boldsymbol{F}$ by taking a good guess.

Solution: $\boldsymbol{c}(t) = (\cos t, \sin t)$.

c) Do the same for $\boldsymbol{F}(x,y) = (2x, -y)$.

Solution: $\boldsymbol{c}(t) = (\exp(2t), \exp(-t))$.

2.

a) Find the arc length of $\boldsymbol{c}(t) = \left(2t, t^2\right)$, $0 \le t \le 1$.

Solution: $\sqrt{2} + \log\left(\sqrt{2} + 1\right)$.

b) Find the arc length of $\boldsymbol{c}(t) = \left(2\sin^2 t, \sin^4 t\right)$, $0 \le t \le \pi/2$.

Solution: $\sqrt{2} + \log\left(\sqrt{2} + 1\right)$.

c) Explain your answers to parts (a) and (b).

Solution: The two parametrizations are the same.

3.

a) Evaluate the following integral over the region D bounded by the lines $x+y=1$, $x+y=-1$, $x-y=1$, and $x-y=-1$:

$$\int\int_D 3(x+y)e^{x-y}\,dx\,dy.$$

Solution: 0.

b) Evaluate

$$\int_0^1 \int_0^x \sin y^2\,dy\,dx.$$

Solution: $(1-\cos(1))/2$.

c) Find the mass of a wall described by $0 \le y \le -x^2 - 2x + 3$, $-3 \le x \le 3$, having mass density $\rho(x,y) = 2|y| + 3$.

Solution: 1848/5.

4. Let $\boldsymbol{F}(x,y) = e^x \cos 3y\,\boldsymbol{i} - 3e^x \sin 3y\,\boldsymbol{j}$.

[Hint: Use shortcuts to quickly find answers to the following questions.]

a) Find an $f(x, y)$ such that $\nabla f = \boldsymbol{F}$ for all (x, y).

Solution: $f(x, y) = e^x \cos(3y) + C$ for any constant C.

b) Evaluate $\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s}$ for the path $\boldsymbol{c}(t) = (\cos t, \sin t)$, $0 \le t \le \pi$.

Solution: $f(-1, 0) - f(1, 0) = e^{-1} - e^1$.

c) Compute $\operatorname{div} \boldsymbol{F}$ and $\operatorname{curl} \boldsymbol{F}$.

Solution: $\operatorname{div} \boldsymbol{F} = -8e^x \cos 3y$, $\operatorname{curl} \boldsymbol{F} = \boldsymbol{0}$.

5.

a) A particle moves in a path $\boldsymbol{c}(t) = (2t, 3t, t)$ in a force field $\boldsymbol{F} = (2x, 2y, 4z)$. What is the work done by $\boldsymbol{F}$ in the time interval $0 \le t \le 5$?

Solution: 375.

b) With the same $\boldsymbol{c}(t)$ and $f(x, y, z) = xyz$, what is $\int_{\boldsymbol{c}} f ds$?

Solution: $\frac{3}{2}\sqrt{14} \cdot 5^4$.

6. A hole of radius 1/2 is drilled through the axis of symmetry of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$.

a) Write (but do not compute) an integral for the volume of the remaining piece in cartesian coordinates.

Solution:

$$4 \int_{1/2}^{1} \int_{\sqrt{\frac{1}{4} - x^2}}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} dz dy dx.$$

b) Write (but do not compute) an integral for the volume of the remaining piece in cylindrical coordinates.

Solution:

$$\int_0^{2\pi} \int_0^{1/2} \int_0^{\sqrt{1-r^2}} r dz dr d\theta.$$

c) Compute the volume.

Solution: $\frac{\pi}{3}\left(1 - \left(\frac{3}{4}\right)^{3/2}\right)$.

7.

a) Use the vector field $\boldsymbol{V} = \nabla f$ with $f(x, y, z) = c_1 x + c_2 y + c_3 z$ to explain the meaning of divergence. For this particular choice of vector field, what is $\nabla \times \boldsymbol{V}$?

Solution: See lecture notes.

b) Use the vector field $\boldsymbol{V}(x, y) = -\omega y \boldsymbol{i} + \omega x \boldsymbol{j}$ to explain the meaning of curl. For this particular choice of vector field, what is $\nabla \cdot \boldsymbol{V}$?

Solution: See lecture notes.

c) Are the divergence and curl at a point local quantities or do they depend on the entire vector field?

Solution: Local quantitites.