

## Chapter 5

# INVERSION

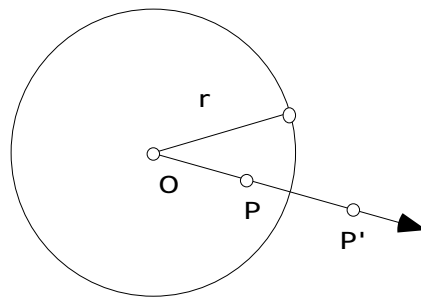
The notion of inversion has occurred several times already, especially in connection with Hyperbolic Geometry. Inversion is a transformation different from those of Euclidean Geometry that also has some useful applications. Also, we can delve further into hyperbolic geometry once we have developed some of the theory of inversion. This will lead us to the description of isometries of the Poincaré Disk and to constructions via Sketchpad of tilings of the Poincaré disk just like the famous ‘Devils and Angels’ picture of Escher.

**5.1 DYNAMIC INVESTIGATION.** One very instructive way to investigate the basic properties of inversion is to construct inversion via a script in Sketchpad. One way of doing this was described following Theorems 3.5.3 and 3.5.4 in Chapter 3, but in this section we’ll describe an alternative construction based more closely on the definition of inversion. Recall the definition of inversion given in section 5 of chapter 3.

**5.1.1 Definition.** Fix a point  $O$  and a circle  $C$  centered at  $O$  of radius  $r$ . For a point  $P$ ,  $P \neq O$ , the *inverse* of  $P$  is the unique point  $P'$  on the ray starting from  $O$  and passing through  $P$  such that  $OP \cdot OP' = r^2$ .

The point  $O$  is called the *center of inversion* and circle  $C$  is called the *circle of inversion*, while  $r$  is called the *radius of inversion*.

$$\begin{aligned} OP &= 0.51 \text{ inches} \\ OP' &= 1.08 \text{ inches} \\ r &= 0.74 \text{ inches} \\ OP \cdot OP' &= 0.55 \text{ inches}^2 \\ r^2 &= 0.55 \text{ inches}^2 \end{aligned}$$



To write a script that constructs the inverse of a point  $P$  given the circle of inversion and its center, we can proceed as follows using the dilation transformation.

- Begin with a new sketch and a new script. Press the record button in the script window.
- In the sketch, draw a circle by center and point. Label the center by  $O$  and label the point on the circle by  $R$ . Construct a point  $P$  not on the circle. Construct the ray from the center of the circle, passing through  $P$ . Construct the point of intersection between the circle and the ray, label it  $D$ .
- Mark the center of the circle - this will be the center of dilation. Then while holding down the Shift key, select the center of the circle, then the point  $P$ , and then the point of intersection of the ray and the circle. While continuing to hold down the shift key go to “Mark Ratio  $OD/OP$ ” under the **Transform** menu. This defines the ratio of the dilation.
- Now select the point of intersection of the ray and the circle, and dilate by the marked ratio. The dilated point is the inverse point to  $P$ . Label the dilated point  $P'$ .
- Hide *everything* except  $P$  and  $P'$ .
- Under the **Work** Menu, find the script, and press the “Stop” button. You may wish to use Auto-Matching for  $O$  and  $R$  as we are about to use our inversion script to explore many examples. Under the **Givens** List for your script, double click on  $O$  and change the name to *Auto-O*. Then double-click on  $R$  and change the name to *Auto-R*. To make use of the Auto-Matching you need to start with a circle that has center labeled by  $O$  and a point on the circle labeled by  $R$ .
- Save your script.

Use your script to investigate the following.

**5.1.2 Exercise.** Where is the inverse of  $P$  if

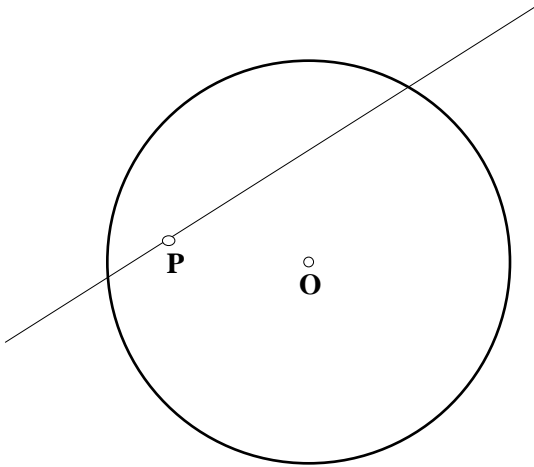
- $P$  is outside the circle of inversion? \_\_\_\_\_
- $P$  is inside the circle of inversion? \_\_\_\_\_
- $P$  is on the circle of inversion? \_\_\_\_\_
- $P$  is the center of the circle of inversion? \_\_\_\_\_

Using our script we can investigate how inversion transforms various figures in the plane by using the construct “Locus” property in the **Construct** menu. Or by using the “trace” feature. For instance, let’s investigate what inversion does to a straight line.

- Construct a circle of inversion. Draw a straight line and construct a free point on the line. Label this free point by  $P$ .

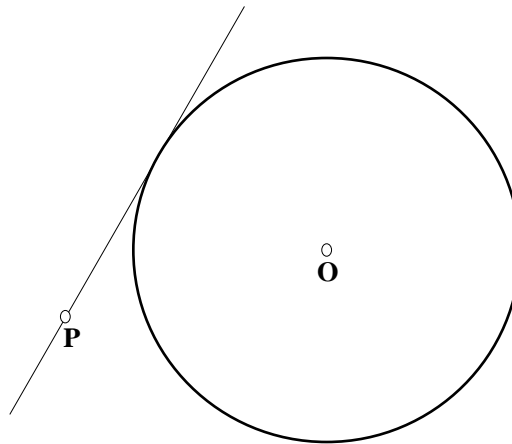
- Use your script to construct the inverse point  $P'$  to  $P$ .
- While holding down the shift and option keys, select in order the point  $P$  and then the point  $P'$ . Then select “Locus” in the **Construct** menu. (Alternatively, one could trace the point  $P'$  while dragging the point  $P$ .)

**5.1.3 Exercise.** What is the image of a straight line under inversion? By considering the various possibilities for the line describe the locus of the inversion points. Be as detailed as you can.



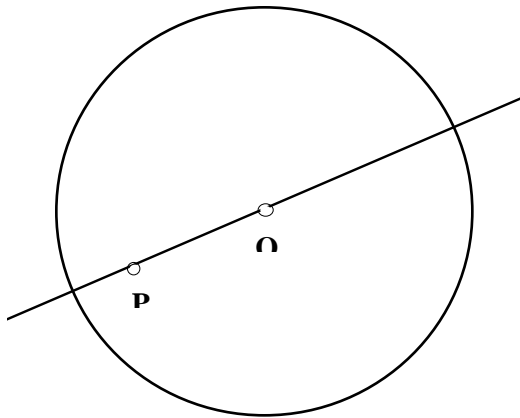
A line, which passes through the circle of inversion

Image: \_\_\_\_\_

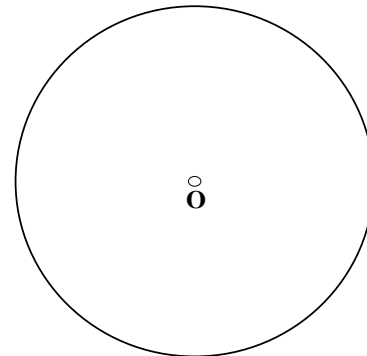


A line, which is tangent to the circle of inversion

Image: \_\_\_\_\_



A line, which passes



A line, which doesn't

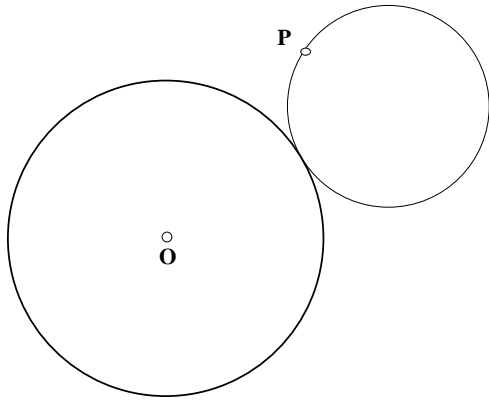
through the center of the circle of inversion

intersect the circle of inversion

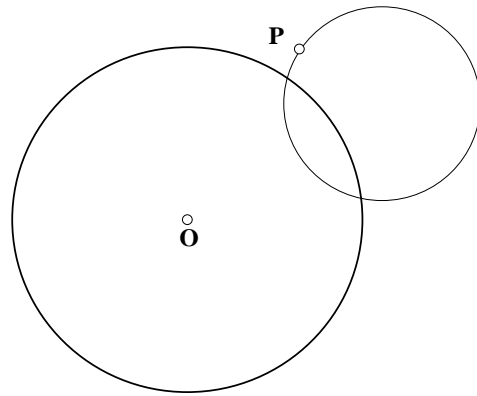
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Image: \_\_\_\_\_

**5.1.4 Exercise.** What is the image of a circle under inversion? By considering the various possibilities for the line describe the locus of the inversion points. Be as detailed as you can.



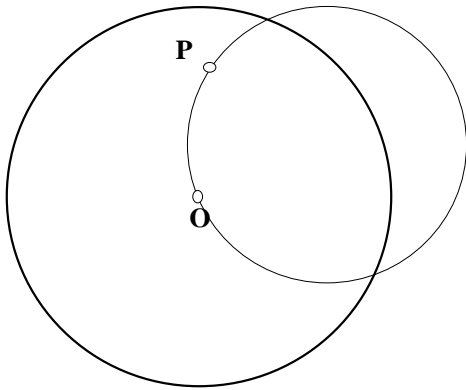
A circle, which is tangent to the circle of inversion



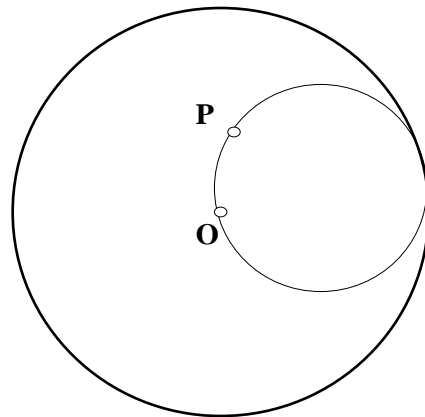
A circle, which intersects the circle of inversion in two points.

Image: \_\_\_\_\_

Image: \_\_\_\_\_



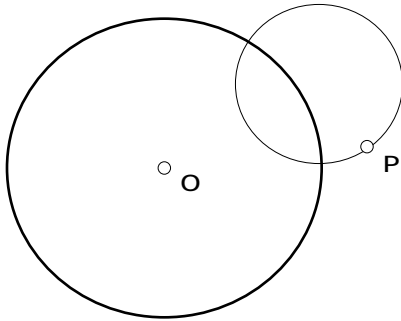
A circle, which passes through the center of the circle of inversion



A circle passing through the center of the circle of inversion, also internally tangent

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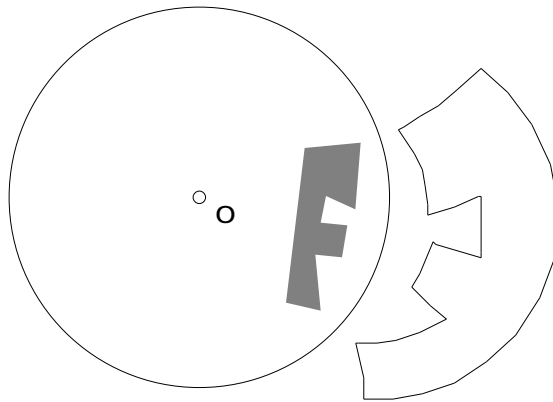
A circle which is orthogonal to the circle of inversion.

Image: \_\_\_\_\_

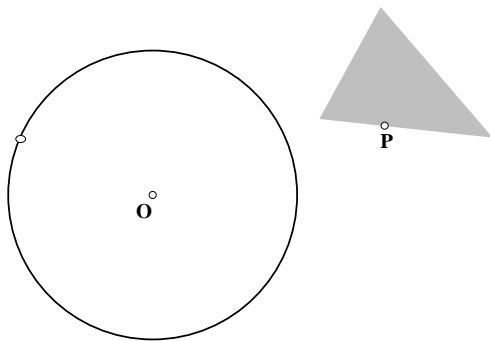
You should have noticed that some circles are transformed into another circle under the inversion transformation. Did you notice what happens to the center of the circle under inversion in these cases? Try it now.

**End of Exercise 5.1.4.**

You can easily construct the inverse image of polygonal figures by doing the following. Construct your figure and its interior. Next hide the boundary lines and points of your figure so that only the interior is visible. Next select the interior and choose “Point on Object” from the **Construct** Menu. Now construct the inverse of that point and then apply the locus construction. Here is an example.

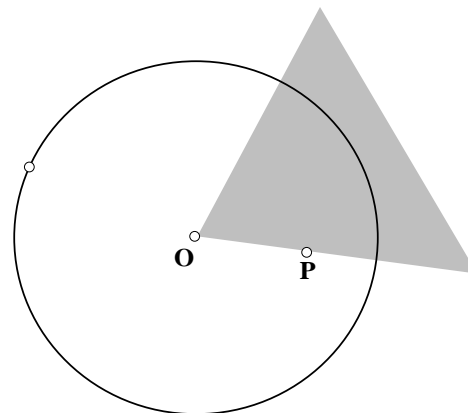


**5.1.5 Exercise.** What is the image of other figures under inversion? By considering the various possibilities for the line describe the locus of the inversion points. Be as detailed as you can.



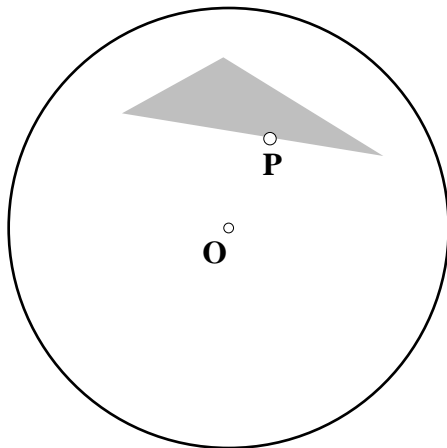
A triangle, external to the circle of inversion

Image: \_\_\_\_\_



A triangle, with one vertex as the center of the circle of inversion

Image: \_\_\_\_\_



A triangle internal to the  
circle of inversion

Image: \_\_\_\_\_

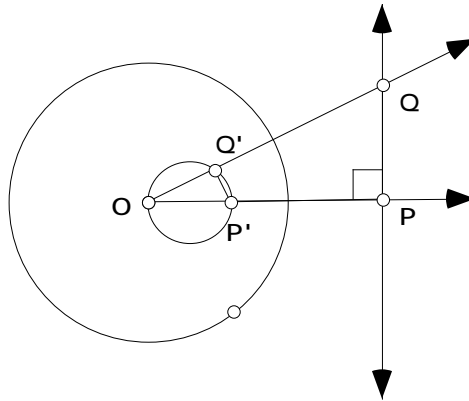
**5.2 PROPERTIES OF INVERSION.** Circular inversion is not a transformation of the Euclidean plane since the center of inversion does not get mapped to a point in the plane. However if we include the point at infinity, we would have a transformation of the Euclidean Plane and this point at infinity. Also worth noting is that if we apply inversion twice we obtain the identity transformation. With these observations in mind we are now ready to work through some of the basic properties of inversion. Let  $C$  be the circle of inversion with center  $O$  and radius  $r$ . Also, when we say “line”, we mean the line including the point at infinity. The first theorem is easily verified by observation.

**5.2.1 Theorem.** Points inside  $C$  map to points outside of  $C$ , points outside map to points inside, and each point on  $C$  maps to itself. The center  $O$  of inversion maps to  $\{ \infty \}$

**5.2.2 Theorem.** The inverse of a line through  $O$  is the line itself.

Again, this should be immediate from the definition of inversion, however note that the line is not pointwise invariant with the exception of the points on the circle of inversion. Perhaps more surprising is the next theorem.

**5.2.3 Theorem.** The inverse image of a line not passing through  $O$  is a circle passing through  $O$ .

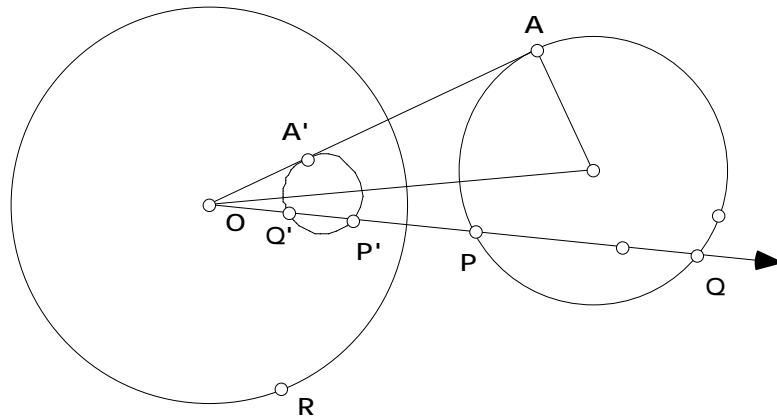


**Proof.** Let  $P$  be the foot of the perpendicular from  $O$  to the line. Let  $Q$  be any other point on the line. Then  $P$  and  $Q$  are the respective inverse points. By the definition of inverse points,  $OP \cdot OP' = OQ \cdot OQ'$ . We can use this to show that  $OPQ$  is similar to  $OQ'P'$ . Thus the image of any  $Q$  on the line is the vertex of a right angle inscribed in a circle with diameter  $OP$ .

The proof of the converse to the previous theorem just involves reversing the steps. The converse states, the inverse image of a circle passing through  $O$  is a line not passing through  $O$ . Notice that inversion is different from the previous transformations that we have studied in that lines do not necessarily get mapped to lines. We have seen that there is a connection between lines and circles.

**5.2.4 Theorem.** The inverse image of a circle not passing through  $O$  is a circle not passing through  $O$ .





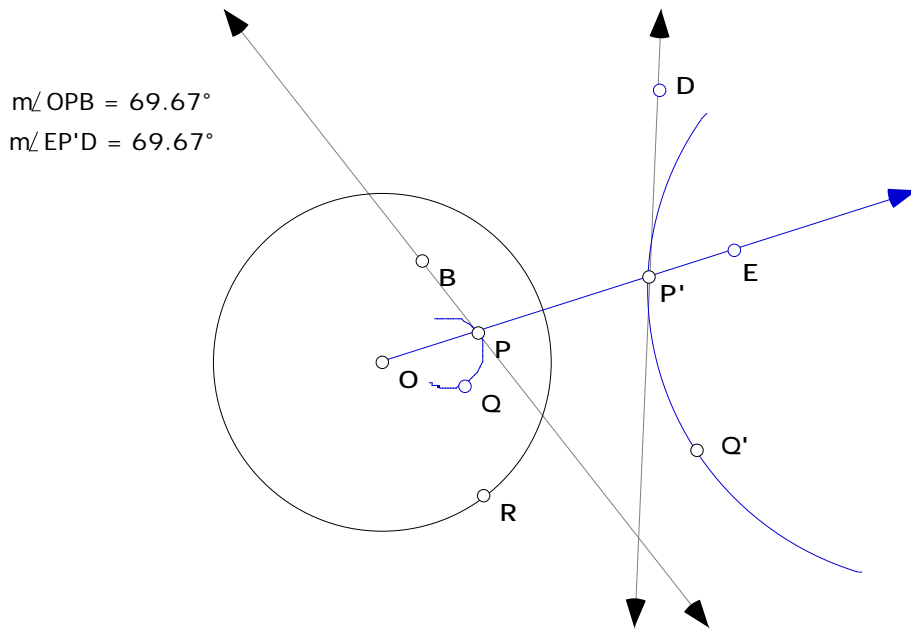
**Proof.** Construct any line through the center of inversion which intersects the circle in two points  $P$  and  $Q$ . Let  $P'$  and  $Q'$  be the inverse points to  $P$  and  $Q$ . We know that

$OP \cdot OP' = OQ \cdot OQ' = r^2$ . Also by Theorem 2.9.2 (Power of a Point),  $OP \cdot OQ = OA^2 = k$ .

Thus  $\frac{OP \cdot OP'}{OP \cdot OQ} = \frac{OQ \cdot OQ'}{OP \cdot OQ} = \frac{r^2}{k}$  or  $\frac{OP'}{OQ} = \frac{OQ'}{OP} = \frac{r^2}{k}$ . In other words, everything reduces to a dilation.

**5.2.5 Theorem.** Inversion preserves the angle measure between any two curves in the plane. That is, inversion is conformal.

**Proof.** It suffices to look at the case of an angle between a line through the center of inversion and a curve. In the figure below,  $P$  and  $Q$  are two points on the given curve and  $P'$  and  $Q'$  are the corresponding points on the inverse curve. We need to show that  $m \angle OPB = m \angle EPD$ . The sketchpad activity below will lead us to the desired result.



- Open a new sketch and construct the circle of inversion with center  $O$  and radius  $r$ . Construct an arc by 3 points inside the circle and label two of the points as  $P$  and  $Q$ . Next construct the inverse of the arc by using the locus construction and label the points  $P'$  and  $Q'$ . Finally construct the line  $OP$  (it will be its own inverse).
- Next construct tangents to each curve through  $P$  and  $P'$  respectively.
- Notice that  $P, Q, P'$ , and  $Q'$  all lie on a circle. Why? Thus  $\angle QPP'$  and  $\angle P'Q'Q'$  are supplementary (Inscribed Angle Theorem).
- Thus  $m\angle OPQ = m\angle P'Q'Q'$ . Check this by measuring the angles.
- Next drag  $Q$  towards the point  $P$ . What are the limiting position of the angles  $\angle OPQ$  and  $\angle P'Q'Q'$ ?
- What result does this suggest?

**5.2.6 Theorem.** Under inversion, the image of a circle orthogonal to  $C$  is the same circle (setwise, not pointwise).

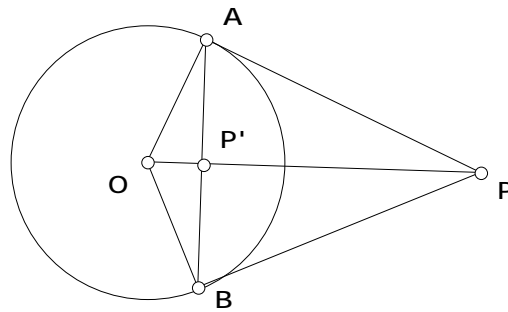
**Proof.** See Exercise 5.3.1.

**5.2.7 Theorem.** Inversion preserves the generalized cross ratio  $\frac{d(P,M)}{d(P,N)} \frac{d(Q,N)}{d(Q,M)}$  of any four distinct points  $P, Q, M$ , and  $N$  in the plane.

**Proof.** See **Exercise 5.3.3**.

Recall our script for constructing the inverse of a point relied on the dilation transformation. A compass and straightedge construction is suggested by the next result.

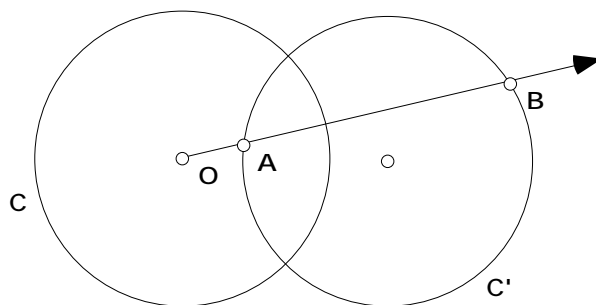
**5.2.8 Theorem.** The inverse of a point outside the circle of inversion lies on the line segment joining the points of intersection of the tangents from the point to the circle of inversion.



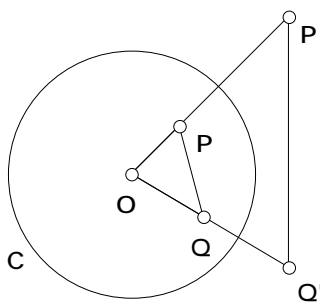
**Proof.** By similar triangles  $OAP$  and  $OP'A$ ,  $\frac{OA}{OP} = \frac{OP'}{OA}$ . Use this to conclude that  $P$  and  $P'$  are inverse points.

**5.3 Exercises.** These exercises are all related to the properties of inversion.

**Exercise 5.3.1.** Prove Theorem 5.2.6. That is if  $C$  is the circle of inversion and  $C$  is orthogonal to it, draw any line through  $O$  which intersects  $C$  in  $A$  and  $B$  and show that  $A$  and  $B$  must be inverse to each other.



**Exercise 5.3.2.** Let  $C$  be the circle of inversion with center  $O$ . Show that if  $P$  and  $Q$  are the inverse images of  $P'$  and  $Q'$  then  $OPQ \sim OQ'P'$ .



**Exercise 5.3.3.** Do the following to prove Theorem 5.2.7. Let  $P, Q, N,$  and  $M$  be any four distinct points in the plane. Use Exercise 5.3.2 to show that  $\frac{PM}{PN} = \frac{OM}{ON}$  and  $\frac{PM}{PN} = \frac{OM}{ON}$ .

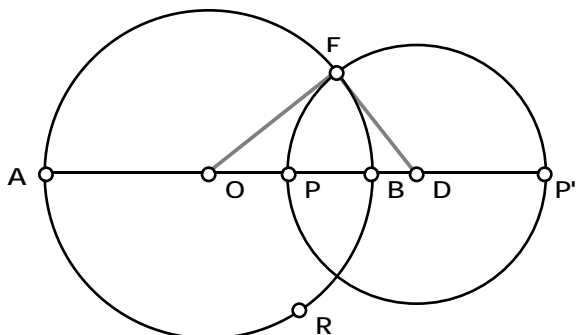
Show that these imply  $\frac{PM}{PN} = \frac{PM}{PN} \cdot \frac{ON}{OM}$ .

Complete the proof that  $\frac{PM}{PN} \cdot \frac{QN}{QM} = \frac{PM}{PN} \cdot \frac{QN}{QM}$ .

**Exercise 5.3.4.** Use Theorem 5.2.8 and Sketchpad to give compass and straightedge constructions for the inverse point of  $P$  when  $P$  is inside the circle of inversion and when  $P$  is outside the circle of inversion.

**Exercise 5.3.5.** Let  $C$  be the circle having the line segment  $\overline{AB}$  as a diameter, and let  $P$  and  $P'$  be inverse points with respect to  $C$ . Now let  $E$  be a point of intersection of  $C$  with the circle

having the line segment  $\overline{PP'}$  as diameter. See the figure below. Prove that  $\angle OED = 90^\circ$ .  
 (Hint: Recall the theorems about the Power of a Point in Chapter 2.)



**Exercise 5.3.6.** Again, let  $C$  be the circle having the line segment  $\overline{AB}$  as a diameter, and let  $P$  and  $P'$  be inverse points with respect to  $C$ . Now let  $C'$  be the circle having the line segment  $\overline{PP'}$  as diameter. See the figure above. Prove that  $A$  and  $B$  are inverse points with respect to  $C'$ .  
 (Hint: Recall the theorems about the Power of a Point in Chapter 2.)

**5.4 APPLICATIONS OF INVERSION** There are many interesting applications of inversion. In particular there is a surprising connection to the Circle of Apollonius. There are also interesting connections to the mechanical linkages, which are devices that convert circular motion to linear motion. Finally, as suggested by the properties of inversion that we discovered there is a connection between inversion and isometries of the Poincaré Disk. In particular, inversion will give us a way to construct “hyperbolic reflections” in h-lines. We will use this in the next section to construct tilings of the Poincaré Disk.

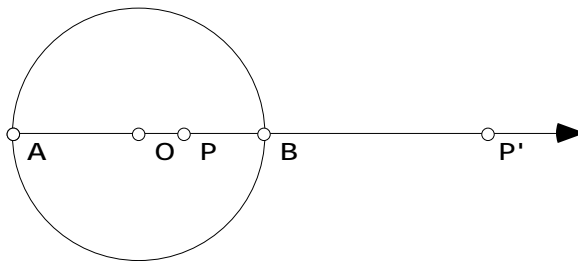
First let's look at the Circle of Apollonius and inversion in the context of a magnet. A common experiment is to place a magnet under a sheet of paper and then sprinkle iron filings on top of the paper. The iron filings line up along circles passing through two points, the North and South poles, near the end of the magnets. These are the Magnetic lines of force. The theory of magnetism then studies equipotential lines. These turn out to be circles each of which is orthogonal to all the magnetic lines of force. The theory of inversion was created to deal with the theory of magnetism. We can interpret these magnetic lines of force and equipotential lines within the geometry of circles.

- Open a new sketch and construct a circle having center  $O$  and a point on the circle labeled  $R$ .
- Next construct any point  $P$  inside the circle and the inverse point  $P'$ . Construct the diameter  $\overline{AB}$  of the circle of inversion that passes through the point  $P$ .
- Finally construct the circle with diameter  $\overline{PP'}$  and construct any point  $Q$  on this circle.
- Construct the segments  $\overline{AQ}$  and  $\overline{BQ}$ . Select them using the arrow tool in that order (while holding down the shift key) and choose “Ratio” from the **Measure** menu. You should be computing the ratio  $\frac{AQ}{BQ}$ .
- Drag the point  $Q$ . What do you notice? What does this tell you about the circle with diameter  $\overline{PP'}$ ?

**5.4.1 Conjecture.** If  $P$  and  $P'$  are inverse points with respect to circle  $C$  and lie on the diameter  $\overline{AB}$  of  $C$  then the circle with diameter  $\overline{PP'}$  is \_\_\_\_\_.

Towards the proof of the conjecture we'll need the following.

**5.4.2 Theorem.** Given  $P$  and  $P'$  which are inverse points with respect to a circle  $C$  and lie on the diameter  $\overline{AB}$  of  $C$ , then  $\frac{AP}{BP} = \frac{AP'}{BP'}$ .

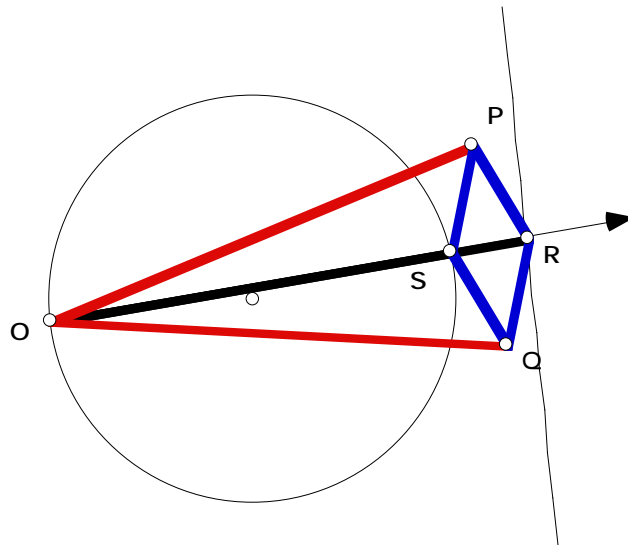


**Proof.** Since  $P$  and  $P'$  are inverse points  $OP \cdot OP' = OB^2$  or  $\frac{OP}{OB} = \frac{OB}{OP'}$ . Now one can check that if  $a/b = c/d$  then  $\frac{a+b}{a-b} = \frac{c+d}{c-d}$  so that  $\frac{OP+OB}{OP-OB} = \frac{OB+OP}{OB-OP}$  or  $\frac{AP}{BP} = \frac{AP'}{BP'}$ . **QED**

The completion of the proof can be found in **Exercise Set 5.6**.

Another interesting application of inversion underlies one possible mechanical linkage that converts circular motion to linear motion. Such a change of motion from circular to linear occurs in many different mechanical settings from the action of rolling down the window of your car to the pistons moving within the cylinders in the engine of the car. The Peaucellier linkage figure below shows the components. The boldface line segments represent rigid rods such that  $PR = PS = QR = QS$  and  $OP = OQ$ . There are hinges at the join of these rods at  $O$ ,  $P$ ,  $Q$ ,  $R$ , and  $S$ . Points  $P$ ,  $Q$ , and  $R$  can move freely while  $S$  is free to move on a circle  $C$  and  $O$  is fixed on that circle. Surprisingly, as  $S$  moves around the circle the point  $R$  traces out a straight line. It is an interesting exercise to try to construct this linkage on Sketchpad. Try it! In case you get stuck, one such construction is given below. The “proof” that  $R$  should trace out a straight line is part of the next assignment.

**5.4.2a Demonstration. Constructing a Peaucellier Linkage.**

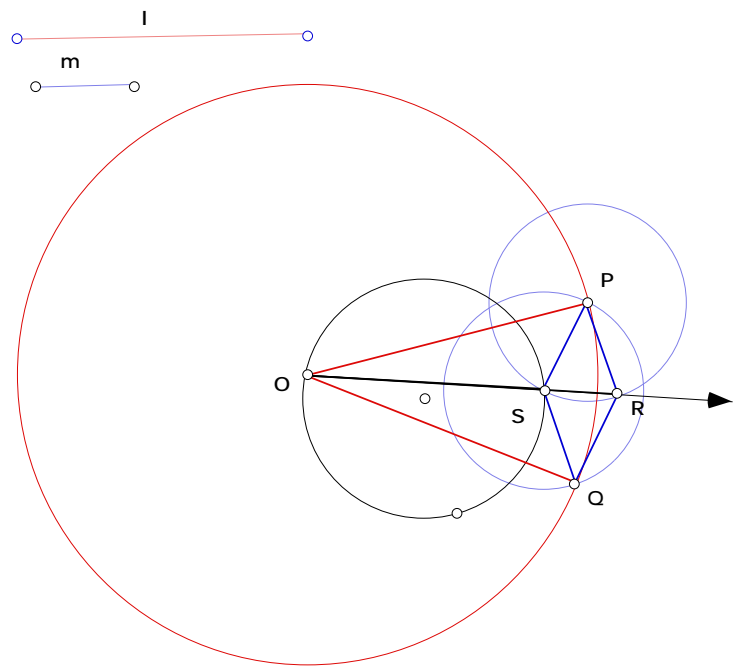


- Open a new sketch and construct a circle. Draw the ray  $\overline{OS}$  where  $O$  and  $S$  are points on the circle.
- In the corner of your sketch construct two line segments  $l$  and  $m$ . (See below. Segment  $l$  will determine the length of  $\overline{OP}$  and segment  $m$  will determine the length of  $\overline{PS}$ .) Color  $l$  red, and color  $m$  blue.
- Construct a circle with center  $O$  and the same length as segment  $l$  and another circle with center  $S$  and the same length as segment  $m$ . Color the circles appropriately. Adjust  $l$  and  $m$

if necessary so that the circles intersect outside of  $C$ . Next construct the intersection points of the circles and label them  $P$  and  $Q$ , respectively.

- Construct a circle with center  $P$  and radius the same as segment  $m$ . Label the intersection point with the ray  $\overrightarrow{OS}$  by  $R$ . Join the points to construct the rhombus  $PRQS$  and color the segments blue.
- Construct the segments  $\overline{OP}$  and  $\overline{OQ}$ , then color them red.
- Finally select the point  $R$  and choose “Trace Points” from the **Display** Menu and then drag  $S$  making sure that  $O$  is staying fixed. (Or alternatively, select the point  $R$  and then the point  $S$  and then choose “Locus” from the **Construct** Menu.)
- Do you notice anything special about the line that is traced out?  
Can you describe it in another way?

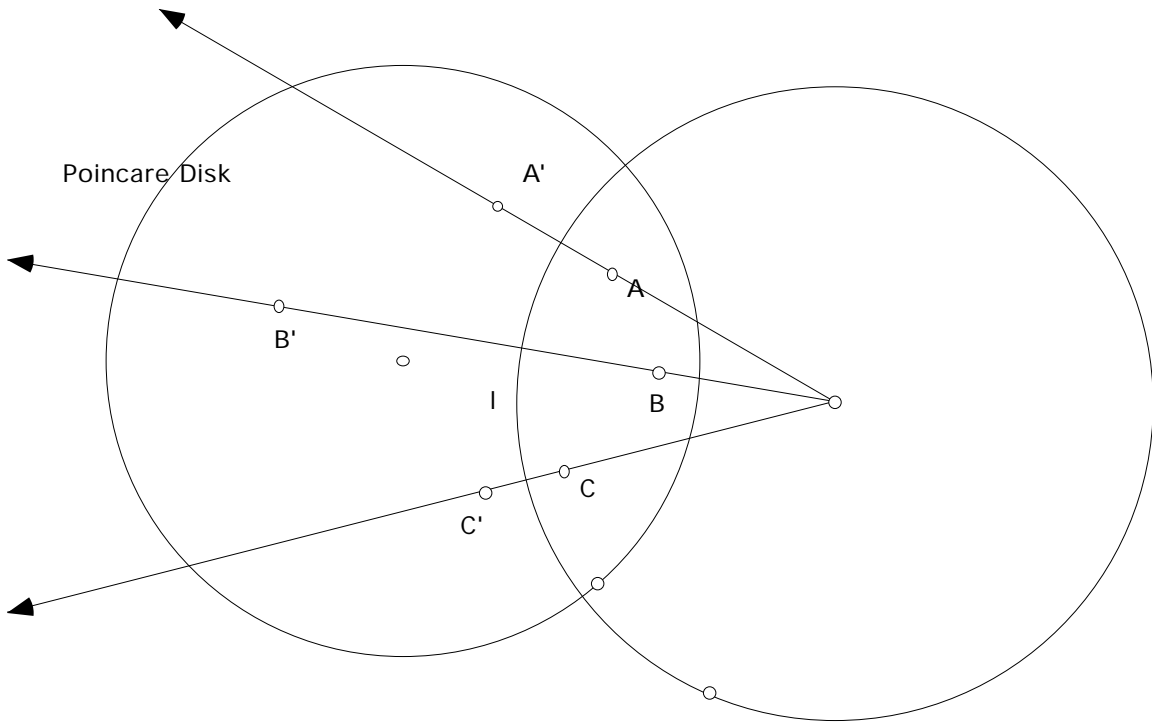
Try various positions for  $O$ .



#### End of Demonstration 5.4.2a.

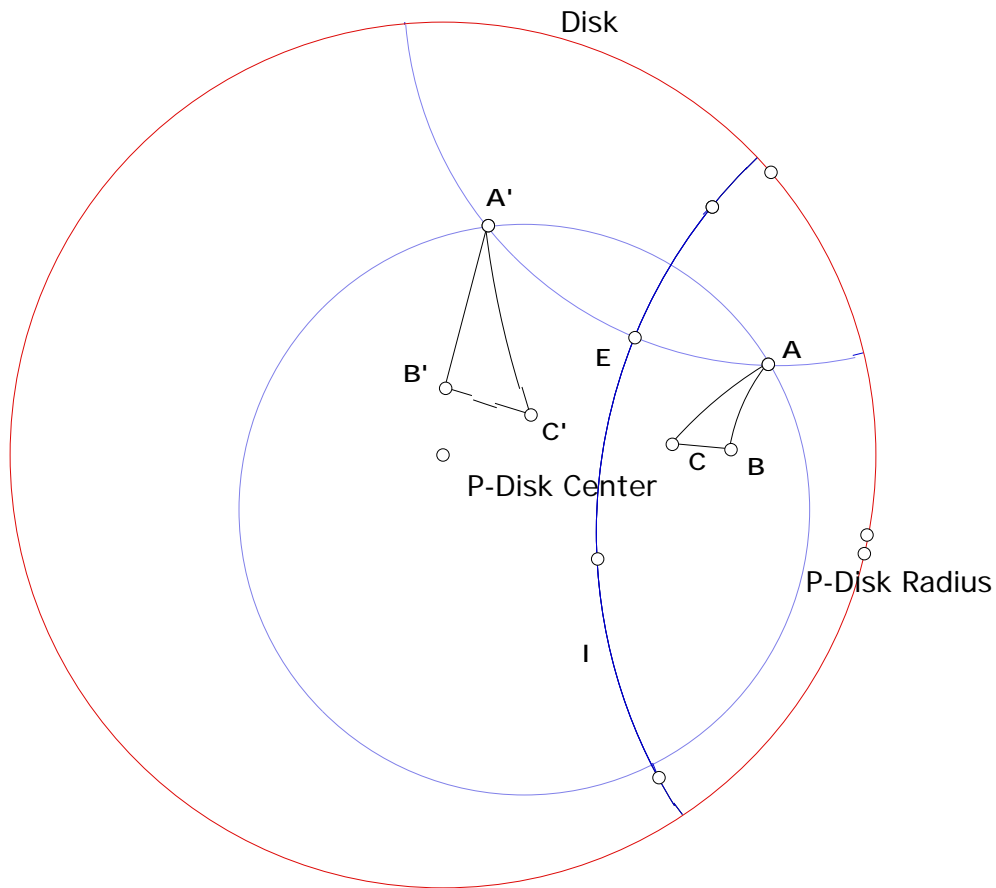
Finally, let's return to the Poincaré disk and Hyperbolic Geometry. We only need to put a few things together to realize that inversion gives us a way to construct h-reflection in an h-line  $l$ . If  $l$  is a diameter of  $C$ , take just the Euclidean reflection in the Euclidean line containing  $l$ . Since this is a Euclidean isometry, cross ratios, h-distance, and h-angle measure are preserved. If  $l$  is the arc of a circle  $C$  orthogonal to the Poincaré Disk, consider inversion with  $C$  as the circle of inversion. This provides the desired h-reflection since  $l$  maps to itself, the half planes of  $l$  map to each other and an inversion is h-distance preserving and h-conformal.





Putting this together our knowledge of inversion we can actually construct specific isometric transformations of the Poincaré Disk. We'll see that there are several useful reasons for doing so. First, let's check this out on Sketchpad.

### 5.4.2b Demonstration. Investigating constructions on the Poincaré Disk.



- Open the Poincaré Disk Starter.
- Construct any h-line  $l$  and then a h-triangle  $ABC$ .
- First, try reflecting the triangle in the h-line using the definition of reflection. First construct the h-line through the vertex  $A$  perpendicular to  $l$ . Then construct the intersection point of  $l$  and the perpendicular line, label it  $E$ . Next construct an h-circle by center  $E$  and point  $A$ . The image point  $A'$  will be the intersection of the circle and the perpendicular line.
- Repeat for  $B$ , and  $C$ . Connect  $A'$ ,  $B'$ , and  $C'$  with h-segments.
- Next, try this again but now using the notion of inversion. Continue with the same sketch.
- Find the “hptref-arc” script. This will allow you to construct the inverse of a point, by only clicking on the arc of the circle of inversion and the point to reflect. (You can easily make this script for yourself. It is just a slightly fancier version of the inverse point

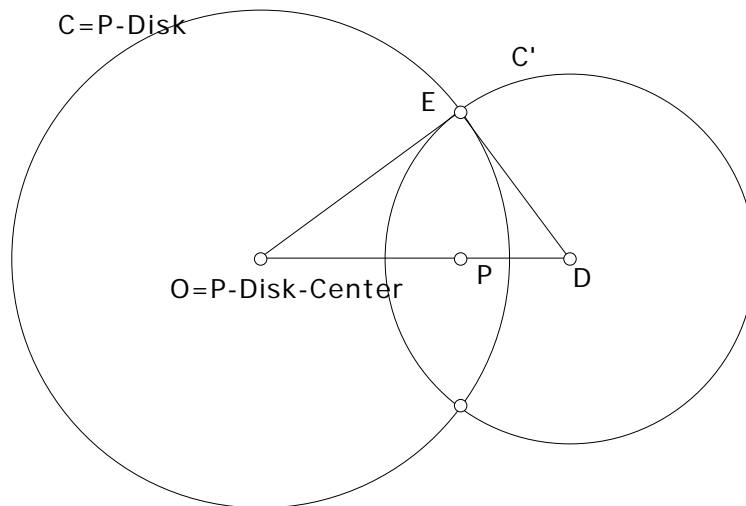
script that we made earlier in the chapter. The steps for the script are given in the next section.)

- Use the script to construct the inverse point for each of  $A$ ,  $B$ , and  $C$ . What do you notice?

**End of Demonstration 5.4.2b.**

**5.4.3 Demonstration. Mapping a point  $P$  to the Origin.** Given a point  $P$  in the Poincaré Disk, describe and then construct the hyperbolic isometry mapping  $P$  onto the origin.

Using Sketchpad we were able to perform an h-reflection, but the question here is to construct a specific h-reflection. What this boils down to is describing the circle  $C$  with respect to which inversion maps  $P$  onto the origin. To ensure that  $C$  is an h-line we also require that  $C$  be orthogonal to the bounding circle  $C$  for the Poincaré Disk.



We have to construct the circle  $C'$  so that  $C'$  is orthogonal to  $C$  and  $DP \cdot DO = DE^2$ . Surprisingly the solution is easy. Let  $D$  be the inverse point to  $P$  with respect to the circle  $C$ . Then  $DO \cdot PO = OE^2$ . Consequently,  $DP \cdot DO = DO(DO - PO) = DO^2 - OE^2 = DE^2$ . Thus,  $D$  is the center of the desired circle as  $O$  and  $P$  will be inverse points. To determine the radius we need to describe  $E$ . The condition  $DP \cdot DO = DE^2$  ensures that  $\triangle OED \sim \triangle EPD$ . Hence the line segment  $EP$  is perpendicular to the line segment  $OD$ . Thus to determine  $E$  we just need to draw the perpendicular to  $OD$  and find the intersection point with  $C$ . The point  $D$  is the intersection of this last perpendicular with the ray from  $O$  passing through  $P$ .

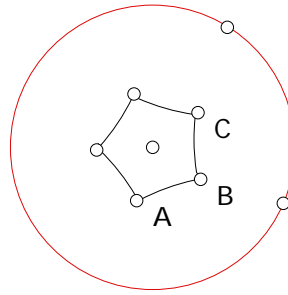
**End of Demonstration 5.4.3.**

Suppose now that we are given **any** two points  $P$  and  $Q$  in the Poincaré Disk. **We can, in fact, construct a hyperbolic isometry of the Disk that maps  $P$  onto  $Q$ .** All we have to do is first construct an isometry mapping  $P$  to the origin, and then construct an isometry mapping the origin to  $Q$ . We can also use this result to prove some results about Hyperbolic geometry. We discovered that the sum of the interior angles of an h-triangle is less than 180 degrees. This is easily seen when the origin is one of the vertices of the triangle for then two of the sides of the h-triangle will be Euclidean Line segments. Given an arbitrary h-triangle we can always map one vertex to the origin using the result above and since inversion is a hyperbolic isometry we can see the result is also true for any triangle.

**5.5 TILINGS OF THE HYPERBOLIC PLANE.** Let's pull together many of the ideas developed in this course by investigating tilings of the hyperbolic plane – in its Poincaré disk model – and then use this to explain the geometry underlying the most sophisticated of Escher's repeating graphic designs. Earlier in Chapter 2 we saw that very few regular polygons could be used to provide edge-to-edge tilings of the Euclidean plane. In fact, only equilateral triangles meeting six at a vertex, squares meeting four times at a vertex, and finally regular hexagons, meeting three times at a vertex. As we have extended to the hyperbolic plane the notion of distance between points and the angle between lines, we can now formulate the notion of a regular h-polygon in exactly the same way as before. A regular h-polygon is a figure in the hyperbolic plane whose edges are h-line segments that have the same length and the same interior angles. What should be noted is that the interior angles of a regular h-polygon can have arbitrary values so long as those values are less than their Euclidean values. Thus for any  $n$ , any regular h-polygon with  $n$  sides will tile the Hyperbolic plane, so long as the interior angle evenly divides 360! The first question we face is the following:

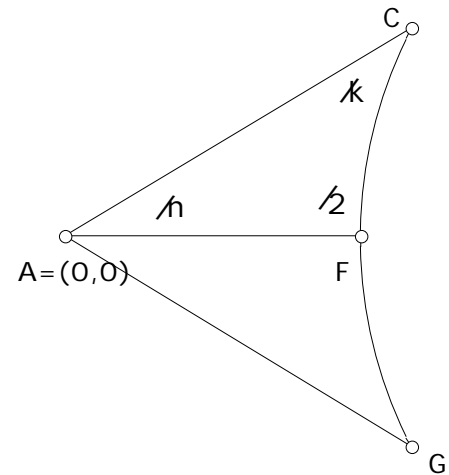
**5.5.1 Demonstration. How do we construct a regular  $n$ -gon that will tile the hyperbolic plane?**

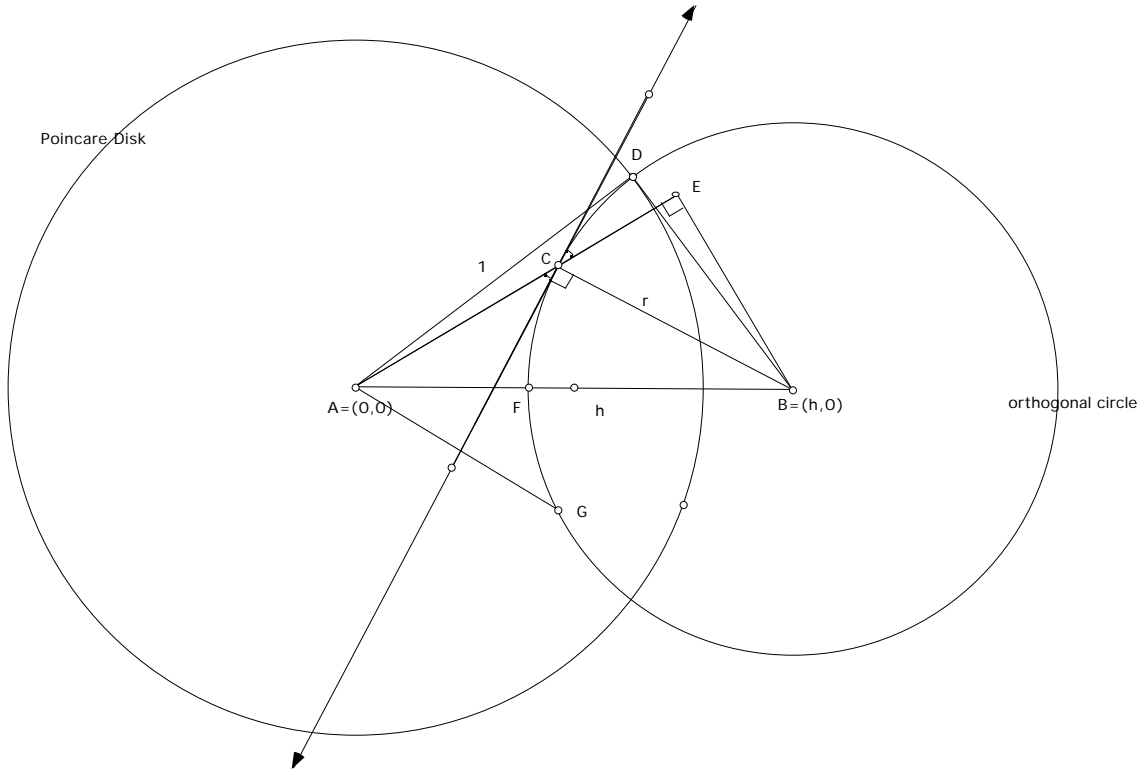
$$\text{h-angle } ABC = 90.0^\circ$$



We will construct our regular  $n$ -gon centered at the center of the Poincaré Disk. The edges of the regular  $n$ -gon are arcs of circles that are orthogonal to the Poincaré Disk. We can find the center of one of those circles by some basic trigonometry. The central h-angles of a regular  $n$ -gon are all equal to  $2\pi/n$ . For our  $n$ -gon to tile the plane the interior h-angles must all be equal to  $2\pi/k$  where  $k$  is an appropriate positive integer. Any regular  $n$ -gon is comprised of  $n$  congruent isosceles triangles.  $AGC$  is one of those isosceles triangles. We will focus our attention on  $AFC$ , where  $\overline{AF}$  is the perpendicular bisector of  $\overline{GC}$ .

Assume the Poincaré Disk has center  $(0,0)$  and radius 1 and that the desired orthogonal circle has center  $(h,0)$  and radius  $r$ . The key step is to extend  $\overline{AC}$  to  $\overline{AE}$  which gives the right  $ABE$ .





We are given that  $m(h - \text{angle } FAC) = \pi/n$  and  $m(h - \text{angle } ACF) = \pi/k$ .

Using trigonometry,

$$\sin(\text{ECB}) = EB/r \text{ and } \sin(\text{CAB}) = EB/h.$$

Thus,

$$r \sin(\text{ECB}) = h \sin(\text{CAB})$$

Now,

- $1 + r^2 = h^2$
- $\sin(\text{ECB}) = \pi/2 - \pi/k$
- $\sin(\text{CAB}) = \pi/n$

yielding  $\sqrt{h^2 - 1} \sin(\pi/2 - \pi/k) = h \sin(\pi/n)$ .

Solving for  $h$  we get,

$$h = \sqrt{\frac{\sin^2(\pi/2 - \pi/k)}{\sin^2(\pi/2 - \pi/k) - \sin^2(\pi/n)}}$$

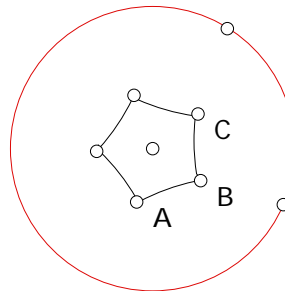
where  $h$  is the center of the orthogonal circle which determines the edge of a regular  $n$ -gon with interior angles equal to  $2\pi/k$ .

**Examples:**

- Regular hexagon meeting 4 at vertex (i.e. interior angles are equal to  $2\pi/4$ ):  $k=4, n=6$  thus  $h = \sqrt{2} \approx 1.414$
- Regular quadrilateral meeting 6 at vertex (i.e. interior angles are equal to  $2\pi/6$ ):  $k=6, n=4$ , thus  $h = \sqrt{3} \approx 1.732$
- Regular pentagon meeting 4 at vertex (i.e. interior angles are equal to  $2\pi/4$ ):  $k=4, n=5$  thus  $h = \sqrt{\sqrt{5} + 1} \approx 1.798$

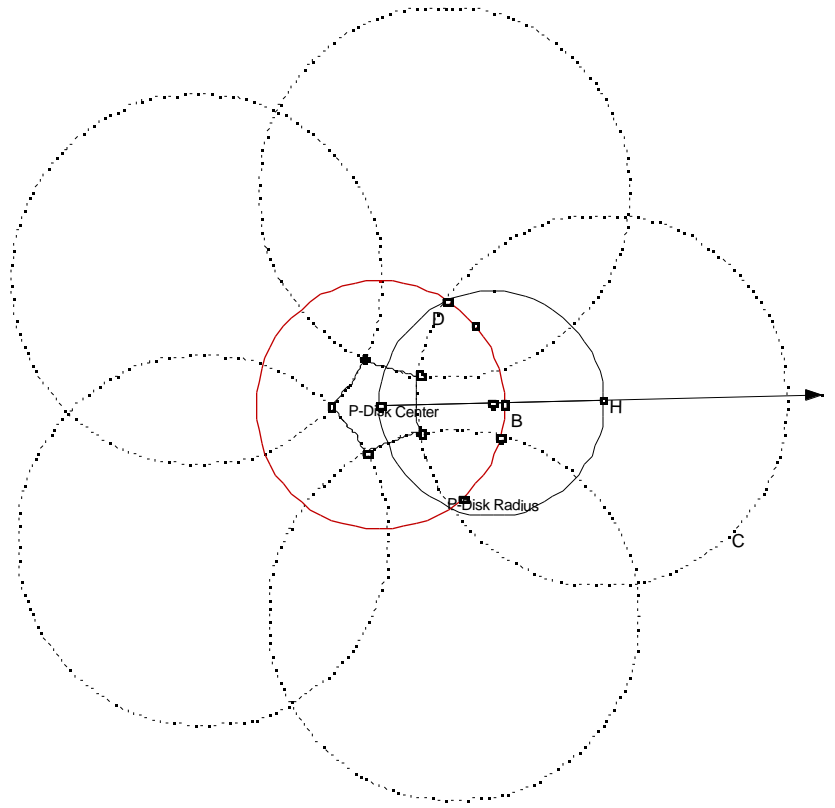
To construct the  $n$ -gon, once we know  $h$ , we can do the following. We'll do the specific case of a regular pentagon,

h-angle ABC = 90.0°



- Open the Poincaré Disk Starter.
- Draw a ray through the disk center. Construct the point of intersection with the Poincaré Disk. Label it  $B$ .
- Select the P-Disk Center and “Mark Center” under the **Transform** menu. Now dilate  $B$  by the scale factor  $= h = 1.798$ . This new point is the center of the desired circle, label it  $H$ .
- Let  $O$  denote the P-Disk Center (do not change the label in your sketch since any script that uses auto-matching will not work). Construct a circle with diameter  $\overline{OH}$ . Then construct one of the points of intersection with the Poincaré Disk, label it  $D$ .

- Construct the circle  $C$  by center  $H$  and point  $D$ .
- Rotate  $C$  about the P-disk Center by  $72$  degrees. Do this  $5$  times.
- Construct the  $5$  points of intersection that are closest to the P-Disk Center. These points are the vertices of the pentagon. Connect them with h-segments. Hide anything that is unwanted.



**End of Demonstration 5.5.1.**



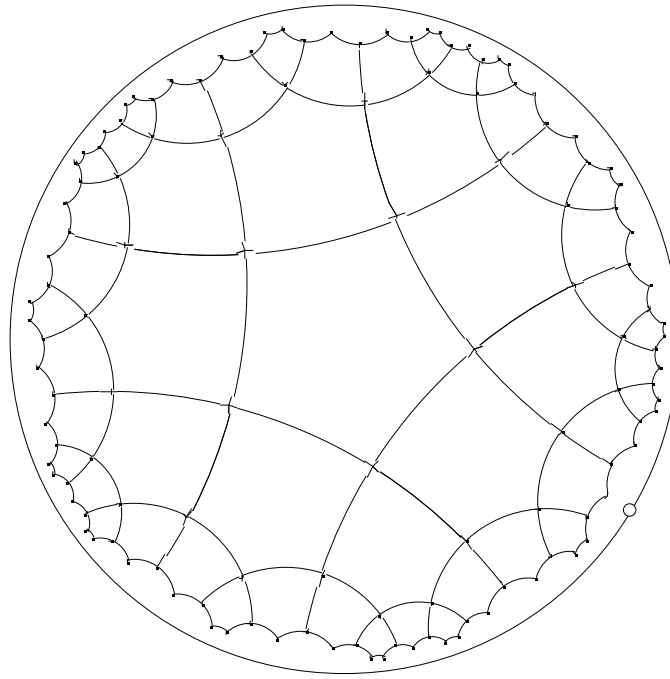
### 5.5.1a Demonstration. Tiling the hyperbolic plane.

Once we have an appropriate starter  $n$ -gon that will tile the hyperbolic plane by meeting  $k$  at a vertex (i.e. the interior angles equal  $2\pi/k$ ) we can tile plane successively h-reflecting the figure. Things will go a little quicker if we also allow ourselves rotations as well. Choose one side of the regular  $n$ -gon and reflect the vertices of the  $n$ -gon across this h-segment (we can accomplish this with an appropriate script since this is equivalent to inverting the vertices with respect to the circle). Then connect the images of these vertices by h-segments. One could continue this process producing a tiling of the plane (up to the memory limitations of SketchPad). To make the process go faster one could also use (Euclidean) rotations about the P-Disk Center of  $2\pi/n$  degrees.

To construct a script that will let you h-reflect an h-point across a given h-segment where you only need to select the h-segment and the h-point do the following. We will make a script that produces the inverse of a given point with respect to a given arc. What will be nice about the script is you will only need to select the arc, not three points that determine the arc.

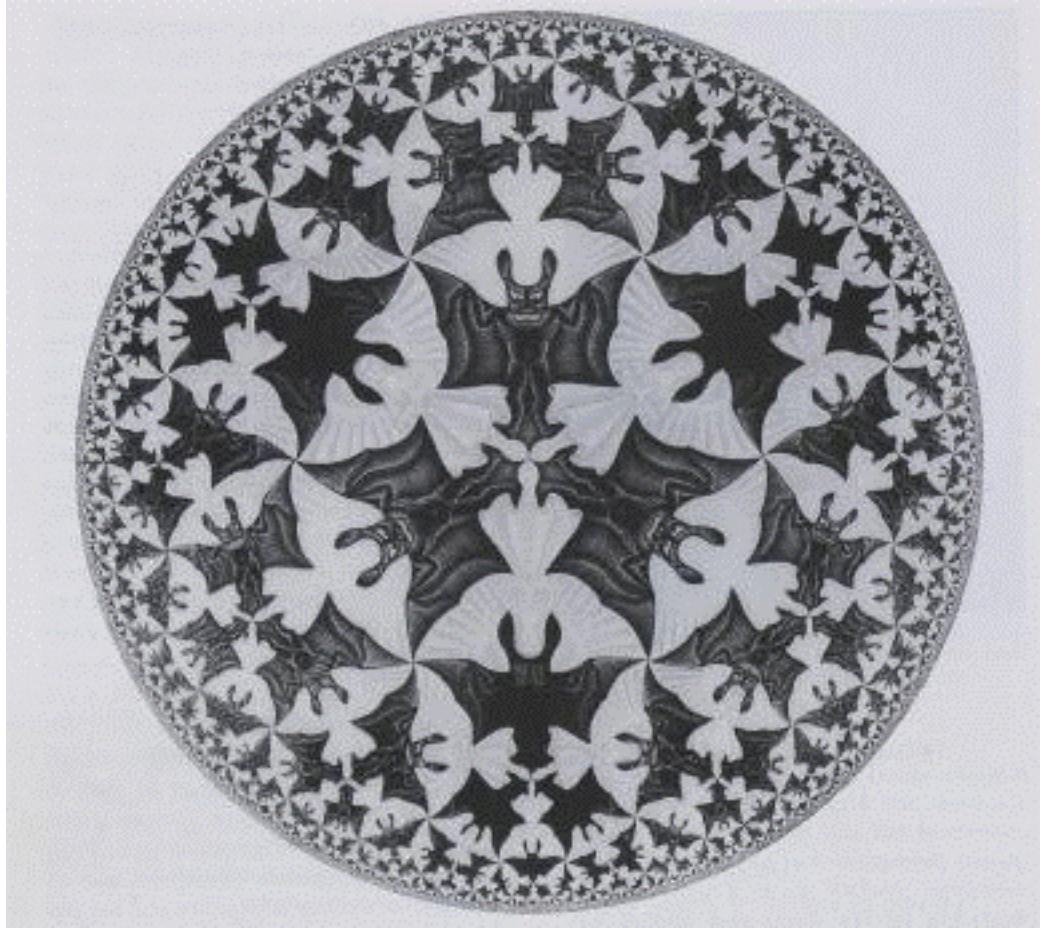
- Open a new sketch and construct an arc through 3 points.
- Open a new script and press record. (It is important to do this after the first step is done.)
- Select 3 points on the arc and construct the circle through the 3 points by any method. Label the center of the circle  $O$  and a point on the circle by  $r$ .
- Construct a point  $P$ . Now, construct the inverse of  $P$  as we did before. You could even run your inverse point script.
- Hide everything except the inverse point (including its label).
- Stop recording and save the script.

Now you can open one the files “quadstr”, “pentstr”, or “hexstr” and start tiling as in the figure below!



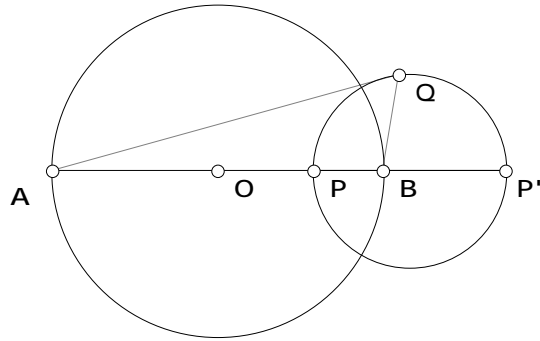
**End of Demonstration 5.5.1a.**

Now go back to Escher's Devils and Angels graphic in the hyperbolic plane (see below). Escher is using "colored" tiles to tile the hyperbolic plane. Can you determine what regular polygon is underlying the tiling? How many are meeting at each vertex?



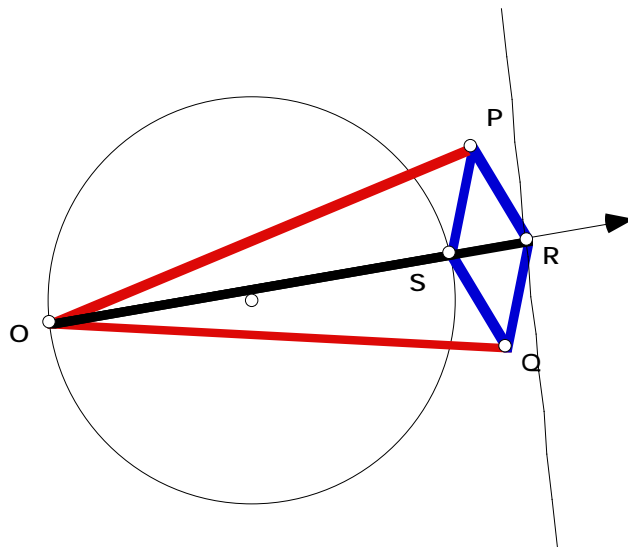
**5.6 Exercises.** These exercises follow up on the connection between inversion and Apollonius' Circle and between inversion and linkages.

**Exercise 5.6.1.** Complete the proof of Conjecture 5.4.2. That is if  $P$  and  $P'$  are inverse points with respect to circle  $C$  and lie on the diameter  $\overline{AB}$  of  $C$  and  $Q$  any point on the circle with diameter  $\overline{PP'}$  then  $\frac{AQ}{BQ} = \frac{AP}{BP}$ . Follow the steps below to give a coordinate geometry proof.



- Let  $A=(-1,0)$ ,  $B=(1,0)$ , and the  $P$  be the point  $(a,0)$ . What are the coordinates of  $P$  ?
- What are the coordinates of the midpoint of the line segment  $\overline{PP'}$  ?
- What is the equation of the circle  $C$  ' ?
- Determine the ratio  $PA/PB$  .
- Determine the ratio  $QA/QB$ .
- Complete the solution by showing  $\frac{AQ}{BQ} = \frac{AP}{BP}$  .

The remaining exercises refer to the Peaucellier linkage and the figure below.



**Exercise 5.6.2.** Using the fact that  $PRQS$  is a rhombus, prove that its diagonals are perpendicular and bisect each other.

**Exercise 5.6.3.** Prove that  $OS \cdot OR$  is a constant by proving that  $OS \cdot OR = OP^2 - PR^2$ . When do  $S, R$  lie on the circle centered at  $O$  having radius  $\sqrt{OP^2 - PR^2}$  ?

**Exercise 5.6.4.** Deduce from Exercise 5.6.3 that the locus of  $R$  is a straight line  $l$  as  $S$  varies over circle  $C$ .

**Exercise 5.6.5.** Prove that  $l$  is perpendicular to the line passing through  $O$  and the center of the circle  $C$ .

**Exercise 5.6.6.** As  $S$  varies over the circle  $C$  does  $R$  vary over all of the (infinite) line  $l$ ? If not, give a precise description of the line segment that  $R$  describes. Can  $S$  go around all of circle  $C$ ? If not, give a precise description of the arc of  $C$  that  $S$  traces.