Congruence and Normal Subgroups, Part II

Recall that we defined the right cosets of a subgroup $H$ in a group $G$ to be the sets $Ha = \{ha | h \in H\}$. We could just as well have defined the left cosets $aH = \{ah | h \in H\}$. Will the left coset $aH$ be the same as the right coset $Ha$? It is certainly true if $G$ is abelian, but what about for a non-abelian group $G$?

Example: In $D_4$, let $r$ be the transformation defined by rotating $\frac{2\pi}{2}$ units about the $z$-axis, let $a$ be rotation $\frac{\pi}{2}$ units about the line $y=x$ in the $x$-$y$ plane, $b$ be rotation $\frac{\pi}{2}$ units about the $x$–axis, and $c$ be rotation $\frac{\pi}{2}$ units about the $y$–axis. Let $H = \{e, a\}$. Since $a$ has order 2, $H$ is a subgroup of $D_4$. Then $rH = \{re, ra\} = \{r, c\}$, and $Hr = \{er, ar\} = \{r, b\}$. So $rH \neq Hr$.

Example: Again in $D_4$, let $K = \{e, r, r^2, r^3\}$. Then $aK = \{ae, ar, ar^2, ar^3\} = \{a, b, d, c\}$ and $Ka = \{ea, ra, r^2a, r^3a\} = \{a, c, d, b\}$. So, for this subgroup $aK = Ka$. It is easy to verify that for any $g \in D_4$, $gK = Kg$.

From these two examples, it is clear that while we cannot assume in general that left and right cosets agree, there are some subgroups $K \triangleleft G$ for which $gK = Kg$ for all $g \in G$. As we shall see shortly, subgroups with this property are particularly nice, and so we give them a name.

Definition: A subgroup $N$ of a group $G$ is called normal if and only if $gN = Ng$ for all $g \in G$.

Caution: It is important to note that the statement $gN = Ng$, does not mean that $gn = ng$ for each $n \in N$. It does mean that for each $n \in N$, there is some $n_1 \in N$ such that $gn = n_1g$.

Problem 1: In $S_3$, let $H = \langle (1 \ 2) \rangle$ and $K = \langle (1 \ 2 \ 3) \rangle$. Show that $K$ is a normal subgroup of $S_3$, but $H$ is not.

Why are we interested in normal subgroups? Recall that what we have done with congruence modulo a subgroup $H$ generalizes what was done with congruence modulo $n$ in the integers. In that case, we were able to define a product on the set of congruence classes modulo $n$, since we were able to show that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$. If you review the proof of this result, you will see that we used the commutative property of the integers under multiplication. For a general group $G$, then, the corresponding result may not hold. In other words, it can happen that $a \equiv b \pmod{H}$ and $c \equiv d \pmod{H}$, but $ac \not\equiv bd \pmod{H}$. 
Example: Let $H = \{e, a\}$ in $D_4$, as in the example above. Since $Hr = \{r, b\}$, $r \equiv b \pmod{H}$. Likewise, since $Hr^2 = \{er^2, ar^2\} = \{r^2, d\}$, $r^2 \equiv d \pmod{H}$, where $d$ is rotation $\frac{\pi}{2}$ units about the line $y=-x$ in the $x$-$y$ plane. But if $rr^2 \equiv bd \pmod{H}$, then $r^3 \equiv r \pmod{H}$, since $bd = r$. But that can only happen if $r^2 \not\in H$, which is clearly not true. So $rr^2 \not\equiv bd \pmod{H}$.

Problem 2: In $S_3$, let $a = (1\ 3)$, and $b = (1\ 3\ 2)$. Let $H = \langle (1\ 2) \rangle$. Show that $a \equiv b \pmod{H}$, but $a^2 \not\equiv b^2 \pmod{H}$.

This means, that for a group $G$ which is non-abelian, we cannot in general define a product on the set of equivalence classes modulo a subgroup $H$. However, in the case that the subgroup is normal, we can as the theorem below states.

Theorem: Let $N$ be a normal subgroup of a group $G$. If $a \equiv b \pmod{N}$ and $c \equiv d \pmod{N}$, then $ac \equiv bd \pmod{N}$.

We will pursue the consequences of this theorem in the next handout, but for now, let’s consider the problem of identifying normal subgroups in a group $G$. Using the definition, as it turns out, is not the easiest way to prove that a subgroup is normal. Other equivalent conditions are given in the theorem below. Using the second condition is often the easiest way to prove that a subgroup is normal.

Theorem: For a subgroup $N$ of a group $G$, the following statements are equivalent:

1. $N$ is a normal subgroup.
2. For each $g \in G$, $gN = \{gn | n \in N\} \subseteq N$.
3. For each $g \in G$, $g^3N = N$.

Caution: As above, it is important to note that the statement $g^3N = N$, does not mean that $g^3n = n$ for each $n \not\in N$. It does mean that for each $n \not\in N$, there is some $n_1 \not\in N$ such that $g^3n = n_1$.

Problem 3: Let $G$ be a group, and define $Z(G) = \{a \in G | ag = ga$ for every $g \in G\}$. $Z(G)$ is called the center of $G$.

(a) Show that $Z(G)$ is a subgroup of $G$.
(b) Show that $Z(G)$ is a normal subgroup.

Problem 4: Show that $N = \{e, r^2\}$ is a normal subgroup of $D_4$. (Hint: show that $r^2 \not\in Z(D_4)$.)

Problem 5: If $M$ and $N$ are normal subgroups of a group $G$, show that $M \cap N$ is normal.